CHAPTER 2

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2.1 INTRODUCTION: In this chapter we deal with some simple commutativity conditions for associative rings. Proofs of most of the results given herein are easy and elementary. Generalizations of some of the theorems were possible by using comparatively heavy machinery but our object throughout the chapter remains that the most of the techniques should depend on mere substitutions.

Section 2.2 begins with a ring-theoretic analogue of the simple group-theoretic result that a group $G$ satisfying the polynomial identity $(xy)^2 = xy^2 x$ is commutative. In section 2.3 we investigate the class of commutative rings which satisfy $[x, y]^2 = [x^2, y^2]$ or $(xoy)^2 = x^2 oy^2$. A result of Herstein [43] asserts that if $R$ is a ring in which the mapping $x \rightarrow x^n$ for a positive integer $n > 1$, is group homomorphism, then every commutator in $R$ is nilpotent. One can easily see that the above rings must satisfy the identity $[x, y^n] = [x^n, y]$ which, of course, does not guarantee the commutativity of rings. In section 2.4 we single out the case when $n = 2$ and study the commutativity of the rings with $[x, y^2] = [x^2, y]$ and also with $x^2 oy = xoy^2$. In the last section we include a result whose proof is though equally simple but depends on a preliminary lemma. This is, in fact,
a warm-up for using heavy and elegant tools in the subsequent text.

2.2 In 1963 Johnsen, Outcalt and Yaqub [61] established the ring-theoretic analogue of an elementary group-theoretic result which states that a group $G$ satisfying $(xy)^2 = x^2y^2$, for all $x$ and $y$ in $G$ is abelian. There are yet many group-theoretic results whose ring-theoretic versions are to be investigated. For example we are not aware of any readily accessible source in literature of the ring-theoretic analogue of the following result in groups. "Let $G$ be a group in which $(xy)^2 = xy^2x$ for all $x$ and $y$ in $G$. Then $G$ is commutative."

We begin with the following result:

**Theorem 2.2.1**: Let $R$ be a ring with unity $1$, satisfying $(xy)^2 = xy^2x$, for all $x$ and $y$ in $R$. Then $R$ must be commutative.

**Proof**: By the hypothesis, we have

1. $(xy)^2 - xy^2x = 0$ for all $x$ and $y$ in $R$.

Replacing $y+1$ for $y$ in (1) and using $(xy)^2 = xy^2x$, we get

2. $x^2y - xyx = 0$ for all $x$ and $y$ in $R$.

Next, putting $x = x + 1$ in (2), we obtain

3. $x^2y - xyx + xy - yx = 0$ for all $x$ and $y$ in $R$.

Thus (3) together with (2) yields $xy = yx$ and $R$ is commutative.
The following example shows that the hypothesis of the existence of the unity in the above theorem is indeed essential.

**EXAMPLE 2.2.1**: Let $R$ be a subring generated by the matrices

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

in the ring of all $3 \times 3$ matrices over $\mathbb{Z}_2$, the ring of integer modulo 2. It can be easily verified that $R$ satisfies $(xy)^2 = xy^2x$, for all $x, y \in R$. However $R$ is not commutative.

2.3 It is well-known that given an associative ring $R$, we can induce on $R$, using its operations, two well-known structures—the Lie structure and the Jordan structure by defining the products $[x, y] = xy - yx$, and $xoy = xy + yx$ respectively. In this section we attempt to replace the associative product of $R$ in the result of Johnsen, Outcalt and Yaqub [61] by either of the induced non-associative structures and investigate the commutativity of the associative product. In fact, we prove the following:

**THEOREM 2.3.2**: Let $R$ be a ring with unity 1 satisfying either of the following identities:

(a) $[x, y]^2 = [x^2, y^2]$ for all $x$ and $y$ in $R$.

(b) $(xoy)^2 = x^2o_2y^2$ for all $x$ and $y$ in $R$.

If, in addition, $\text{char } (R) \neq 2$, then, in each case, $R$ must be
Proof (a): For all $x$ and $y$ in $R$, trivially we have \[(xy-yx)^2 = (yx-xy)^2\] i.e. \([x,y]^2 = [y,x]^2\) and so by the hypothesis \([x^2,y^2] = [y^2,x^2]\) which gives \(2[x^2,y^2] = 0\), implying that \([x^2,y^2] = 0\). Now putting $x = x + 1$ in the identity \([x^2,y^2] = 0\) and using the fact that the char $(R) \neq 2$, we get \([x,y^2] = 0\). Again we set $y = y + 1$, to obtain \([x,y] = 0\). Hence $R$ is commutative.

(b): Replacing $x + 1$ for $x$ in the identity and using \((xoy)^2 = x^2oy^2\), we get \(2(y^2+2xy) = 0\) which implies that \(y^2 + 2xy = 0\), because char $(R) \neq 2$. Now replacing $x + y$ for $y$ the identity $y^2 + 2xy = 0$ yields,

\[(1) \quad 2(x^2y+yx^2) = -(xy+yx) \quad \text{for all } x \text{ and } y \text{ in } R.\]

Putting $y = y + 1$ in (1), we obtain \(2(2x^2+x) = 0\) which gives \(2x^2 + x = 0\). Now set $x = x + y$ in this to get,

\[(2) \quad xy + yx = 0 \quad \text{for all } x \text{ and } y \text{ in } R.\]

Replacing $x + x^2$ for $x$ in (2), we have

\[(3) \quad x^2y + yx^2 = 0 \quad \text{for all } x \text{ and } y \text{ in } R.\]

Putting $y = y + y^2$ in (3), we get

\[(4) \quad x^2y^2 + y^2x^2 = 0 \quad \text{for all } x \text{ and } y \text{ in } R.\]
Now, combining (3) and (4), we obtain

\[(5) \quad (x^2 y - yx^2)y = 0 \quad \text{for all } x \text{ and } y \text{ in } R.\]

Replacing \( y + 1 \) for \( y \) in (5) and using \( x^2 y^2 = yx^2 y \), we get

\[(6) \quad x^2 y - yx^2 = 0 \quad \text{for all } x \text{ and } y \text{ in } R.\]

Again, putting \( x = x + 1 \) in (6) yields \( 2(xy - yx) = 0 \), forcing \( xy = yx \). Hence \( R \) is commutative.

The Example 2.2.1 is enough to show that the unity in the hypothesis of the above result is essential.

Remark 2.2.1: The torsion condition imposed on the hypothesis of our Theorem 2.3.2 is natural also. Indeed, with these torsion restrictions the conditions (a) and (b) which are proved above to be sufficient for commutativity become necessarily as well.

2.4 A result of Herstein [43] asserts that if \( R \) is a ring in which there exists a positive integer \( n > 1 \) such that

\[(1) \quad (x+y)^n = x^n + y^n \quad \text{for all } x \text{ and } y \text{ in } R\]

then every commutator in \( R \) is nilpotent and nilpotent elements of \( R \) form an ideal. We note that in this case \( (x+y) \) and \( (x^n + y^n) \) must commute so that

\[(\beta) \quad [x, y^n] = [x^n, y].\]
It is obvious that the identity (a) can not guarantee the commutativity. In this section we singleout the case for which n = 2 and investigate the conditions for commutativity of rings. We also replace Lie product in (β) by Jordan product. In fact, we prove the following:

**Theorem 2.4.5**: Let $R$ be an associative ring with identity 1 satisfying either of the following identities:

(a) $[x, y^2] = [x^2, y]$ for every $x$ and $y$ in $R$.

(b) $xoy^2 = x^2oy$ for every $x$ and $y$ in $R$.

If $\text{char}(R)$ is not 2, then, in each case $R$ must be commutative.

**Proof of Theorem 2.4.5 (a)**: Proof of this part is straightforward. Just put $x + 1$ for $x$ in the hypothesis and we get the result.

(b) By the hypothesis, we have

(1) $xoy^2 - x^2oy = 0$ for every $x$ and $y$ in $R$.

Replacing $x + y$ for $x$ in (1) and using $xoy^2 = x^2oy$, we get

(2) $y( (x+1)y + (x+1)y ) = 0$ for every $x$ and $y$ in $R$.

Putting $y = x$ for $y$, (2) yields:

$$y( (x+1)y + (x+1)y ) - ( (x+1)y + (x+1)y ) y^2 - 2(x^2y + xy^2) = 0.$$
Now using (2) this becomes

(3) \[ 2(x^2y + yx^2) = 0, \text{ for every } x \text{ and } y \text{ in } R. \]

Since \( \text{char } (R) \neq 2 \), (3) gives

(4) \[ x^2y + yx^2 = 0, \text{ for every } x \text{ and } y \text{ in } R. \]

Hence afterwards proceed in the same way as in the latter part of the proof of Theorem 2.3.2 (b) and get the required result.

2.5 In [50] Derstein proved that if for given elements \( a, b \) in a ring \( R \), there exist integers \( m = m(a,b) > 1, n = n(a,b) > 1 \) such that \( a^m b^n = b^n a^m \), then the commutator ideal of \( R \) is nil. This study was extended by many research workers in different directions. Some of the results can be looked into [1], [2] and [8]. Recently Chang [24] proved the following theorem:

THEOREM 2.5.4 [24] : Let \( R \) be a semi prime ring and suppose that a positive integer \( m \) exists such that for any \( x \) and \( y \) in \( R \) there is a positive integer \( n = n(x,y) \) with \( x^m y^n = y^n x^m \) and \( x^m y^{n+1} - y^{n+1} x^m \) then \( R \) is necessarily commutative.

This naturally gives rise to the following question: "What additional conditions are needed to force the commutativity of \( R \) when \( R \) is an arbitrary ring?" With this
motivation, we establish the following result:

**Theorem 2.5.5**: Let $R$ be a ring with unity and suppose that $m = k$, $k+1$ and $n$ are fixed positive integers greater than $1$, such that for all $x, y \in R$, $x^m y^n = y^n x^m$ and $x^{m+1} y^n = y^{n+1} x^m$. Then $R$ must be commutative.

In preparation for the proof of the above theorem we need the following lemma:

**Lemma 2.5.1**: Suppose $x$ and $y$ are elements of $R$ satisfying $x^p [x, y] = 0$ and $(x+1)^p [x, y] = 0$, for some positive integer $p$. Then $[x, y] = 0$.

**Proof**: By our hypothesis, we have

(1) \((x+1)^p [x, y] = 0\) for all $x, y$ in $R$.

Expanding $(x+1)^p$ in (1), we get

(2) \(x^p [x, y] + pC_{p-1} x^{p-1} [x, y] + \ldots + pC_1 [x, y] + [x, y] = 0\).

If $p = 1$. Then by the assumption $x^p [x, y] = 0$, the result follows immediately from (2). Suppose $p > 1$. Multiply equation (2) by $x^{p-1}$ from the left and use the hypothesis $x^p [x, y] = 0$ to get $x^{p-1} [x, y] = 0$ and consequently:

(3) \(pC_{p-2} x^{p-2} [x, y] + \ldots + pC_1 x [x, y] + [x, y] = 0\).

If $p = 2$, then since $x^{p-1} [x, y] = 0$ and the result follows
at once from (3). Suppose \( p > 2 \). Again multiplying equation (3) by \( x^{p-2} \) from the left, and using \( x^{p-1} \) \([x,y] = 0 \) to get \( x^{p-2} \) \([x,y] = 0 \). Continue this process until we get \([x,y] = 0 \) for all \( x \) and \( y \) in \( R \). This proves the Lemma 2.5.1.

**Proof of Theorem 2.5.5**: By our hypothesis, we have

\[
y^{n+1}x^m = x^ny^{n+1} = x^my^n = y^nx^m.
\]

This yields

\[
(y+1)^n[y,x^m] = 0 \quad \text{for all } x, y \in R.
\]

Replacing \( y+1 \) for \( y \) in (1) we also get,

\[
(y+1)^n[y,x^m] = 0 \quad \text{for all } x, y \in R.
\]

Combining (1) and (2) and using Lemma 2.5.1 we obtain

\[
[y,x^m] = 0 \quad \text{for all } x, y \in R.
\]

Now for \( m = k, k+1 \), we have

\[
x^ky = yx^k
\]

and

\[
x^{k+1}y = yx^{k+1}
\]

Combining (4) and (5), we obtain

\[
x^k[x,y] = 0 \quad \text{for all } x, y \in R.
\]

Replacing \( x + 1 \) for \( x \) in (6) we get
(7) \( (x+1)^k [x,y] = 0 \) for all \( x \) and \( y \) in \( R \).

Now by Lemma 2.5.1, (6) and (7) yield that \( xy = yx \). Hence \( R \) is commutative.