CHAPTER-0

NOTE ON CONVENTIONS
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Here we state a few conventions, which will be used throughout the thesis, other definitions will be introduced as they become necessary.

0.1. THE SYMBOLS $\mathbb{Z}^+$, $\mathbb{R}$ AND $\mathbb{C}$:

$\mathbb{Z}^+$, $\mathbb{R}$ and $\mathbb{C}$ will be used to denote, respectively the set of all positive integers, the set of all real numbers and the set of all complex numbers.

0.2. LIMIT, SUP AND INF:

By 'lim', 'sup' and 'inf' we mean 'lim' 'sup $n \to \infty$ n=0,1,2,...' and 'inf $n=0,1,2$' respectively.

0.3. CONSTANT $K$:

Throughout we write $K$ to denote an absolute constant, not necessarily the same at each occurrence.
0.4. **SUMMATION CONVENTIONS:**

By \( \sum_{\alpha}^{\beta} f(n) \), we mean the sum of all values of \( f(n) \) for which \( \alpha \leq n \leq \beta \); if \( \beta < \alpha \), this is zero. Summations are over 0, 1, 2, ..., when there is no indication to the contrary. If \((x_n) = (x_1, x_2, \ldots)\) is a sequence of terms, then by \( \sum x_n \) we mean \( \sum_{n=1}^{\infty} x_n \), and we shall sometimes write as \( \sum x_n \), where no possible confusion can arise.

0.5 **MATRIX A:**

Throughout this thesis \( A \) denotes an infinite matrix \((a_{nk})\).

0.6 **FINITE DIFFERENCES:**

For any sequence \((f_n)\), we write

\[
\Delta^0 f_n = f_n;
\]

\[
\Delta f_n = \Delta^1 f_n = f_n - f_{n+1};
\]

0.7 **SEQUENCES** \(x, p; e\) AND \(e_{(k)}\):

\(x = (x_k)\) denotes any sequence whose \(k\)-th term is \(x_k\) and \(p = (p_k)\) a sequence of strictly positive numbers with \(\sup p_k < \infty\). \(e_{(k)}\) and \(e\), denote the sequences
\( e^{(k)} := \{0,0,\ldots,0,1(\text{kth place}),0,0,\ldots, \text{ for all } k \in \mathbb{Z}^+ \} \),

and

\( e := \{1,1,\ldots\} \),

respectively.

0.8. **SEQUENCE SPACES:**

\( W \) denotes the space of all sequences real, or complex.

We denote by

\[ l_{\infty} := \left\{ x : \sup_{k} |x_k| < \infty \right\}, \]

\[ c := \left\{ x : |x_k - l| \longrightarrow 0 \text{ for some } l \right\}, \]

\[ c_0 := \left\{ x : |x_k| \longrightarrow 0 \right\}, \]

respectively the Banach spaces of bounded, convergent and null sequences with the usual norm:

\[ \| x \| = \sup_{k} |x_k|. \]

We note that

\[ c_0 \subset c \subset l_{\infty}. \]

\[ BV := \left\{ x : \Sigma_{k} |x_k - x_{k-1}| < \infty, x_0 = 0 \right\}, \]

the space of sequences of bounded variation is a Banach space.
with the usual norm:

$$\| x \| = \sum_{k} |x_k - x_{k-1}|.$$  

Also, we define

\[ cs : = \{ x : \{ \sum_{n=0}^{k} x_n \} \text{ is convergent} \}, \]

\[ bs : = \{ x : \sup_{k} |\sum_{n=0}^{k} x_n| < \infty \}, \]

\[ \ell_p : = \{ x : \sum_{k} |x_k|^p < \infty \}, 0 < p < \infty \; \text{and} \]

\[ W_p : = \{ x : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell|^p = 0, \text{for some } \ell \in C \}. \]

Let us write

\[ x_n = z_0 + z_1 + \cdots + z_n, \text{ for all } z = (z_n) \in W; \]

\[ t_{m,n}(x) = \frac{1}{m+1} \sum_{j=n}^{m+n} x_j, \]

and

\[ \phi_{m,n}(x) = t_{m,n}(x) - t_{m-1,n}(x) = \frac{1}{m(m+1)} \sum_{j=1}^{m} j(x_{n+j} - x_{n+j-1}). \]

Then

\[ \hat{c} : = \{ x : \lim_{m} t_{m,n}(x) \text{ exists uniformly in } n, x \in \ell_\infty \}. \]
A \chi : \lim_{m \to \infty} t_{m,n}(x) > 0, \text{ as } m \to \infty, \text{ uniformly in } n, x \in l_{\infty} \}
\hat{c}_0 : = \left\{ x : \lim_{m \to \infty} t_{m,n}(x) \to 0, \text{ as } m \to \infty, \text{ uniformly in } n, x \in l_{\infty} \right\}
\hat{\ell}_\infty : = \left\{ x : \sup_{m,n} |\phi_{m,n}(x)| < \infty, x \in l_{\infty} \right\}

\text{and}
\hat{B}V : = \left\{ x : \sum_{m} |\phi_{m,n}(x)| < \infty, \text{ uniformly in } n, x \in l_{\infty} \right\}

Again, let us write
\sigma_{t_{m,n}(x)} = \frac{1}{m+1} \sum_{j=0}^{m} T_{x_{n}}, T_{x_{n}} = x_{\sigma(n)};

where \sigma is a 1-1 mapping of \mathbb{Z}^+ into itself, such that \sigma^{m}(n) \neq n, \text{ for all } n \in \mathbb{Z}, \text{ where } \sigma^{m}(n) \text{ denotes the } m^{th} \text{ iterate of the mapping } \sigma \text{ at } n; \text{ and }
\phi_{m,n}(x) = \sigma_{t_{m,n}(x)} - \sigma_{t_{m-1,n}(x)}
\phi_{m,n}(x) = \frac{1}{m(m+1)} \sum_{j=1}^{m} j[x_{\sigma^{j}(n)} - x_{\sigma^{j-1}(n)}].

We define the spaces [(p_k) is the same as defined in 0.7]:
\sigma_{l_{\infty}} : = \left\{ x : \sup_{m,n} |\phi_{m,n}(x)| < \infty, x \in l_{\infty} \right\}
\sigma_{c} : = \left\{ x : \lim_{m} \sigma_{t_{m,n}(x)} \text{ exists, uniformly in } n, x \in l_{\infty} \right\}
\[
\begin{align*}
\mathcal{C}(p) & := \left\{ x : \lim_{k \to \infty} |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathcal{C} \right\}, \\
\mathcal{C}_0(p) & := \left\{ x : \lim_{k \to \infty} |x_k|^{p_k} = 0 \right\}, \\
\ell(p) & := \left\{ x : \sum_k |x_k|^{p_k} < \infty \right\}, \\
\ell_\infty(p) & := \left\{ x : \sup_k |x_k|^{p_k} < \infty \right\}, \\
\hat{\mathcal{C}}(p) & := \left\{ x : \lim_m \left| t_{m,n}(x) - \ell e \right|^{p_k} = 0, \text{ uniformly in } n, \text{ for some } \ell \in \mathcal{C} \text{ and } x \in \ell_\infty \right\}, \\
\hat{\mathcal{C}}_0(p) & := \left\{ x : \lim_m \left| t_{m,n}(x) \right|^{p_k} = 0, \text{ uniformly in } n, x \in \ell_\infty \right\}, \\
\hat{\ell}_\infty(p) & := \left\{ x : \sup_{m,n} \left| \phi_{m,n}(x) \right|^{p_k} < \infty, x \in \ell_\infty \right\}, \\
\sigma(p) & := \left\{ x : \lim_m \left| t_{m,n}(x) - \ell' e \right|^{p_k} = 0, \text{ uniformly in } n, \text{ for some } \ell' \in \mathcal{C} \right\}, \\
\sigma_\infty(p) & := \left\{ x : \sup_{m,n} \left| \psi_{m,n}(x) \right|^{p_k} < \infty, x \in \ell_\infty \right\}, \\
M_0(p) & := \bigcup_{N>1} \left\{ x : \sum_k |x_k|^{-1/p_k} < \infty \right\}, \\
W(p) & := \left\{ x : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathcal{C} \right\}, \\
\text{ces}(p) & := \left\{ x : \sum_{r=0}^{\infty} \left( \frac{1}{2^r} \sum_{k=2^r}^{2^{r+1}} |x_k| \right)^{p_k} < \infty \right\}.
\end{align*}
\]
If $p_k = p$, for all $k$, we have $c(p) = c$, $c_0(p) = c_0$, $l(p) = l_p$, $l_\infty(p) = l_\infty$, $c(p) = c$, $l_\infty(p) = l_\infty$, $c_0(p) = c_0$, $M_0(p) = l_1$ and $W(p) = W_p$.

0.9. CONVERGENT AND ABSOLUTELY CONVERGENT SERIES:

If $\sum a_n$ be a given series of complex terms, then $\sum a_n < \infty$ and $\sum |a_n| < \infty$ symbolize respectively the convergence and absolute convergence of the series $\sum a_n$.

0.10. KÖTHE-TOEPLITZ AND CONTINUOUS DUALS OF SEQUENCE SPACES:

Let $E$ be a set of complex sequences, then its Köthe-Toeplitz dual denoted by $E^+$, is defined as,

$$E^+ := \left\{ a = (a_k) : \sum |a_k x_k| < \infty, \text{ for all } x \in E \right\}.$$

$E^+$ will denote the set of all continuous linear functionals on $E$.

0.11. CLASS OF MATRICES:

Let $X$ and $Y$ be any two non-empty subsets of the space $\mathcal{W}$. Let $A = (a_{nk})$, $(n, k = 1, 2, \ldots)$ be an infinite matrix of real, or complex numbers. We write
Then $A_n(x) = \sum_k a_{nk} x_n$. Then $Ax = \left\{ A_n(x) \right\}$ is called the $A$-transform of $x$. Also

$$\lim_n Ax = \lim_{n \to \infty} A_n(x),$$

whenever it exists. If $x \in X$ implies $Ax \in Y$, we say that $A$ defines a matrix transformation from $X$ into $Y$, denoted by $A : X \longrightarrow Y$. By $(X,Y)$ we mean the class of matrices $A$ such that $A : X \longrightarrow Y$. By $(X,Y;P)$ we mean the subset of $(X,Y)$ for which limits or sums are preserved.