CHAPTER 3

FRACTIONAL INTEGRAL FORMULA AND TRANSFORMATIONS OF
HYPERGEOMETRIC SERIES

3.1 INTRODUCTION

Saran (1954), Srivastava (1964), Exton (1976), Karlsson and Srivastava (1985) have discussed many transformations and interesting instances of the reducibility of triple hypergeometric functions. These results are obtained mainly by manipulation of series. Of the various methods available for obtaining transformations of multiple hypergeometric functions, the manipulations of integrals representing the functions may often be employed to good effect, and this will be the approach adopted in this chapter. The main interest of the result to be given in this chapter is that the Eulerian integral formula (obtained in this chapter in section 3.2) is evaluated in terms of Lauricella's functions $F_5$ and $F_D$ and Srivastava's function $F^{(3)}$. Certain transformations of Appell's functions $F_1$ and $F_3$ and Kampé de Fériet's function $F_{p:q;k}^{f:m;n}$ are readily obtainable from our main results listed in section 3.3. On account of
the large number of such transformations and reduction formulas, we restrict ourselves here to those functions which involve more than one variable.

As is well known, the task of unifying and collecting transformations relating to various multiple hypergeometric functions is considerably simplified by means of appropriate literature from which we should single out the encyclopedic work of Erdelyi et al. [26], [27], Exton [34], Karlsson and Srivastava [130] and Prudnikov et al. [99]. However they contained only formulae up to the end of 80's and this has led to the need for creating and collecting more transformations and reduction formulae in which new results are reflected. This chapter is an attempt to tabulate a number of results on double and triple hypergeometric series which are not included in the references given above. Also such results are stated as a fractional integral formulas involving the operator

3.2 MAIN TRANSFORMATIONS

In this section, the following eight transformations are derived
\[
F_S[\alpha, \beta, \gamma, \lambda, \delta; \alpha+\beta, \alpha+\beta, \alpha+\beta, x, y, z] = (1-Z) \left(3 \begin{array}{c}
\alpha+\beta \\
\alpha+\beta \\
\alpha+\beta \\
\lambda, \beta; \delta; \\
x, y, \frac{z}{z-1}
\end{array} \right)
\] (3.2.1)

\[
F \left(3 \begin{array}{c}
\alpha+\beta-\delta \\
\alpha+\beta-\delta \\
\alpha+\beta-\delta \\
\alpha+\beta-\delta \\
\lambda, \alpha, \gamma, \lambda;
\end{array} \right)
\] (3.2.2)

\[
= (1-Z) \left(3 \begin{array}{c}
\alpha+\beta-\delta \\
\alpha+\beta-\delta \\
\alpha+\beta-\delta \\
\alpha+\beta-\delta \\
\lambda, \alpha, \gamma, \lambda;
\end{array} \right)
\] (3.2.3)

\[
F_D \left[\beta, \gamma, \lambda, \delta; \alpha+\beta; \frac{x}{x-1}, y, z\right] = (1-X) \left(3 \begin{array}{c}
\alpha+\beta \\
\alpha+\beta \\
\alpha+\beta \\
\lambda, \beta; \\
x, y, \frac{z}{z-1}
\end{array} \right)
\] (3.2.4)

\[
F \left(3 \begin{array}{c}
\alpha+\beta-\gamma-\delta \\
\alpha+\beta-\gamma-\delta \\
\alpha+\beta-\gamma-\delta \\
\alpha+\beta-\gamma-\delta \\
\lambda, \beta;
\end{array} \right)
\] (3.2.5)
In order to obtain the above transformations we establish an integral in the form

\[ I = \int_{a}^{b} \left( \frac{t-a}{b-t} \right)^{\alpha-1} \left( \frac{ut+v}{yt+z} \right)^{\gamma} \left( \frac{pt+q}{\lambda} \right)^{\sigma} \, dt \]

\[ = (bp + q) \left( au + v \right)^{\gamma} \left( by + z \right)^{\delta} \left( b-a \right)^{\alpha+\beta-1} B(\alpha, \beta) \]
To obtain (3.2.9), express \((ut + v), (yt + z)\) and \((pt + q)\) in terms of series with the help of the result [129; p. 76 (2.4)]

\[
(p + q)^{\lambda} = (bp + q)^{\lambda} \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} \left\{ \frac{(b-t)p}{bp+q} \right\}^m
\]

and integrate term by term with the help of the result [129; p. 78 (3.3)]

\[
\int_{a}^{b} (t-a)^{\alpha-1} (b-t)^{\beta-1} (ut + v) (yt + z)^{\delta} \, dt
\]

\[
= (b-a)^{\alpha+\beta-1} (au+v)^{\gamma} (by+z)^{\delta} F_3(\alpha, \beta, \gamma, \delta; \alpha+\beta; \frac{(b-a)u}{au+v}, \frac{(b-a)v}{by+z})
\]
to get

\[ I = (bp + q)^\lambda (by + z)^\delta (au + v)^\gamma (b - a)^{\alpha + \beta - 1} B(\alpha, \beta) \]

\[ \sum_{m=0}^{\infty} \frac{(-\lambda)_m (\beta)_m}{(\alpha + \beta)_m m!} \left\{ \frac{(b-a)p}{bp+q} \right\}^m \]

\[ F_3[\alpha, \beta + m, -\gamma, -\delta; \alpha + m; -\frac{(b-a)u}{au + v}, \frac{(b-a)v}{by + z}] \quad (3.2.12) \]

Again expressing \( F_3 \) in series and interpreting by (1.4.7), we arrive at (3.2.9).

If we apply a relation [5; p. 25 (34)]

\[ F_3[\alpha, \gamma - \alpha, \beta, \beta'; \gamma; x, y] \]

\[ = (1-y)^{-\beta'} F_1[\alpha, \beta, \beta'; \gamma; x, \frac{y}{y-1}] \quad (3.2.13) \]

in (3.2.12) then we have

\[ I = (bp + q)^\lambda (au + v)^\gamma (by + z)^\delta (b-a)^{\alpha + \beta - 1} B(\alpha, \beta) \left\{ \frac{ay+z}{by+z} \right\}^\delta \]

\[ \sum_{m=0}^{\infty} \frac{(-\lambda)_m (\beta)_m}{(\alpha + \beta)_m m!} \left\{ \frac{(b-a)p}{bp+q} \right\}^m \]
Again expressing $F_1$ in series and interpreting by (1.4.11), we obtain

\[
F_1[\alpha, -\gamma, -\delta; \alpha + \beta + m; \frac{(b-a)u}{au + v}, \frac{(b-a)v}{ay + z}] \quad (3.2.14)
\]

Equating (3.2.9) with (3.2.15) and adjusting the parameters, we get the transformation (3.2.1).

Similarly, if we apply \[131; pp. 300 (82) and 300 (84)\]

\[
F_3[\alpha, \alpha', \beta, \beta'; \gamma; x, y] = (1-y)^{-\alpha'} F_{1:2;1}^{1:1;0} \left[ \begin{array}{c} \gamma - \beta' \quad \alpha', \beta' \quad \alpha' \\ \alpha + \beta \quad \gamma \quad \frac{y}{y-1} \\ \gamma \quad \gamma - \beta \quad - \\ x \end{array} \right],
\]

and

\[
= (1-y)^{\gamma - \alpha' - \beta'} F_{1:2;0}^{1:2;0} \left[ \begin{array}{c} \gamma - \alpha', \gamma - \beta' \quad \alpha, \beta \quad - \\ \gamma \quad \gamma - \alpha' \quad \gamma - \beta' \quad - \\ \gamma \quad \gamma - \alpha', \gamma - \beta' \quad - \end{array} \right],
\]
to (3.2.12) and interpret with the help of (1.4.11) and (3.2.9), we get the transformations (3.2.2) and (3.2.3).

Also if we apply the relations [26; pp. 239 (1,2 and 3) and 240 (4 and 5)]

\[ F_1[\alpha, \beta, \beta'; \gamma; x, y] \]

\[ = (1-x)^{-\beta} (1-y)^{-\beta'} F_1[\gamma-\alpha, \beta, \beta'; \gamma; \frac{x}{x-1}, \frac{y}{y-1}] \]

\[ = (1-x)^{-\alpha} F_1[\alpha, \gamma-\beta-\beta', \beta'; \gamma; \frac{x}{x-1}, \frac{y}{y-1}] \]

\[ = (1-y)^{-\alpha} F_1[\alpha, \beta, \gamma-\beta-\beta'; \gamma; \frac{y-x}{y-1}, \frac{y}{y-1}] \]

\[ = (1-y)^{-\gamma-\alpha-\beta} (1-y)^{-\beta'} F_1[\gamma-\alpha, \gamma-\beta-\beta', \beta'; \gamma; x, \frac{y-x}{y-1}] \]

\[ = (1-x)^{-\beta} (1-y)^{-\gamma-\alpha-\beta'} F_1[\gamma-\alpha, \beta, \gamma-\beta-\beta'; \gamma; \frac{x-y}{x-1}, y] \]

to (3.2.14) and interpret with the help (1.4.11) and (3.2.9), we get the transformations (3.2.4), (3.2.5), (3.2.6), (3.2.7) and (3.2.8).
3.3 SPECIAL CASES

In this section, certain transformations of Appell's functions $F_1$ and $F_3$ and Kampé de Fériet's function $F_{1:q:k}$ are obtained from our main results. Some of these special cases of formulas of section (3.2) would provide useful generalizations of the known results of hypergeometric functions and are listed below:
Each of our transformation formulas (3.2.1) to (3.2.8), and indeed their special cases listed in the table given above can be suitably applied to derive a number of results involving a remarkably wide variety of hypergeometric functions of one and more variables. For example, if we choose \( \alpha = \beta \), \( \gamma = \delta \) and \( z = -x \); \( \alpha = \beta \), \( \gamma = \lambda \) and \( \gamma = -x \); \( \gamma = 0 \) and \( z = 0 \), then the table listing various special cases can be employed with a view to deriving transformations and reductions involving simpler special functions.

Next we turn to the applications of our main integral formulas (3.2.9), (3.2.12), (3.2.14) and (3.2.15). For \( b = x \), each of the Eulerian integral formula involving the (fractional) differintegral operator \( \alpha^D_x f(x) \) defined by (cf. e.g., Oldham and Spanier [88], Samko et al. [104], and Miller and Ross [77])

\[
\alpha^D_x \{ f(x) \} = \begin{cases} 
\frac{1}{\Gamma(-\nu)} \int_0^x (x-t)^{-\nu-1} f(t) \, dt \quad (a \in \mathbb{R}; \, \text{Re}(\nu) < 0) \\
\frac{d^m}{dx^m} \alpha^D_x \{ f(x) \} \quad (0 \leq \text{Re}(\nu) < m; \, m \in \mathbb{N} = \{1, 2, \ldots\}) 
\end{cases}
\]

(3.3.1)
provided the integral exists. In fact when \( \alpha = 0 \), the operator

\[
D_\alpha^\nu = _0D_\alpha^\nu, \quad \nu \in \mathcal{F},
\]

(3.3.2)
corresponds to the classical Riemann–Liouville fractional derivative (or integral) of order \( \nu \) (or \(-\nu\)). Thus for example, our integral formulas (3.2.9), (3.2.12), (3.2.14) and (3.2.15) yield the following results which are valid under the conditions stated already (with, of course, \( b = x \)).

\[
F_3[\alpha, \beta, -\gamma, -\delta; \alpha + \beta, \alpha + \beta, \alpha + \beta; \frac{(x-a)p}{au + v}, \frac{(x-a)y}{xy + z}] = \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta)} \frac{\Gamma(\lambda)(\lambda + \gamma + \delta)}{\Gamma(\lambda + \gamma + \delta + 1)} \frac{\Gamma(\lambda + \gamma + \delta)(xy + z)}{\Gamma(\lambda + \gamma + \delta + 1)}
\]

(3.3.3)

\[
F_3[\alpha, \beta + m, -\gamma, -\delta; \alpha + \beta + m; \frac{(x-a)p}{au + v}, \frac{(x-a)y}{xy + z}] = \sum_{m=0}^{\infty} \left( \frac{\Gamma(-\lambda)}{\Gamma(\lambda + m)} \left( \frac{\Gamma(\beta)}{\Gamma(\beta + m)} \right) \right) m \frac{(x-a)p}{x^m + q^m}
\]

(3.3.4)
\[\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \left( \frac{x+p}{x+y+z} \right)^{\beta} \left( \frac{a+u+v}{x+y+z} \right)^{\beta} \delta^{a+\beta-1} \left[ \frac{a+y+z}{x+y+z} \right]^{\delta}\]

\[\sum_{m=0}^{\infty} \frac{(-\lambda)^{m}}{(\alpha+\beta)^{m}m!} \left\{ \frac{(x-a)^{p}}{x+p+q} \right\}^{m}\]

\[F_{1}[\alpha,-\gamma,-\delta;\alpha+\beta+m; -\frac{(x-a)u}{au+v}, -\frac{(x-a)v}{ay+z}] \quad (3.3.5)\]

\[\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \left( \frac{x+p}{x+y+z} \right)^{\beta} \left( \frac{a+u+v}{x+y+z} \right)^{\beta} \delta^{a+\beta-1} \left[ \frac{a+y+z}{x+y+z} \right]^{\delta}\]

\[F(3) \left[ \alpha : \gamma; \lambda, \beta ; -\delta ; \alpha+\beta : \beta ; -\beta ; -\beta ; -\beta ; -\beta ; \right.\]

\[-\frac{(x-a)u}{(au+v)}, \frac{(x-a)p}{x+p+q}, -\frac{(x-a)v}{ay+z} \left.\right]. \quad (3.3.6)\]