CHAPTER-VI

MATRIX TRANSFORMATIONS OF SEQUENCES OF $\sigma$-BOUNDED VARIATION
6.1. **Definitions and Notations:**

In addition to the notations and definitions given in the preceding chapters, we follow the following:

We know that (see Schaefer [58]),

\[ \sigma_c := \left\{ x : \lim_{m} t_{mn} (x) \text{ exists, uniformly in } n \right\}, \]

where

\[ t_{mn}(x) = \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n. \]

For \( m,n \geq 0 \), put

\[ \psi_{mn}(x) = t_{mn}(x) - t_{m-1,n}(x). \]

When \( m \geq 1 \), we have

\[ \psi_{mn}(x) = \frac{1}{m(m+1)} \left[ m \sum_{j=0}^{m} T^j x_n - (m+1) \sum_{j=0}^{m-1} T^j x_n \right] \]

\[ = \frac{1}{m(m+1)} \left[ m T^m x_n - (x_n + T x_n + \ldots + T^{m-1} x_n) \right] \]

\[ = \frac{1}{m(m+1)} \sum_{j=1}^{m} j[T^j x_n - T^{j-1} x_n]. \]
$= \frac{1}{m(m+1)} \sum_{j=1}^{m} j[ x_{j\sigma}(n) - x_{j\sigma-1}(n) ].$

Now, we define (See Mursaleen [39]):

$\text{BV}_\sigma := \left\{ x : \sum_{m} |\psi_{mn}(x)| < \infty, \text{ uniformly in } n \right\}.$

When $\sigma(n) = n+1$, the space $\text{BV}_\sigma$ is same as the space $\hat{\text{BV}}$ of almost bounded variation (see Nanda [45]).

Also, we write

$\psi_{mn}(Ax) = \sum_{k} \sum_{j=1}^{m} j[ a(\sigma(n),k) - a(\sigma^j(n),k) ] x_k / m(m+1)$

$= \sum_{k} g(n,k,m) x_k,$

where

$g(n,k,m) = \frac{1}{m(m+1)} \sum_{j=1}^{m} j[ a(\sigma(n),k) - a(\sigma^{j-1}(n),k) ].$

6.2. INTRODUCTION:

It is natural to accept that $\sigma$-convergence must be related to some concept of $\text{BV}_\sigma$ in the same vein as convergence is related to the concept of BV. Mursaleen [39], has characterized the classes of matrices $(\text{BV},\text{BV}_\sigma)$ and $(\ell_\infty,\text{BV}_\sigma)$. The object of this chapter is to determine the necessary and
sufficient conditions for the classes of matrices \((c_o, BV_o), (c(p), BV_o), (M_o(p), BV_o)\) and \((\ell_1, BV_o)\).

6.3. Here we state the following result which will be used in the proof of our Theorems.

**Lemma 6.1.** (See Mursaleen [39]):

\[BV_o \subset c\] and this inclusion is proper.

6.4. We prove the following theorems:

**Theorem 6.1.** \(A \in (c_o, BV_o)\) if and only if \((6.4.1), (6.4.2)\) hold.

\[(6.4.1) \quad \sum_{m} \sum_{k} |g(n,k,m)| \leq K, \text{ for all } n;\]

\[(6.4.2) \quad \lim_{m} g(n,k,m) = u_k, \text{ uniformly in } n, \text{ for all } k.\]

**Proof. Necessity.** Suppose that \(A \in (c_o, BV_o)\). Write

\[q_n(x) = \sum_{m} |\psi_{mn}(Ax)|.\]

Now \((q_n)\) is a sequence of continuous seminorms on \(c_o\) such that \(\sup_n q_n(x) < \infty\) for all \(x \in c_o\). Therefore, by Banach–Steinhaus theorem there exists a constant \(K > 0\), such that
\[ q_n(x) \leq K \| x \| \quad \text{(for all } x \in c_0, \text{ for all } n). \]

For each \( r \in \mathbb{Z}^+ \), define a sequence

\[ x = (x_k) = \begin{cases} \text{sgn } g(n,k,m), & 0 \leq k \leq r; \\ 0, & k > r. \end{cases} \]

Then \( x \in c_0, \| x \| = 1 \) and

\[ q_n(x) = \sum_{m} \sum_{k=0}^{r} |g(n,k,m)|. \]

Therefore,

\[ \sum_{m} \sum_{k} |g(n,k,m)| \leq K \quad \text{(for all } n). \]

Now, since \((c_0, BV_0) \subset (c_0, c)\) by Lemma 6.1, the condition (6.4.1) must hold (see Schaefer [58]).

**SUFFICIENCY.** Suppose that the conditions (6.4.1), (6.4.2) hold and \( x \in c_0 \). Now,

\[ \sum_{m} |\psi_{mn}(Ax)| \leq \sum_{m} \sum_{k} |g(n,k,m)| (\sup_{k} |x_k|) \leq K \| x \|, \]

\(< \infty, \text{ uniformly in } n. \)
Now, \( Ax \in \sigma \text{BV}_\sigma \). Consequently \( Ax \in \sigma \) by Lemma 6.1, and therefore (see Schaefer [58])

\[
\lim \sum_{m,k} g(n,k,m) x_k = \Sigma u_k x_k,
\]

uniformly in \( n \).

This completes the proof of Theorem 6.1.

**Theorem 6.2**

(a) \( A \in (c(p), BV_\sigma) \) if and only if (6.4.3), (6.4.4), (6.4.5) hold.

(6.4.3) There exists an integer \( B > 1 \), such that

\[
\sum_{m,k} |g(n,k,m)|^{1/p_k} \leq K \quad \text{(for all } n),
\]

(6.4.4) \( \lim_{m,k} g(n,k,m) = u_k \), uniformly in \( n \), for each \( k \),

(6.4.5) \( \lim_{m,k} \sum g(n,k,m) = u \), uniformly in \( n \).

(b) \( A \in (c(p), BV_\sigma ; P) \) if and only if (6.4.3), (6.4.4) with \( u_k = 0 \), for each \( k \) and (6.4.5) with \( u = 1 \), hold.

**Proof.** (a). **Necessity.** Suppose that \( A \in (c(p), BV_\sigma) \).

Since \( (c(p), BV_\sigma) \subset (c(p), c) \), (6.4.4) and (6.4.5) must hold [2].
Put
\[ f_n(x) = \sum_m |\psi_{mn}(Ax)|. \]

Since \((c(p),BV_\sigma)^0(c_0(p),BV_\sigma), \{f_n\}\) is a sequence of continuous linear functionals on \(c_0(p)\) such that \(\sup_n f_n(x) < \infty\), therefore, by uniform boundedness principle, for \(0 < \delta < 1\), there exists a closed sphere \(S_\delta[0] \subset c_0(p)\) and a constant \(K\) such that, for every \(x \in S_\delta[0]\),
\[(6.4.6) \quad f_n(x) \leq K \quad (\text{for all } n, \text{ for all } x \in c(p)).\]

In particular, put
\[ x_k = \begin{cases} 
\delta^{K/p} & \text{sgn } g(n,k,m), 0 \leq k \leq r; \\
0 & k > r,
\end{cases} \]
in (6.4.6), where \(r\) is arbitrary. Then the condition (6.4.3) holds for \(B = \delta^{K} \).

**Sufficiency.** Suppose that the conditions (6.4.3), (6.4.4), (6.4.5) hold and \(x \in c(p)\). Then, there exists \(\ell \in c\), such that \(|x_k - \ell|^{p_k} \to 0\) \((k \to \infty)\). Hence, for a given \(0 < \varepsilon < 1\), there exists an integer \(k_0\) such that, for all \(k > k_0\),
and therefore, for $k > k_0$,

$$\frac{1}{p_k} \left| x_k - \ell \right| < B \left| x_k - \ell \right|, \quad \left( \frac{\varepsilon}{(K+1)} \right)^{M/p_k} < \frac{\varepsilon}{K+1},$$

where $M = \max \{1, \sup_k p_k\}$.

Therefore, we have

$$\sum_m |\psi_{mn}(Ax)| = \sum_m \left| \sum_k g(n,k,m)x_k \right| = \sum_m \left| \sum_k g(n,k,m)(x_k - \ell + \ell) \right|$$

$$(6.4.7) \quad \leq \sum_m \left| \sum_k g(n,k,m)(x_k - \ell) \right| + \sum_m \left| \sum_k g(n,k,m) \ell \right|. $$

Now,

$$\sum_m \sum_{k > k_0} g(n,k,m) (x_k - \ell) \left| x_k - \ell \right| B^{1/p_k} \quad - \frac{1}{p_k} \quad \frac{1}{p_k}$$
(6.4.8) \[ \sum_{m} \sum_{k}^{\frac{\varepsilon}{K+1}} |g(n,k,m)| B^{-1/p_k} \]

and

\[ \sum_{m} |\sum_{k} g(n,k,m)| \ell B^{-1/p_{k}^{1/p_{k}}} \]

\[ \leq B^{1/\ell} \]

(6.4.9) \[ \leq B \]

\[ \leq K \]

\[ < \infty, \text{ uniformly in } n, \]

where \( \inf_{k} p_k = \Theta \) and \( 0 < p_k \leq 1 \).

Hence, combining (6.4.7), (6.4.8) and (6.4.9) we have

\[ \sum_{m} |\psi_{mn}(Ax)| < \infty, \text{ uniformly in } n. \]

Now, \( Ax \in BV_{g} \) and consequently \( Ax \in c \). Therefore, we have [2],

\[ \lim_{m} \sum_{k} g(n,k,m) x_{k} = \ell u + \sum_{k} u_{k} (x_{k} - \ell), \]

uniformly in \( n \).

Proof of (b) is immediate if we observe that \( u_{k} = 0 \), for each \( k \), and \( u = 1 \).

This completes the proof of Theorem 6.2.
COROLLARY 6.1. \( A \in (c_0(p), BV_\sigma) \) if and only if (6.4.3) and (6.4.4.) hold.

PROOF. Since \( \ell = 0 \) in this case the proof is immediate.

THEOREM 6.3. \( A \in (M_0(p), BV_\sigma) \) if and only if (6.4.10) and (6.4.11) hold.

(6.4.10) For every \( N > 1 \),
\[
\sum_{m} \sum_{k} \frac{1}{p_k} |g(n,k,m)| N < \infty, \text{ uniformly in } n.
\]

(6.4.11) \( \lim_{m} g(n,k,m) = u_k, \text{ uniformly in } n, \text{ for each } k. \)

PROOF. NECESSITY. Suppose that \( A \in (M_0(p), BV_\sigma) \). Since \( (M_0(p),BV_\sigma) \subseteq (M_0(p),c) \) the condition (6.4.11) must hold [38]. If (6.4.10) is not true, then \( B = (b_{nk}) = (a_{nk} N 1/p_k) \notin (M_0(p), BV_\sigma) \) for some integer \( N > 1 \). So, there exists \( x \in M_0(p) \) such that \( Bx \notin BV_\sigma \). Now, \( y = (y_k) = (x_k N 1/p_k) \in M_0(p) \), but \( Ay = Bx \notin BV_\sigma \), which contradicts the fact that \( A \in (M_0(p), BV_\sigma) \). Hence, (6.4.10) is true.

SUFFICIENCY. Suppose that the conditions (6.4.10), (6.4.11) hold and \( x \in M_0(p) \). Then,
\[ \sum_{m,k} \left| g(n,k,m) x_k \right|^\frac{1}{p_k} \leq \sum_{m,k} \left| g(n,k,m) \right| \left| x_k \right|^N < \infty. \]

Now, \( Ax \in BV_\sigma \) and consequently \( Ax \in c_0 \), by Lemma 6.1. Therefore, we have (see Mursaleen [38])

\[ \lim_{m,k} \sum_{n} g(n,k,m) x_k = \sum_{k} u_k x_k, \]

uniformly in \( n \).

This completes the proof of Theorem 6.3.

**Theorem 6.4.** \( A \in (\ell_2, BV_\sigma) \) if and only if

(6.4.12) \( \sum_{m,k} \left| g(n,k,m) \right| \leq \infty \), for all \( n \),

(6.4.13) \( \lim_{m} g(n,k,m) = u_k \), uniformly in \( n \), for each \( k \).

**Proof.** Sufficiency and Necessity of (6.4.13) are trivial. The necessity of (6.4.12) can be obtained by an analysis similar to Theorem 6.3.