1.1 INTRODUCTION

The theory of differential equations has originated a large number of new functions. We call them as special functions, special in the sense that they arise as the solutions of special problems. Most of these applications lie in the field of applied mathematics, statistics, physics, engineering, quantum mechanics, Lie theory and theory of elasticity et cetera.

Theory of special functions plays a basic role in the formalism of mathematical physics and applied mathematics. It covers extremely wide domain of study, firstly formulated by the pioneering works of Euler, Gauss, Laplace, Bessel, Legendre, Jacobi, Hermite, Laguerre, Riemann and many others, secondly by Whittaker, Watson, Ramanujan, Appell, Ragab, Hardy, Lebdev, Erdelyi, Chaundy, Meijer, Bailey, Fox et cetera and continuously refined by new achievements and suggestions within the context of applied sciences.
Generating functions, summations, transformations and reduction formulae have been studied, stimulated by pure mathematical curiosity as well as by specific problems. In the theory of special functions, summation, transformations and reduction formulae have received some attention (little in author's opinion) during the last years. To quote a few recent research papers where transformations play a crucial role in the study of special functions, we recall the work of Abiodun [1], Antone [3], Bawaja[6], [7], Bhonsle [10], Carlitz [15], [16], [17], Carlson [18] Chandel [19], Chhabra and Rusia [20], Dange [21], De Oliveira [23], Deshpande [24], Erdelyi [29], [30], Exton [31], [32], [33], [35], Gottschalk [38], Grosjean and Sharma [40], Grosjean and Srivastava [41], Joshi [51], Kalnins and Manocha [53] Kalia [52], Kant and Koul [55], Karlsson [56], [57], [58], Khan and Pathan [60], [61], Kita [63], Lavoie and Grondin [67], Mahajan [68], [69], Manihar and Banarji [70], Mendas [74], Miller [75], Miller and Srivastava [76], Misra and Paliwal [81] Wali and Qureshi [82], Mukherjee [81], Nguyen [84], Okamoto [86], Daalhuis [87], Pathan [90], [92], [93], [94], [95], Pathan,
Qureshi and Khan [97], Qureshi and Wali [101], Sala [103], Saran [106], Sharma [108], [109], Sharma and Ahmad [110], Sharma [111], Singh and Bhatt [113], [114], Singh and Singh [115], Srivastava [118], [119], [123], [125], Srivastava and Damjanovic [127], Srivastava and Exton [128], Srivastava and Panda [130], Tuan and Kalla [134], Viskov [135] and Yan [137].

This Chapter aims at introduction of several classes of special function which occur rather more frequently in the study of applied science. We present some basic definitions and properties of special functions needed for the presentation of subsequent chapters.

We first give the definitions of Gamma functions (and related functions) and then proceed to hypergeometric functions (and their generalization) in section 1.2, while a brief account of other hypergeometric functions of two and several variables are presented in section 1.3 and 1.4 respectively. In 1.5 we present the definitions of orthogonal polynomial and some hypergeometric representations of polynomials. A concept of generating functions (and their classifications) is given in the last section 1.6.
1.2 GAUSSIAN HYPERGEOMETRIC SERIES AND ITS GENERALIZATIONS

With a view to introducing the Gaussian hypergeometric series and its generalizations, we recall here some definitions and identities involving Pochhammer's symbol \((a)_n\), Gamma function \(\Gamma(z)\) and the related functions.

**The Gamma Function**

One of the simplest but very important special functions is the Gamma function \(\Gamma(z)\) defined by

\[
\Gamma(z) = \begin{cases} 
\int_0^\infty e^{-t} t^{z-1} \, dt, & \text{Re}(z) > 0 \\
\frac{\Gamma(z+1)}{z}, & z \neq 0, -1, -2, \ldots 
\end{cases}
\]  

(1.2.1)

In fact, the Gamma function \(\Gamma(x)\) is a generalization of the factorial function \(x!\) from the domain of positive integers to the domain of all real numbers except as \(0, -1, -2, \ldots\).

**The Pochhammer's Symbol**

The Pochhammer symbol \((a)_n\) is defined by
In particular, $(1)_n = n!$, hence the symbol $(\alpha)_n$ is also referred to as the factorial function.

In terms of Gamma functions, we have

$$ (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} , \quad \alpha \neq 0,-1,-2,\ldots \quad (1.2.3) $$

Furthermore, the binomial coefficient may now be expressed as

$$ \binom{\alpha}{n} = \frac{\alpha(\alpha+1)\ldots(\alpha-n+1)}{n!} = \frac{(-1)(-\alpha)_n}{n!} . \quad (1.2.4) $$

Also, we have

$$ (\alpha)^{m+n} = (\alpha)^{m} (\alpha^{m}) , \quad (1.2.5) $$

$$ (\alpha)^{-n} = \frac{(-1)^n}{(1-\alpha)^n} , \quad n \geq 1; \quad \alpha \neq 0, \pm 1, \pm 2, \ldots \quad (1.2.6) $$

and

$$ (\alpha)^{n-k} = \frac{(-1)^k (\alpha)^{n-k}}{(1-\alpha-n)_k} , \quad 0 \leq k \leq n. \quad (1.2.7) $$
For \( \alpha = 1 \), equation (1.2.7) yields
\[
(n-k)! = \frac{(-1)^{n-k}}{(-n)_k}, \quad 0 \leq k \leq n,
\] (1.2.8)

which may alternatively be written in the form
\[
(-n)_k = \begin{cases} 
\frac{k}{(-1)^{n-k}} \frac{n!}{(n-k)!}, & 0 \leq k \leq n \\
0, & k > n.
\end{cases}
\] (1.2.9)

Gauss's multiplication formula:

For every positive integer \( m \), we have
\[
(\alpha)_{mn} = m \prod_{p=1}^{m} \left( \frac{\alpha+p-1}{m} \right)_n, \quad n = 0,1,2,\ldots.
\] (1.2.10)

which reduces to Legendre duplication formula when \( m = 2 \), viz.
\[
(\alpha)_{2n} = 2^{2n} \left( \frac{\alpha}{2} \right)_n \left( \frac{\alpha+1}{2} \right)_n, \quad n = 0,1,2,\ldots.
\] (1.2.11)

In particular, we have
\[
(2n)! = 2^{2n} \left( \frac{1}{2} \right)_n n! \quad \text{and} \quad (2n+1)! = 2^{2n} \left( \frac{3}{2} \right)_n n!.
\] (1.2.12)
Also [131; problem 2, p.86]

If \( m \) being a positive integer, then

\[
(\alpha)_{n-mk} = \frac{(1/k) (\alpha)_n}{\prod_{j=1}^{m} \left( \frac{j-a-n}{m} \right)_{k}}, \quad 0 \leq k \leq \lfloor n/m \rfloor. \quad (1.2.13)
\]

For \( \alpha = 1 \), (1.2.13) gives

\[
(n-mk)! = \frac{(-1/m)^n}{\prod_{j=0}^{m-1} \left( \frac{j-n}{m} \right)_{k}}, \quad 0 \leq k \leq \lfloor n/m \rfloor. \quad (1.2.14)
\]

The Gaussian Hypergeometric Function

The hypergeometric series

\[
_{2}F_{1}(a,b;c;z) = 1 + \frac{a \cdot b}{c} \frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)} \frac{z^2}{2!} + \ldots \nonumber
\]

\[
= \sum_{n=0}^{\infty} \frac{(a)_n \cdot (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (c \neq 0,-1,-2,\ldots) \quad (1.2.15)
\]

was introduced by a German mathematician C.F. Gauss in 1812,
and is a generalization of ordinary geometric series. Here \((a)_n\) denotes the Pochhammer's symbol, defined by (1.2.3).

If \(c\) is zero or a negative integer, the series (1.2.15) does not exist and hence the function \(F(a,b;c;z)\) is not defined unless one of the parameter \(a\) or \(b\) is also a negative integer such that \(-c < -a\). If either of the parameters \(a\) or \(b\) is a negative integer \(m\) then in this case (1.2.15) reduces to the hypergeometric polynomial defined by

\[
F(-m,b;c;z) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n}{(c)_n n!} z^n, \quad -\infty < z < \infty. \tag{1.2.16}
\]

The series given by (1.2.15), converges absolutely within the limit circle \(|z| < 1\) and \(|z| = 1\), provided that \(\text{Re}(c-a-b) > 0\) for \(z = 1\) and \(\text{Re}(c-a-b) > -1\) for \(z = -1\).

**Generalized Hypergeometric Functions**

A natural generalization of the hypergeometric function \(2F_1\) is the generalized hypergeometric function, so called \(pF_q\), which is defined as
\[
\begin{align*}
\sum_{n=0}^{\infty} \frac{x^n}{(b_1)_n \cdots (b_q)_n} = \\
\sum_{n=0}^{\infty} \frac{[\{a\}]_n \frac{z^n}{n!}}{[\{b\}]_n}
\end{align*}
\]

(1.2.17)

where, as usual,

\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad \text{and} \quad [a]_n = \frac{\Gamma[p]}{\Gamma(a)_n}
\]

Here \(p\) and \(q\) are positive integers or zero, the numerator parameters \(a_1, \ldots, a_p\) and the denominator parameters \(b_1, \ldots, b_q\) take on complex values, provided that

\(b_j \neq 0, -1, -2, \ldots; \quad j = 1, 2, \ldots, q.\)

Convergence of \(\sum_{n=0}^{\infty} \frac{x^n}{(b_1)_n \cdots (b_q)_n}\)

(i) If \(p \leq q\), the series converges for all finite \(z\).

(ii) If \(p = q+1\), the series converges for \(|z| < 1\) and diverges for \(|z| > 1\).

(iii) If \(p > q+1\), the series converges only when \(z = 0\) and diverges when \(z \neq 0\).
(iv) If \( p = q + 1 \), the series is absolutely convergent on circle \(|z| = 1\) if

\[
\text{Re} \left( \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \right) > 0 \quad \text{for} \quad z = 1
\]

and \( \text{Re} \left( \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \right) > -1 \) for \( z = -1 \)

An important special case when \( p = q = 1 \), (1.2.17) reduces to the confluent hypergeometric series \( 1^F_1 \) named as Kummer's series \([64]\) and is given by

\[
1^F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}
\]

\[
= \lim_{|b| \to \infty} 2^F_1(a, b; c; z/b) \quad (1.2.18)
\]

\( 1^F_1(a; c; z) \) is also denoted by Humbert's symbol \( \Phi(a; c; z) \) and it is known as confluent hypergeometric function of first kind.

We note that a differential property for \( 1^F_1[2; p, 300] \) is

\[
\frac{d^k}{dx^k} 1^F_1(a; c; x) = \frac{(a)_k}{(c)_k} 1^F_1(a+k; c+k; x) \quad (1.2.19)
\]

\( K = 1, 2, 3, \ldots \).
The Gauss hypergeometric function \( {}_2 F_1 \) and the confluent hypergeometric function \( {}_1 F_1 \) form the core of special functions and include as special cases most of the commonly used functions. The \( {}_2 F_1 \) includes as special cases, most of the classical orthogonal polynomials, Legendre function, the incomplete Beta function etc. On the other hand, \( {}_1 F_1 \) includes as its special cases, the Bessel functions, coulomb wave equation etc.

Note:- If the homogeneous linear differential equation of second order has almost three singularities, we may assume that these are at 0, 1. If all these singularities are 'regular' (Cf. Poole [98]), the equation reduces to the form

\[
\frac{d}{dz} z(1-z) \frac{d}{dz} + [c-(a+b+1)z] \frac{du}{dz} - abu = 0 , \tag{1.2.20}
\]

where \( a, b, c \) (independent of \( z \)) are parameters of the equation. Equation (1.2.20) is called hypergeometric having the solution (1.2.15)
1.3 HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

Appell Functions

In 1880, Appell [4] introduced four hypergeometric series, which are given below

\[ F_1[a, b, b'; c; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)^{m+n}(b)^{m}(b')^{n}}{(c)^{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.1) \]

where \(\max \{|x|, |y|\} < 1\)

\[ F_2[a, b, b'; c, c'; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)^{m+n}(b)^{m}(b')^{n}}{(c)^{m}(c')^{n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.2) \]

where \(|x| + |y| < 1\)

\[ F_3[a, a', b, b'; c; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)^{m}(a')^{n}(b)^{m}(b')^{n}}{(c)^{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.3) \]

where \(\max \{|x|, |y|\} < 1\)

\[ F_4[a, b; c, c'; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)^{m+n}(b)^{m+n}}{(c)^{m}(c')^{n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.4) \]

where \(\gamma|x| + \gamma|y| < 1\).
Here, as usual, the denominator parameter $c$ and $c'$ are neither zero nor a negative integer.

The standard work on the theory of Appell series is the monograph by Appell and Kampé de Fériet [5]. See Erdelyi et al. [26; pp. 222-245] for a review of the subsequent work on the subject; see also Slater [117; Chapter 8] and Exton [34; pp.23-28].

**Horn Functions**

In 1931, Horn [45] defined 10-hypergeometric functions of two variables and denoted them by $G_1, G_2, G_3, H_1, \ldots, H_7$; he thus completed the set of all possible complete hypergeometric functions of two variables. See also [131; pp. 56-57] and Erdelyi et al. [26; pp. 224-228].

Here we need $H_4$ only which is defined as follows:

$$H_4[a,b;c,d;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.5)$$

$$|x| < r, \ |y| < s, \ ur = (s-1)^2.$$
The Kampé de Fériet Function

The four Appell series were unified and generalized by Kampé de Fériet [54], who defined a general hypergeometric series in two variables. (See Appell and Kampé de Fériet [5; p. 150 (29)].

The notation introduced by Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy [13; p. 112]. We recall here the definition of a more general double hypergeometric function in slightly modified notation (see [13; p. 423 (26)]).

\[
\sum_{r,s=0}^{\infty} \frac{[\prod_{j=1}^{p} (a_j)^{r+s} \prod_{j=1}^{q} (b_j)^{r} \prod_{j=1}^{k} (c_j)^{s}]_{r+s} \frac{x^r}{r!} \frac{y^s}{s!}}{[\prod_{j=1}^{r} (a_j)^{r+s} \prod_{j=1}^{s} (b_j)^{r} \prod_{j=1}^{s} (c_j)^{s}]_{r+s} \frac{x^r}{r!} \frac{y^s}{s!}}
\]

(1.3.6)
where, for convergence,

(i) \( p+q < l+m+1, \ p+k < l+n+1, \ |x| < \infty, \ |y| < \infty. \)

or

(ii) \( p+q = l+m+1, \ p+k = l+n+1, \) and

\[
\begin{align*}
\left\{ \begin{array}{c}
|x|^{1/(p-1)} + |y|^{1/(p-1)} < 1, \quad \text{if} \quad p > 1, \\
\max \{|x|, |y|\} < 1, \quad \text{if} \quad p \leq 1.
\end{array} \right.
\]

Also, we note that

\[
\begin{align*}
1:1;1 & = F_1 \\
1:0;0 & = F_2 \\
0:2;2 & = F_3 \\
1:0;1 & = F_4.
\end{align*}
\] (1.3.7)

1.4 OTHER HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

**Lauricella Function of \( n \)-variables**

Lauricella [66] further generalized the four Appell functions \( F_1, \ldots, F_4 \) to functions of \( n \)-variables and defined
his function as follows (see [131; p.60]).

\[ F^{(n)}_A \left[ a, b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_n; x_1, x_2, \ldots, x_n \right] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a_1)^{m_1} \cdots (b_1)^{m_1} \cdots (b_n)^{m_n}}{(c_1)^{m_1} \cdots (c_n)^{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \quad (1.4.1) \]

\[ |x_1| + \ldots + |x_n| < 1; \]

\[ F^{(n)}_B \left[ a_1, \ldots, a_n; b_1, b_2, \ldots, b_n; c; x_1, \ldots, x_n \right] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a_1)^{m_1} \cdots (a_n)^{m_n} (b_1)^{m_1} \cdots (b_n)^{m_n}}{(c)^{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \quad (1.4.2) \]

\[ \max \{|x_1|, \ldots, |x_n|\} < 1; \]

\[ F^{(n)}_C \left[ a, b; c_1, \ldots, c_n; x_1, \ldots, x_n \right] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)^{m_1+\ldots+m_n} (b)^{m_1+\ldots+m_n}}{(c_1)^{m_1} \cdots (c_n)^{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \quad (1.4.3) \]

\[ \forall |x_1| + \ldots + |x_n| < 1; \]
Clearly, we have,

\[(2) \quad F_A = F_2 \quad ; \quad F_B = F_3 \quad ; \quad F_C = F_4 \quad \text{and} \quad F_D = F_1\]

and

\[(1) \quad F_A = F_B = F_C = F_D = {}_2F_1.\]  \hspace{1cm} (1.4.5)

Lauricella [66; p.114] introduced 14-complete hypergeometric functions of three variables and of second order, denoted by the symbols

\[F_1, F_2, F_3, \ldots, F_{14}\]

of which \(F_1, F_2, F_5\) and \(F_9\) correspond to the three variables \(F_A, F_B, F_C\) and \(F_D\) defined by (1.4.1) to (1.4.4) with \(n = 3\). The remaining ten functions
$F_3, F_4, F_6, F_7, F_8, F_{10}, \ldots, F_{14}$ of Lauricella's set apparently fell into oblivion (except that there is an isolated appearance of triple hypergeometric function $F_8$ in a paper by Mayr [71; p.265]. Saran [105] initiated a systematic study of these ten triple hypergeometric functions of Lauricella's set. He denoted his triple hypergeometric function by the symbols (see [131; pp. 66-68])

$F_E, F_F, \ldots, F_T$

Here we need $F_G$ and $F_S$ only which is defined as follows:

$$F_G[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z]$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha_1)_m (\beta_1)_m (\beta_2)_n (\beta_3)_p}{(\gamma_1)_m (\gamma_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}$$ (1.4.6)

$|x| < r, |y| < s, |z| < t, r+s = 1 = r+t$;

$$F_S[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z]$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_n (\beta_1)_m (\beta_2)_n (\beta_3)_p}{(\gamma_1)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}$$ (1.4.7)

$|x| < r, |y| < s, |z| < t, r+s = rs, s = t.$
Srivastava's Triple Series $H_A$, $H_B$ and $H_C$

In 1964, Srivastava [122] gave three additional triple hypergeometric functions of second order, which bear his name. Their series definitions are given below:

$$H_A(\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z)$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha)_{m+p} (\beta)_{m+p} (\beta')_{n+p}}{(\gamma)_m (\gamma')_{n+p}} \frac{x^m y^n z^p}{m! n! p!}$$

$$|x| < r, \ |y| < s, \ |z| < t, \ r+s+t = 1+st;$$

$$H_B(\alpha, \beta, \beta'; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha)_{m+p} (\beta)_{m+n} (\beta')_{n+p}}{(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p} \frac{x^m y^n z^p}{m! n! p!}$$

$$|x| < r, \ |y| < s, \ |z| < t, \ r+s+t + 2\gamma_1 s t = 1;$$

and

$$H_C(\alpha, \beta, \beta'; \gamma; x, y, z)$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha)_{m+p} (\beta)_{m+n} (\beta')_{n+p}}{(\gamma)_{m+n+p}} \frac{x^m y^n z^p}{m! n! p!}$$

$$|x| < 1, \ |y| < 1, \ |z| < 1.$$
The General Triple Hypergeometric Series \( F^{(3)} [x,y,z] \)

A unification of Lauricella 14-hypergeometric functions \( F_1, \ldots, F_{14} \) of three variables [66; p.114] and the additional functions \( H_A, H_B, H_C \) [126; pp. 99-100] was introduced by Srivastava [121; p. 428] who defined a general triple hypergeometric series \( F^{(3)} [x,y,z] \):

\[
F^{(3)} [x,y,z] = F^{(3)} \left[ \begin{array}{c} (a) ; (b) ; (b') ; (b'') ; (c) ; (c') ; (c'') ; \\
(e) ; (g) ; (g') ; (h) ; (h') ; (h'') ; \\
\end{array} \right]_{x,y,z}^{x+y+z}
\]

\[
= \sum_{m,n,p=0}^{\infty} \Delta (m,n,p) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}
\]

(1.4.11)

where, for convenience,

\[
\Delta (m,n,p) = \frac{[(a)]_{m+n+p}[b]_{m+n}[b']_{n+p}[b'']_{p+m}[c]_{m}[c']_{n}[c'']_{p}}{[(e)]_{m+n+p}[g]_{m+n}[g']_{n+p}[g'']_{p+m}[h]_{m}[h']_{n}[h'']_{p}}
\]

(1.4.12)

as usual, (a) abbreviates the array of A-parameters \( a_1, \ldots, a_A \), with similar interpretations for (b), (b'), (b''), et cetera and
\[ [(a)]_m = \prod_{j=1}^{\infty} (a_j)_m. \]

For convergence of series (1.4.11), see [13; p.70].

In 1963, Panday [89] gave two triple hypergeometric series of Horn's type and their series definition is given below (see also [13; p.70])

\[
G_A(\alpha, \beta, \beta'; \gamma; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(\alpha)_{n+p-m} (\beta)_{m+p} (\beta')_n}{(\gamma)_{n+p-m}} \frac{x^m y^n z^p}{m! n! p!}, \quad (1.4.13)
\]

which provides a generalization of Appell's function \( F_1 \) and Horn's function \( G_1, G_2 \):

\[
G_B(\alpha, \beta_1, \beta_2, \beta_3; \gamma; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(\alpha)_{n+p-m} (\beta_1)_m (\beta_2)_n (\beta_3)_p}{(\gamma)_{n+p-m}} \frac{x^m y^n z^p}{m! n! p!}, \quad (1.4.14)
\]

which is generalization of Appell's function \( F_1 \) and the Horn function \( G_2 \).
1.5 CLASSICAL POLYNOMIALS

Associated Laguerre Polynomials

An associated Laguerre polynomial \( L_n^{(\alpha)}(x) \) is defined by the generating relation

\[
(1-t)^{-1-\alpha} \exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n, \tag{1.5.1}
\]

or

\[
\alpha \exp(-xt) = \sum_{n=0}^{\infty} L_n^{(-\alpha)}(x) t^n. \tag{1.5.2}
\]

For \( \alpha = 0 \), equation (1.5.1) reduces to generating function for simple Laguerre polynomial.

\[
(1-t)^{-1} \exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} L_n(x) t^n. \tag{1.5.3}
\]

In particular, for \( x = 0 \)

\[
L_n^{(0)}(o) = \binom{n+\alpha}{n} \quad \text{and} \quad (1+t)^{-\alpha} = \sum_{n=0}^{\infty} \binom{n}{n} t^n, \quad |t| < 1. \tag{1.5.4}
\]

For \( \alpha = 0 \), (1.5.2) reduces to

\[
L_n(x) = \frac{(-1)^n x^n}{n!}. \tag{1.5.5}
\]
A series representation of \( L_n^{(\alpha)}(x) \), for non negative integer \( n \) is

\[
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+\alpha)!}{k! (n-k)! (\alpha+k)!} x^k.
\]

(1.5.6)

When \( \alpha = 0 \), it readily follows that

\[
L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! x^k}{(k!)^2 (n-k)!}.
\]

(1.5.7)

**Jacobi Polynomials**

The Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) are defined by the generating function

\[
2^{1/2} \frac{\alpha + \beta}{\omega} - 1 (1 - t + \omega)^{-\alpha} (1 + t + \omega)^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t^n,
\]

(1.5.8)

\( \text{Re}(\alpha) > -1, \text{Re}(\beta) > -1 \)

where \( \omega = (1 - 2xt + t^2) \) and \( n \) be a non-negative integer.

The Jacobi polynomials have the three finite series representations, see [102; p. 255]; one of them is given below

\[
P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\alpha+\beta)_n}{k! (n-k)! (1+\alpha)_k (1+\alpha+\beta)_n} \frac{x^k}{2^{k}}.
\]

(1.5.9)
We note that

\[ p_n^{(\alpha,\beta)}(-x) = (-1)^n p_n^{(\beta,\alpha)}(x). \]  

(1.5.10)

**Special Cases:**

(i) When \( \alpha = \beta = 0 \), the polynomial becomes the Legendre polynomial

\[ p_n^{(0,0)}(x) = p_n(x). \]  

(1.5.11)

(ii) If \( \beta = \alpha \), then \( p_n^{(\alpha,\alpha)}(x) \) is called the ultraspherical polynomial. The Gegenbauer polynomial \( C_n^{\alpha}(x) \) [2; p. 185] is connected with the ultraspherical polynomial by relation (cf. see [2; p. 191] or [102; p. 283]).

\[ C_n^{\alpha}(x) = \frac{(2\alpha)_n p_n^{(\alpha-1/2,\alpha-1/2)}(x)}{(\alpha + \frac{1}{2})_n}, \]  

or

\[ p_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n C_n^{\alpha+1/2}(x)}{(1 + 2\alpha)_n}. \]  

(1.5.12)
(iii) The Laguerre polynomials are, in fact, limiting cases of the Jacobi polynomials [102; p.74]

\[ L_n^{(\alpha)}(x) = \lim_{|\beta| \to \infty} P_n^{(\alpha, \beta)}(1-2x/\beta). \quad (1.5.14) \]

**Generalized Humbert Polynomials**

Gould [39] presented a systematic study of an interesting generalization of Humbert, Gegenbauer and several other polynomials system defined by

\[ (c-mxt + yt) = \sum_{n=0}^{\infty} P_n(m,x,y,p,c) t^n, \quad (1.5.15) \]

where \( m \) is positive integer and other parameters are unrestricted in general. For the table of main special cases of (1.5.15), including Gegenbauer, Legendre, Tchebycheff, Pincherle, Kinney and Humbert polynomials, see Gould [39].

**Some Useful Hypergeometric Representations**

Hyper Bessel function (see [12; p.176]) is defined as

\[ J_{\nu}(z) = \left( \frac{z}{n+1} \right)^{1/2} \frac{1}{\Gamma(\nu+1)} \frac{1}{\nu!} \sum_{i=1}^{n} \chi_i \left( \frac{z}{n+1} \right)^{n+1}. \quad (1.5.16) \]
In case $n = 1$ these functions coincide with hypergeometric representation of Bessel function of the first kind $J_v$

$$J_v(z) = \frac{(z/2)^v}{\Gamma(v+1)} \, _0F_1\left(\frac{1}{2}; v+1; -\frac{z^2}{4}\right). \quad (1.5.17)$$

Laguerre polynomial (see [102, p. 200])

$$L_n^\alpha(x) = \frac{(1+\alpha)_n}{n!} \, _1F_1(-n; 1+\alpha; x). \quad (1.5.18)$$

Legendre polynomial (see [102; p.166])

$$P_n(x) = \frac{(-1)^n}{2^n} \, _2F_1\left[\begin{array}{c} -n, n+1; \\ \frac{1-x}{2} \end{array} \; 1 \right]. \quad (1.5.19)$$

Gegenbauer polynomial (see [102; p. 279])

$$c_n^\mu(x) = \frac{(2\mu)_n}{n!} \, _2F_1\left[\begin{array}{c} -n, 2\mu+n; \\ \frac{1-x}{2} \end{array} \; \frac{1}{\mu+\frac{1}{2}} \right]. \quad (1.5.20)$$

Hermite polynomial (see [102; p. 40])

$$H_n(x) = (2x)^n \, _2F_0\left[\frac{n}{2}, \frac{1}{2} - \frac{n}{2}; -x^2 \right]. \quad (1.5.21)$$
Jacobi polynomial (see [102; p. 255])

\[ p_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{n!} 2F_1 \left[ \begin{array}{c} -n, 1+\alpha+\beta+n; \\ 1+\alpha; \\ \frac{1-x}{2} \end{array} \right] \]  \hspace{1cm} (1.5.22)

\[ = \frac{(1+\beta)_n}{n!} \left( \frac{x-1}{2} \right)^n 2F_1 \left[ \begin{array}{c} -n, -\alpha-n; \\ 1+\beta; \\ \frac{x+1}{x-1} \end{array} \right] \]  \hspace{1cm} (1.5.23)

Generalized Rice polynomial (see [131; p. 140])

\[ H_n^{(\alpha, \beta)}(k, p; x) = \frac{\alpha+n}{n!} 3F_2 \left[ \begin{array}{c} -n, 1+\alpha+\beta+n, k; \\ 1+\alpha; \\ p; \\ x \end{array} \right] \]  \hspace{1cm} (1.5.24)

Clearly, we note that

\[ H_n^{(\alpha, \beta)}(k, k; x) = p_n^{(\alpha, \beta)}(1-2x). \]  \hspace{1cm} (1.5.25)

1.6 GENERATING FUNCTIONS

The name 'generating function' was first introduced by Laplace in 1812. We define a generating function \( F_n(x) \) as follows [72].
**Definition:** Let \( G(x,t) \) be a function that can be extended in power of \( t \) such that

\[
G(x,t) = \sum_{n=0}^{\infty} C_n f_n(x) t^n,
\]  
(1.6.1)

where \( C_n \) is a function of \( n \) and independent of \( x \) and \( t \).

Then \( G(x,t) \) is called a generating function of the set \( \{f_n(x)\} \).

**Remark:** A set of function may have more than one generating function. However if

\[
G(x,t) = \sum_{n=0}^{\infty} h_n(x) t^n
\]

then \( G(x,t) \) is unique generator for the set \( \{h_n(x)\} \) as the coefficient set.

Let us define a generating function of more than one variable.

**Definition:** Let \( G(x_1,x_2,\ldots,x_p,t) \) be function of \( (p+1) \) variables. Suppose \( G(x_1,x_2,\ldots,x_p,t) \) has formal expansion in powers of \( t \) such that

\[
G(x_1,x_2,\ldots,x_p,t) = \sum_{n=-\infty}^{\infty} C_n f_n(x_1,x_2,\ldots,x_p) t^n,
\]  
(1.6.2)
where $C_n$ is independent of the variables $x_1, x_2, \ldots, x_p$ and $t$. Then we say that $G(x_1, x_2, \ldots, x_p; t)$ is a generating function for the $f_n(x_1, x_2, \ldots, x_p)$ corresponding to non-zero $C_n$. In particular, if

$$G(x, y, t) = \sum_{n=0}^{\infty} C_n f_n(x) g_n(y) t^n. \quad (1.6.3)$$

The expansion determines the set of constant $\{C_n\}$ and two sets of function $\{f_n(x)\}$ and $\{g_n(y)\}$. Then $G(x, y, t)$ can be considered as a generator of any of these three sets and as unique generator of co-efficient set $\{C_n f_n(x) g_n(y)\}$.

**Application of Generating Functions**

A generating function may be used to define a set of functions to determine a differential recurrence relation, to evaluate certain integral etc. In this section, some example [28] are summarized [50]. In each case the generating function is given, so as the partial differential equations from which the fundamental relations can be found, by identifying the $n$ term $w$ in the power expansion of the whole equation.
Hermite polynomials:

\[ G(x,w) = e^{-w/4} x^w e^w = \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n!} \]

\[
\begin{aligned}
\frac{\partial G(x,w)}{\partial w} &= (x - \frac{1}{2} w) G, \quad H_{n+1} = 2xH_n - 2n H_{n-1} \\
\frac{\partial G(x,w)}{\partial x} &= wG, \quad H_n' = 2n H_{n-1}.
\end{aligned}
\]

Laguerre polynomials:

\[ G(x,w) = \frac{1}{1-w} e^{-xw/(1-w)} = \sum_{n=0}^{\infty} L_n(x) w^n \]

\[
\begin{aligned}
\frac{\partial G(x,w)}{\partial w} &= \frac{1-w-x}{(1-w)^2} G, \quad (n+1) L_{n+1} = (2n+1-x)L_n - n L_{n-1}, \\
\frac{\partial G(x,w)}{\partial x} &= \frac{-w}{1-w} G, \quad L_n' - L_{n-1}' = -L_{n-1}.
\end{aligned}
\]
Appell polynomials: \( G(x, w) = A(w)^xw \)

\[
G(x, w) = \frac{1}{(x-w)^\lambda} e^{xw} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} p_n(x) w^n,
\]

\[
p_n = (-1)^n L_n^{(\lambda-n)},
\]

\[
\begin{align*}
\frac{\partial G(x, w)}{\partial w} &= \frac{\lambda + x-xw}{1-w} G, \\
\frac{\partial G(x, w)}{\partial x} &= wG, \\
\end{align*}
\]

Bernoulli polynomials:

\[
G(x, w) = \frac{w}{e^w-1} e^{xw} = \sum_{n=0}^{\infty} \frac{B_n(x) w^n}{n!}, \\
\]

\[
\begin{align*}
\frac{\partial G(x, w)}{\partial w} &= \frac{w(1-w-xw)}{e^w-1} G, \\
\frac{\partial G(x, w)}{\partial x} &= wG, \\
\end{align*}
\]

\[
p_n = p_{n-1}.
\]
The first equation gives only rise to an expansion of $p_{n+1}$ in terms of $p_0, \ldots, p_n$.

Sheffer (or generalized Appell) polynomials:

$$G(x,w) = A(w) e^x g(w).$$

Let the function $G$ be $G(x,w) = e^x e^w x \log(1+w)$.

$$\left\{ \begin{array}{l}
\frac{\partial G(x,w)}{\partial w} = \frac{1+w+x}{1+w} G, 
(n+1)p_{n+1} = (1-n+x)p_n + p_{n-1}, \\
\frac{\partial G(x,w)}{\partial x} = \log(1+w)G,
\end{array} \right.$$\]

$\sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{n-k-1} p_k(x).$

Here because $\log(1+w)$ is not a rational function, there is no recurrence differential relation but an expansion of $p_n$ in terms of $(p_k)$.

Generalized Hermite polynomials:

$$G(x,w) = e^{x^4 - (x-w)^4} = \sum_{n=0}^{\infty} \frac{H_{n,2}}{n!} w^n.$$
\[ \frac{\partial G(x,w)}{\partial w} = 4(x-w) G \]
\[ H_{n+1} = 4x H_n - 12nx H_{n-1} + 12n(n-1)xH_{n-2} - 4n(n-1)(n-2)H_{n-3} \]
\[ \frac{\partial G(x,w)}{\partial x} = 4(x-(x-w)) G, \]
\[ H_n = 12nx H_{n-1} - 12n(n-1)xH_{n-2} + 4n(n-1)(n-2)H_{n-3}. \]

**A Bilinear Generating Functions**

If a function \( G(x,y,t) \) can be expanded in the form
\[ G(x,y,t) = \sum_{n=0}^{\infty} g_n f_n(x) f_n(y) t^n, \quad (1.6.4) \]
where \( g_n \) is independent of \( x \) and \( y \) then \( G(x,y,t) \) is called a bilinear generating function. For example the Laguerre polynomials satisfies the following bilinear generating relation [72; p. 17] or [102; p. 212]
\[ \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)}(y)}{(1+\alpha)_n} t^n = (1-t)^{-1-\alpha} \exp \left\{ -\frac{t(x+y)}{1-t} \right\} \]
\[ x \cdot {}_{1}F_{1}(-\alpha; 1+\alpha; \frac{xyt}{(1-t)^2}). \quad (1.6.5) \]
Bilateral Generating Functions

If \( H(x,y,t) \) can be expanded in power of \( t \) in the form

\[
H(x,y,t) = \sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n,
\]

(1.6.6)

where \( h_n \) is independent of \( x \) and \( y \) and \( f_n(x) \) and \( g_n(y) \) are different functions, then \( H(x,y,t) \) is called a bilateral generating function. For example [72; p. 12].