Chapter 1
BASIC DEFINITIONS AND NOTATIONS
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In this chapter we give notations, definitions and some results which are already known in the literature which form the background of the thesis.

1.1. NOTATIONS

Throughout the present work we shall use the following notations which are conventional (cf. Cook [8], Maddox [23], Mursaleen [38]).

\[ \mathbb{N} := \text{The set of all natural numbers} \]

\[ \mathbb{R} := \text{The set of all real numbers} \]

\[ \mathbb{C} := \text{The set of all complex numbers} \]

\[ \lim_{k \to \infty} \]

\[ \inf_{k \geq 1} \]

\[ \sup_{k \geq 1} \]

\[ \sum_{k} : \text{means summation over } k = 1 \text{ to } k = \infty, \text{unless otherwise stated} \]

\[ x := (x_k) \text{ or } \{x_k\}, \text{the sequence whose } k\text{-th term is } x_k \]

\[ e_k := (0, 0, \cdots, 0, 1, 0, 0, \cdots), \text{the sequence whose } k\text{-th component is 1 and others zeros, for all } k \in \mathbb{N} \]

\[ e := (1, 1, 1, \cdots) \]

\[ \omega := \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}, \text{the space of all sequences, real or complex numbers} \]

\[ \ell_\infty := \{x \in \omega : \sup_{k} |x_k| < \infty\}, \text{the space of all bounded sequences} \]

\[ c := \{x \in \omega : \lim_{k} x_k = \ell \text{ for some } \ell \in \mathbb{C}\}, \text{the space of all convergent sequences} \]

\[ c_0 := \{x \in \omega : \lim_{k} x_k = 0\}, \text{the space of all null sequences} \]
\( \ell_\infty, c \) and \( c_0 \) are Banach spaces with the norm

\[ \|x\|_\infty = \sup_k |x_k|. \]

\( \ell_p := \{x \in \omega : \sum_k |x_k|^p < \infty\} \), the space of all absolutely \( p \)-summable sequences

\( \ell_p \) is a Banach space with the norm

\[ \|x\|_p = \left( \sum_k |x_k|^p \right)^{1/p}, \quad 1 \leq p \leq \infty. \]

Note that \( \ell_2 \) is a Hilbert space.

\( A := (a_{nk}) \), \( n, k = 1, 2, 3 \ldots \), denote the infinite matrix of real or complex numbers.

If \( g \) is a function of a variable which tends to a limit, then we write

\begin{align*}
\text{f} = O(g) \text{ : means } |f| \leq M g, \text{ where } M \text{ is a constant} \\
\text{f} = o(g) \text{ : means } f/g \longrightarrow 0.
\end{align*}

### 1.2. Matrix Transformation

**Definition 1.1.** Let \( A = (a_{nk}) \) be an infinite matrix of real or complex numbers and \( x = (x_k) \) be a sequence of real or complex numbers. Then we write

\[ A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k \]

which is called the \( A \)-transform of the sequence \( x = (x_k) \) whenever the series on the right converges for each \( n = 1, 2, \ldots \). The sequence \( x \) is said to be \( A \)-summable to \( \ell \) if \( A_n(x) \) converges to \( \ell \) as \( n \rightarrow \infty \).

**Definition 1.2.** Let \( X \) and \( Y \) be two nonempty subsets of the space \( \omega \). If \( x \in X \) implies \( Ax = (A_n(x)) \in Y \) then we say that \( A \) defines a matrix transformation from \( X \) into \( Y \), and we denote by \((X, Y)\) the class of matrices \( A \) which transform \( X \) into \( Y \). By \((X, Y)_{r, g}\) we denote the subset of \((X, Y)\) for which limit or sum is preserved.
1.3. CONSERVATIVE AND REGULAR MATRICES

**Definition 1.3.** A matrix $A = (a_{nk})$ is said to be conservative if $Ax \in c$ for $x = (x_k) \in c$, and we denote this by $A \in (c, c)$.

**Definition 1.4.** A matrix $A = (a_{nk})$ is said to be regular if it is conservative and $\lim Ax = \lim x$, and we denote this by $A \in (c, c)_{\text{reg}}$.

The following are well-known Silverman-Toeplitz [8, 23] conditions for the regularity of $A$.

**Theorem 1.1.** A matrix $A = (a_{nk})$ is regular, i.e. $A \in (c, c)_{\text{reg}}$ if and only if

(i) $\sup_n \sum_k |a_{nk}| < \infty$;

(ii) $\lim_{n \to \infty} a_{nk} = 0$, for each $k$;

(iii) $\lim_{n \to \infty} \sum_k a_{nk} = 1$.

The following is a generalization of the above theorem by Kojima-Schur [8, 23].

**Theorem 1.2.** The matrix $A = (a_{nk})$ is conservative, i.e. $A \in (c, c)$ if and only if

(i) $\sup_n \sum_k |a_{nk}| < \infty$;

(ii) $\lim_{n \to \infty} a_{nk} = a_k$ for each $k$;

(iii) $\lim_{n \to \infty} \sum_k a_{nk} = a$.

1.4. Schur Matrix

**Definition 1.5.** A matrix $A$ is called Schur matrix if $Ax \in c$ for all $x \in \ell_\infty$, and we denote this by $A \in (\ell_\infty, c)$.

The following are necessary and sufficient conditions for a matrix $A$ to be Schur [48].

**Theorem 1.3.** The matrix $A \in (\ell_\infty, c)$ if and only if

(i) $\lim_n a_{nk}$ for each $k$: 3
\[(ii) \sum_k |a_{nk}| \text{ converges uniformly in } n.\]

### 1.5. Banach Limit and Almost Convergence

In 1948, Lorentz \[22\] introduced a new method of summation which assigns a general limit to certain bounded sequences. This method is narrowly connected with the Banach limit. The sequences which are summable by this method are called almost convergent sequences.

**Definition 1.6.** A linear functional \( L \) on \( \ell_\infty \) is said to be a **Banach limit** (see \([3]\)) if it has the following properties:

\[
(i) \quad L(x) \geq 0 \text{ if } x \geq 0 \text{ (i.e., } x_n \geq 0 \text{ for all } n); \\
(ii) \quad L(e) = 1, \text{ where } e = (1, 1, 1, \cdots); \\
(iii) \quad L(Sx) = L(x);
\]

where the shift operator \( S \) is defined by

\[(Sx)_n = x_{n+1}.

**Definition 1.7.** A sequence \( x \in \ell_\infty \) is said to be **almost convergent** to the number \( \ell \) if each Banach limit of \( x \) is \( \ell \). The class \( f \) of almost convergent sequences was introduced by Lorentz \([22]\), who proved that a sequence \( x = (x_k) \) is almost convergent to \( \ell \) if and only if

\[
\lim_{k \to \infty} t_{kn}(x) = \lim_{k \to \infty} \frac{x_n + x_{n+1} + \cdots + x_{n+k}}{k - 1} = \ell,
\]

uniformly in \( n \), and \( \ell \) is called the \( f \)-limit of \( x \).

A convergent sequence is almost convergent and its limit and its generalized limit are identical. But an almost convergent sequence need not be convergent, e.g. \( x = (1, -1, 1, -1, \cdots) \) is not convergent but it is almost convergent to 0.

**Definition 1.8.** A matrix \( A = (a_{nk}) \) is said to be **strong regular** if \( Ax \in c \) for \( x \in f \) with \( \lim Ax = f-\lim x \), and we denote this by \( A \in (f, c) \). These matrices were defined and characterized by Lorentz \([22]\).

### 1.6. Almost Summability

In 1966 King \([21]\), using the concept of almost convergence introduced almost summability and defined almost conservative and almost regular matrices.
**Definition 1.9.** A sequence $x$ is said to be *almost $A$-summable* if the $A$-transform of $x$ is almost convergent.

**Definition 1.10.** A matrix $A = (a_{nk})$ is said to be *almost conservative* if $Ax \in f$ for $x \in c$, and we denote this by $A \in (c, f)$.

**Definition 1.11.** A matrix $A = (a_{nk})$ is said to be *almost regular* if $Ax \in f$ for $x \in c$ with $f$-lim $Ax = \lim x$, and we denote this by $A \in (c, f)_{reg}$.

King [21] gave the following characterization:

**Theorem 1.4.** The matrix $A = (a_{nk})$ is almost conservative if and only if

(i) $\sup \left\{ \frac{1}{p} \sum_{j=n}^{n+p-1} a_{jk} : p \in \mathbb{N}^+ \right\} < \infty$, $n = 0, 1, 2, \ldots$;

(ii) there exists $\alpha_k \in \mathbb{C}$, $k = 0, 1, 2, \ldots$, such that

$$\lim_{p \to \infty} \frac{1}{p} \sum_{j=n}^{n+p-1} a_{jk} = \alpha_k,$$ for each $k$ (uniformly in $n$);

(iii) there exists $\alpha \in \mathbb{C}$ such that

$$\lim_{p \to \infty} \frac{1}{p} \sum_{j=n}^{n+p-1} \sum_{k=0}^{\infty} a_{jk} = \alpha$$ (uniformly in $n$).

**Remark 1.1.** If we take $\alpha = 1$, $\alpha_k = 0$ for each $k$ in Theorem 1.4; then these conditions are reduced to the necessary and sufficient conditions for almost regular matrices.

### 1.7. Almost Coercive Matrices

In 1969 Eigen-Laush [17] defined and characterized the class of almost coercive matrices.

**Definition 1.12.** A matrix $A = (a_{nk})$ is said to be *almost coercive* if $Ax \in f$ for $x \in \ell_\infty$, and we denote this by $A \in (\ell_\infty, f)$.

**Theorem 1.5.** The matrix $A = (a_{nk})$ is almost coercive if and only if

(i) $\sup \left\{ \frac{1}{p} \sum_{j=n}^{n+p-1} a_{jk} : p \in \mathbb{N}^+ \right\} < \infty$, $n = 0, 1, 2, \ldots$.
(ii) there exists \( \alpha_k \in \mathbb{C} \), \( k = 0, 1, 2, \cdots \), such that
\[
\lim_{p \to \infty} \frac{1}{p} \sum_{j=n}^{n+p-1} a_{jk} = \alpha_k \text{ uniformly in } n;
\]

(iii) \[
\lim_{p \to \infty} \frac{1}{p} \sum_{k=0}^{\infty} \sum_{j=n}^{n+p-1} |a_{jk} - \alpha_k| = 0 \text{ uniformly in } n.
\]

1.8. Invariant Means

In 1972 Schaefer [47] defined the notions of \( \sigma \)-conservative, \( \sigma \)-regular and \( \sigma \)-coercive matrices analogous to the notions of almost conservative, almost regular and almost coercive matrices (by using the concept of invariant mean) and obtained conditions which characterize them.

**Definition 1.13.** Let \( \sigma \) be a one-to-one mapping from \( \mathbb{N} \) into itself. A continuous linear functional \( \varphi \) on \( \ell_\infty \) is said to be an *invariant mean* or a \( \sigma \)-mean [44] if and only if

(i) \( \varphi(x) \geq 0 \) when the sequence \( x = (x_k) \) has \( x_k \geq 0 \) for all \( k \);

(ii) \( \varphi(e) = 1 \);

(iii) \( \varphi(x) = \varphi((x_{\sigma(k)})) \) for all \( x \in \ell_\infty \).

By \( V_\sigma \) we denote the set of bounded sequences all of whose \( \sigma \)-means are equal. We say that a sequence \( x = (x_k) \) is \( \sigma \)-convergent if and only if \( x \in V_\sigma \). For \( \sigma(n) = n + 1 \), the set \( V_\sigma \) is reduced to the set \( f \) of almost convergent sequences. Note that \( c \subset V_\sigma \subset \ell_\infty \).

If \( x = (x_k) \), write \( Tx = (x_{\sigma(k)}) \). It can be shown that the set \( V_\sigma \) can be characterized as the set of all bounded sequences \( x \) for which

\[
\lim_{p \to \infty} (x + Tx + \cdots + T^p x)/(p + 1)
\]

exists in \( \ell_\infty \) and has the form \( L_\sigma L \), \( L \) being the common value of all \( \sigma \)-means at \( x \), we write \( L = \sigma \text{-lim } x \), where \( T^p x = (x_{\sigma^p(k)}) \) is the \( p \)–th iterate of \( T \) on \( x \).
1.9. MATRIX TRANSFORMATIONS IN $V_\sigma$

The following definitions and characterizations are due to Schaefer [47]:

**DEFINITION 1.14.** A matrix $A = (a_{nk})$ is said to be $\sigma$-conservative if and only if $Ax \in V_\sigma$ for $x \in c$, and we denote this by $A \in (c, V_\sigma)$.

**DEFINITION 1.15.** A matrix $A = (a_{nk})$ is said to be $\sigma$-regular if and only if it is $\sigma$-conservative and $\sigma\lim Ax = \lim x$ for $x \in c$, and we denote this by $A \in (c, V_\sigma)_{reg}$.

**DEFINITION 1.16.** A matrix $A = (a_{nk})$ is said to be $\sigma$-coercive if and only if $Ax \in V_\sigma$ for $x \in \ell_\infty$, and we denote this by $A \in (\ell_\infty, V_\sigma)$.

**THEOREM 1.6.** The matrix $A = (a_{nk})$ is $\sigma$-conservative if and only if

(i) $\|A\| = \sup_n \left\{ \sum_{k=0}^{\infty} |a_{nk}| \right\} < \infty$;

(ii) $a_k = \{a_{nk}\}_{n=1}^{\infty} \in V_\sigma$ for each $k$;

(iii) $a = \left\{ \sum_{k=0}^{\infty} a_{nk} \right\}_{n=1}^{\infty} \in V_\sigma$;

when $A$ is $\sigma$-conservative, the $\sigma$-limit of $Ax$ is

$$\lim_{n \to \infty} x \left[ u - \sum_{k=0}^{\infty} u_k \right] + \sum_{k=0}^{\infty} x_k u_k,$$

for every $x = (x_k) \in c$, where $u = \sigma\lim a$ and $u_k = \sigma\lim a_{nk}$, $k = 1, 2, \ldots$.

**THEOREM 1.7.** The matrix $A = (a_{nk})$ is $\sigma$-regular if and only if

(i) $\|A\| = \sup_n \left\{ \sum_{k=0}^{\infty} |a_{nk}| \right\} < \infty$;

(ii) $a_k = \{a_{nk}\}_{n=1}^{\infty} \in V_\sigma$ with $\sigma$-limit zero for each $k$;

(iii) $a = \left\{ \sum_{k=0}^{\infty} a_{nk} \right\}_{n=1}^{\infty} \in V_\sigma$ with $\sigma$-limit +1.

An application of $\sigma$-regular matrices is shown in [1].

**THEOREM 1.8.** The matrix $A = (a_{nk})$ is $\sigma$-coercive if and only if

(i) $\|A\| = \sup_n \left\{ \sum_{k=0}^{\infty} |a_{nk}| \right\} < \infty$. 

7
(ii) \( a_k = \{a_{nk}\}_{n=1}^{\infty} \in V_\sigma \) for each \( k \);

(iii) \( \lim_{p \to \infty} \frac{1}{p+1} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} |a(\sigma^j(n), k) - u_k| = 0 \) uniformly in \( n \);

where \( u_k = \sigma \)-lim \( a_k \). In this case, the \( \sigma \)-limit of \( Ax \) is \( \sum_{k=1}^{\infty} u_k a_k \) for every \( x = (x_k) \in \ell_\infty \).

### 1.10. Double Sequences

**Definition 1.17.** A double sequence \( x = (x_{jk}) \) is said to be **convergent** in the Pringsheim sense (or \( P \)-convergent) if for given \( \epsilon > 0 \) there exists an integer \( N \) such that \( |x_{jk} - \ell| < \epsilon \) whenever \( j, k > N \). We shall write this as

\[
\lim_{j,k \to \infty} x_{jk} = \ell,
\]

where \( j \) and \( k \) tending to infinity independent of each other (cf. [43]). We denote by \( c_2 \), the space of \( P \)-convergent sequences. Throughout the thesis limit of a double sequence means limit in the Pringsheim sense.

**Definition 1.18.** A double sequence \( x \) is **bounded** if

\[
\| x \| = \sup_{j,k \geq 0} |x_{jk}| < \infty.
\]

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. By \( c_2^\infty \), we denote the space of double sequences which are bounded convergent, and by \( \ell_2^\infty \) the space of bounded double sequences. Obviously \( c_2^\infty \subset \ell_2^\infty \).

**Definition 1.19.** A double sequence \( x = (x_{jk}) \) is said to converge **regularly** [30] if it converges in Pringsheim’s sense and, in addition, the following finite limits exist:

\[
\lim_{k \to \infty} x_{jk} = \ell_j, \quad (j = 1, 2, 3, \ldots),
\]

\[
\lim_{j \to \infty} x_{jk} = h_k, \quad (k = 1, 2, 3, \ldots).
\]

Obviously, the regular convergence of \( x \) implies the convergence in Pringsheim’s sense as well as the boundedness of the terms of \( x \), but the converse implication fails.

Let \( c_2^N \) denotes the space of regularly convergent sequences \( x = (x_{jk}) \) and \( c_2^{N'} \) the space of continuous linear functionals on \( c_2^N \).
1.11. **Bounded-Regular Matrices**

Bounded-regular matrices have been studied by Robinson [45] and Hamilton [20].

**Definition 1.20.** A matrix $A = (a_{mnjk})$ is said to be bounded-regular if $Ax \in c_2^\infty$ for all $x \in c_2^\infty$ with $\lim Ax = \lim x$.

**Theorem 1.9.** A matrix $A = (a_{mnjk})$ is bounded-regular if and only if

(i) $\lim_{m,n \to \infty} a_{mnjk} = 0 \ (j, k = 0, 1, \ldots)$;

(ii) $\lim_{m,n \to \infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{mnjk} = 1$;

(iii) $\lim_{m,n \to \infty} \sum_{j=0}^{\infty} |a_{mnjk}| = 0 \ (k = 0, 1, \ldots)$;

(iv) $\lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{mnjk}| = 0 \ (j = 0, 1, \ldots)$;

(v) $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{mnjk}| \leq C < \infty, \ (m, n = 0, 1, \ldots)$.

1.12. **Cores of Real Bounded Sequences**

Let us consider the following functionals defined on $\ell_\infty$:

$$\ell(x) = \lim \inf x, \ L(x) = \lim \sup x, \ q_\sigma(x) = \lim \sup \sup t_{pn}(x),$$

$$L(x) = \lim \sup \sup \frac{1}{p + 1} \sum_{n=0}^{p} x_{n+1}.$$

We define K-core, B-core and $\sigma$-core for single sequences.

**Definition 1.21.** The Knopp core (or K-core) (see [8], [26]) of real number sequence $x = (x_k)$ is defined to be closed interval $[\ell(x), L(x)]$.

**Definition 1.22.** The Banach core (or B-core) (see [12], [39]) of real bounded number sequence $x = (x_k)$ is defined to be closed interval $[-L(-x), L(x)]$.

**Definition 1.23.** The $\sigma$-core (see [9], [27]) of a real bounded number sequence $x = (x_k)$ is defined to be closed interval $[-q_\sigma(-x), q_\sigma(x)]$.  

9
When \( \sigma(n) = n + 1, \quad q_\sigma(x) = \hat{L}(x), \) the \( \sigma \)-core of \( x \) is reduced to the Banach core of \( x \).

Now we define the following cores for double sequences:

**Definition 1.24.** The Pringsheim core (or P-core) (see [40]) of a real bounded double sequence \( x = (x_{jk}) \) is defined to be the closed interval \( [\text{P-lim inf } x, \text{P-lim sup } x] \).

**Definition 1.25.** The M-core (see [31], [37]) of a real bounded double sequence was defined as the closed interval \( [-L^*(-x), L^*(x)] \), where \( L^*(x) \) is a sublinear functional on \( \ell_2^\infty \) defined by

\[
L^*(x) = \limsup_{m,n \to \infty} \sup_{p,q} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk}.
\]

**1.13. Introduction**

In this thesis we study various concepts for double sequences, e.g. almost convergence, almost regular, almost conservative matrices, \( \sigma \)-convergence, \( \sigma \)-multiplicative, \( \sigma \)-conservative, \( \sigma \)-coercive, absolutely \( \sigma \)-conservative (regular), \( V^\sigma \)-regular matrices and \( \sigma \)-core theorems, etc. We also define some new spaces for double sequences involving the idea of almost convergence and \( \sigma \)-convergence and study their properties and some inclusion relations.