CHAPTER-I

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1.1. HISTORICAL BACKGROUND:

A sequence space is defined to be a linear space with elements in another space. Summability can be roughly considered as the study of linear transformations on sequence spaces. The theory originated from the attempts of mathematicians to assign limits to divergent sequences. The earliest idea of summability theory was perhaps contained in a letter written by Leibnitz to C. Wolf (1713) in which he attributed the sum \( \frac{1}{2} \) to the oscillatory series \( 1-1+1-1+1 \ldots \ldots \ldots \). Frobenius (1880) introduced the method of summability by arithmetic means, which was generalized by Cesàro (1890) as the \((C,k)\) method of summability. Towards the end of nineteenth century, study of the general theory of sequences and transformations on them attracted mathematicians, who were chiefly motivated by problems such as those in summability theory, Fourier series, Power series and systems of equations with infinitely many variables.
Toeplitz ([68]) was the first person to study summability method as a class of transformations of complex sequences by complex infinite matrices. Subsequently it was studied by many outstanding mathematicians like Köjima, Steinhaus, Schur, Mazur, Knopp, Agnew, Cooke, Petersen and many others.

With the emergence of Functional Analysis, sharper techniques were available and sequence spaces were studied with renewed vigour, greater insight and motivation. Earliest applications of Functional Analysis to Summability was made by Banach, Hahn, Mazur, Köthe and Toeplitz during the third decade of the present century, followed by a line of many enthusiastic workers such as Lorentz, Zeller, Borwein, Sargent, Russel and so on. A large literature has grown up concerning the characterization of all classes of matrices which transform one given sequence space into another (see Cooke [9], Hardy [13], Ruckle [53], Peyerimhoff [49], Stieglitz [66], Wilansky [69,70], Maddox [30,31] and Malkowsky [33,34] and the chain continues endlessly.

The object of the present thesis is to study characterization problems of classes of matrices in some sequence spaces.
In Chapter II, we introduce the sequence spaces $c_0(p,s)$ and $\ell_\infty(p,s)$. We also determine their K"othe-Toeplitz duals. Finally we determine the necessary and sufficient conditions on the matrix sequence $A = (A_j)$ such that $A \in (X,Y)$, where $X = \ell_\infty c_0(p,s)$, $\ell_\infty(p,s)$ and $\ell(p,s)$ and $Y = m(\phi)$.

In Chapter III, we introduce the sequence space $bs(p,s)$ and determine the necessary and sufficient conditions on the matrix sequence $A = (A_j)$ such that $A \in (X,Y)$, where $X = \ell_\infty bs(p,s)$ and $Y = \ell_\infty bs, c$ and $\hat{cs}$ respectively.

In Chapter IV, we determine the necessary and sufficient conditions on the matrix sequence $A = (A_j)$ such that $A \in (X,Y)$, where $X = \ell(p,s)$ and $Y = \ell_\infty c, bs, \hat{cs}$ and $cs$.

Chapter V, has been devoted to characterize the matrices of the classes $(\ell(p,s), F_B)$ and $(ces(p,s), F_B)$.

In Chapter VI, we determine the matrices of the classes $(c_0, BV_{\sigma})$, $(c(p), BV_{\sigma})$, $(c_0(p), BV_{\sigma})$, $(M_0(p), BV_{\sigma})$ and $(\ell_1, BV_{\sigma})$.

In Chapter VII, we give by compilation, a systematic account of various existing characterization theorems.
1.2. Here we recall the following well known definitions.

**Definition 1.2.1. Characteristic of a Conservative Matrix**

Let $A \in (c,c)$. Then $A$ is called a conservative (or convergence preserving) matrix. If, in addition,

$$\lim_{n} A_n(x) = \lim_{n} x_n,$$

then $A$ is called regular and we write $A \in (c,c;P)$. For a conservative matrix $A$,

$$\chi(A) = \lim_{n} \sum_{k} a_{nk} - \sum_{k} (\lim_{n} a_{nk}),$$

is called the characteristic of $A$. The numbers $\lim_{n} a_{nk}$ $(k = 1,2,...)$ and $\lim_{n} \sum_{k} a_{nk}$ are referred to as the characteristic numbers of $A$.

**Definition 1.2.2. Co-regular and Co-null Matrices**

Let $A \in (c,c)$. Then $A$ is co-regular if and only if $\chi(A) \neq 0$, and $A$ is co-null otherwise.

**Definition 1.2.3. Coercive Matrix**

The matrix $A$ is coercive if and only if $Ax \in c$ for all $x \in l_\infty$, i.e., $A \in (l_\infty,c)$. 
Definition 1.2.4. **BK-Space.** (See Zeller [75]):

A BK-space $X$ is a linear Banach space in which the coordinate maps are continuous, i.e.,

$$
\lim_{m \to \infty} |x_k^{(m)} - x_k^{(m)}| \to 0,
$$

whenever $\|x^{(m)} - x\| \to 0$, as $m \to \infty$ and $x = (x_k), x^{(m)} = (x_k^{(m)})$.

Definition 1.2.5. **Paranormed Space**

A linear topological space $X$ is called a paranormed space if there exists a subadditive function $g : X \to \mathbb{R}$, such that $g(0) = 0$, $g(x) = g(-x)$ and the multiplication is continuous, that is, $\lambda_n \to \lambda$ and $g(x_n - x) \to 0$, implies that $g(\lambda_n x_n - \lambda x) \to 0$, for $\lambda \in \mathbb{C}$ and $x \in X$.

For results in this context references may be made to Chapter VII.

1.3. **BANACH LIMIT AND ALMOST CONVERGENCE:**

Definition 1.3.1. Banach Limit (See [5], P.33-34).

Let $x = (x_n)$ be a bounded sequence, and

$$
P(x_n) = \inf_{n_1, n_2, \ldots, n_k} \limsup_{j \to \infty} \frac{1}{k} \sum_{p=1}^{k} x_{n_p+j},
$$

(1.3.1)
where \( k \) is a positive integer and \( n_1, n_2, \ldots, n_k \) is an arbitrary subset of integers.

Then a linear functional \( L \) which satisfies the condition

\[
L(x_n) \leq P(x_n),
\]

for all bounded sequences \( x \), is called a Banach limit.

The following theorem on Banach limit is well-known,

(See Petersen [48], Theorem 3.1.5).

**THEOREM 1.3.1.** A Banach limit \( L \), satisfies the following conditions:

1. \( L(ax_n) = a \ L(x_n) \), for all \( a \),
2. \( L(x_n+y_n) = L(x_n) + L(y_n) \),
3. \( L(x_{n+1}) = L(x_n) \),
4. \( L(e) = 1 \),
5. \( x_n \geq 0 \ (n = 1, 2, \ldots) \) implies \( L(x_n) \geq 0 \).

1) For the definition of linear functionals and other concepts of Functional Analysis, reference may be made to Maddox [30].
In 1948, Lorentz [23] introduced a new concept of convergence, which is narrowly connected with the limit of S. Banach. The sequences which are summable by this method are called almost convergent sequences.

Definition 1.3.1. (See [48]):

A sequence $x \in l_\infty$ is said to be almost convergent to $\ell$ if each Banach limit of $x$ is $\ell$.

The space of almost convergent sequences is denoted by $c^\wedge$.

Lorentz also characterized the space $c^\wedge$ in the form of the following theorem.

Theorem 1.3.2. ([23], Theorem 1):

A sequence $x = (x_n)$ is almost convergent to $\ell$ if and only if

$$\lim_{k \to \infty} \frac{x_n + x_{n+1} + \cdots + x_{n+k-1}}{k} = \ell,$$

uniformly in $n$.

The space of all almost convergent series is denoted by $cs^\wedge$ and defined by:
\[ \hat{c}_s : = \left\{ x : \left( \sum_{n=0}^{k} x_n \right) \in \hat{c} \right\}, \]

and

\[ \hat{c}_o s : = \left\{ x : \left( \sum_{n=0}^{k} x_n \right) \in \hat{c}_o \right\}. \]

will denote the space of all almost convergent series, whose generalized sum is zero.

Using this new concept of convergence King [19], and Eizen and Laush [11] have introduced more general classes of matrices than the conservative, regular and coercive matrices.

**Definition 1.3.2. Almost Conservative Matrix**

The matrix \( A \) is said to be almost conservative if and only if \( Ax \in \hat{c} \), for all \( x \in c \), that is, \( A \subseteq (c, \hat{c}) \).

**Definition 1.3.3. Almost Regular Matrix**

The matrix \( A \) is said to be almost regular if \( A \subseteq (c, \hat{c}) \) and \( L(A_n(x)) = \lim_{n} x_n \); and we write \( A \subseteq (c, \hat{c};P) \).

**Definition 1.3.4. Almost Coercive Matrix**

The matrix \( A \) is said to be almost coercive if and only if \( Ax \in \hat{c} \), for all \( x \in \ell_\infty \), and we write \( A \subseteq (\ell_\infty, \hat{c}) \).

For results in this context reference may be made to Chapter VII.
1.4. **INVARIANT MEANS AND $\sigma$-CONVERGENCE**:

Schaefer [58], generalized the concept of Banach limits and almost convergence to those of Invariant Means (or $\sigma$-means) and $\sigma$-convergence.

**Definition 1.4.1. Invariant Means:**

Let $\sigma$ be a mapping of $\mathbb{Z}^+$ into itself. A continuous linear functional $\phi$ on $\ell_\infty$ into itself is said to be an invariant mean (or a $\sigma$-mean) if and only if

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$, for all $n$,
2. $\phi(e) = 1$,
3. $\phi(x_{\sigma(n)}) = \phi(x)$, for all $x \in \ell_\infty$.

For $x = (x_n)$, we write

$Tx = (x_{\sigma(n)})$.

Analogous to almost convergence, $\sigma$-convergence has also been characterized as:
THEOREM 1.4.1. (See [58]):

A sequence $x = (x_n) \in l_\infty$ is $\sigma$-convergent if and only if

$$\lim_{m \to \infty} \frac{x + T x + \ldots + T^m x}{m+1} = l' e,$$

uniformly in $n$, $l'$ being the common value of all $\sigma$-means $x$ (see [50]), that is $l' = \sigma - \lim x$.

Schaefer [58], gave the analogous notions of $\sigma$-conservative, $\sigma$-regular and $\sigma$-coercive matrices as generalizations of almost conservative, almost regular and almost coercive matrices.

**Definition 1.4.2. $\sigma$-Conservative Matrix**

The matrix $A$ is said to be $\sigma$-conservative if and only if $Ax \in c$, for all $x \in c$, and we write $A \in (c,c)$.

**Definition 1.4.3. $\sigma$-Regular Matrix**

The matrix $A$ is said to be $\sigma$-regular if and only if $A$ is $\sigma$-conservative and $\sigma$-$\lim Ax = \lim x$, for all $x \in c$, and we write $A \in (c,c;\sigma)$.

**Definition 1.4.4. $\sigma$-Coercive Matrix**

The matrix $A$ is said to be $\sigma$-coercive if and only if
Ax \in C$ for all $x \in \ell_\infty$, and we write $A \in (\ell_\infty, C)$.

For results in this context references may be made to Chapter VII.

1.5. Here we state the Banach-Steinhaus theorem and some other results which are suited to dealing with many problems in the theory of matrix transformations and sequence spaces.

**THEOREM 1.5.1. Banach-Steinhaus Theorem (See [30]):**

If $(A_n)$ is a sequence of bounded linear operators each defined on a Banach space $X$ into a normed space $Y$, and

$$\limsup_n \|A_n(x)\| < \infty, \text{ on } X,$$

then $\sup_n \|A_n\| < \infty$, that is the sequence $(\|A_n\|)$ of norms is bounded.

**THEOREM 1.5.2. Uniform boundedness principle (see [30]):**

Let $P$ be a collection of lower semicontinuous functions $p$ defined on the second category matric space $X$, and suppose $p(x) \leq M(x) < \infty$, for each $x \in X$, all $p \in P$,

then, there exists a sphere $S$ in $X$ and a constant $M$ such that
\[ p(x) \leq M, \text{ for each } x \in S \text{ and all } p \in P. \]

S. Banach and Steinhaus have proved a principle of condensation of singularities given below:

**THEOREM 1.5.3.** (See [72]):

Given a sequence of bounded linear operators \( \{ T_n \} \) defined on a Banach space \( X \) into a normed linear space \( Y \), then the set

\[
B := \left\{ x \in X : \lim_{n \to \infty} \| T_n x \| < \infty \right\},
\]

either coincides with \( X \) or is a set of the first category of \( X \).

**THEOREM 1.5.4.** (See [21], P.100):

For any \( C > 0 \) and any two complex numbers \( a \) and \( b \),

\[
|ab| \leq C (|a|^p + |b|^q),
\]

where

\[
p > 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.
\]