CHAPTER-III
QUASI-VARIATIONAL INEQUALITIES
3.1 INTRODUCTION

Variational inequality theory has emerged as elegant and fascinating branch of applicable mathematics in recent years because it describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics and engineering sciences, see for instance [9,12,42,63,75]. In recent years, variational inequality theory has been extended and generalized in several directions, using new and powerful methods, to study a wide class of unrelated problems in a unified and general framework. An important and useful generalization is the general variational inequality problems. One of the most important and difficult problem in this theory is the development of an efficient and implementable iteration methods for solving variational inequalities in the literature. Among the most efficient methods is the projection technique and its variant forms. In a variational inequality formulation, the underlying convex set does not depend upon the solution. If the underlying convex set does depend upon the solution itself then the variational inequality is called quasi-variational inequality which was introduced and studied by Bensoussan, Goursat and Lions [12]. Also, see Baiocchi and Capelo [9], Bensoussan and Lions [13], Ding [33,35], Noor [88], Siddiqi and Ansari [126] and Mosco [75] for the related work and applications of quasi-variational inequalities. Recently Ahmad, Kazmi and Siddiqi [1] and Kazmi [61] proposed
iterative algorithm with errors for finding the better approximate solutions to the variational inequalities.

Motivated by above cited works, in this chapter, we consider a generalized nonlinear quasi-variational inequalities (in short GSNQVIP) and propose a perturbed Ishikawa iterative algorithm (in short, PIIA) for finding the approximate solution for GSNQVIP. We prove the existence solutions and discuss the convergence criteria for the sequences generated by PIIA. Further we also consider a more general strongly nonlinear quasi-variational inequality problem involving fuzzy mappings (in short, SNQVIPF) and study its existence theory and the convergence criteria for the sequence generated by iterative algorithm proposed for SNQVIPF.

In Section 3.2, we review some definition needed in the sequel and introduce GSNQVIP. While Section 3.3 deals with the existence of solution and with convergence criteria for the iterative sequences generated by PIIA.

In Section 3.4, we consider a more general generalized strongly nonlinear quasivariational inequality problem for fuzzy mappings (GSNQVIPF). We prove the existence of solutions for GSNQVIPF and discuss the convergence criteria for iterative sequences generated by the iterative algorithms.

3.2 PRELIMINARIES AND FORMULATION

Let $H$ be a Hilbert space with its dual $H^*$ whose norm and inner product are denoted by $\| \cdot \|$ and $(\cdot, \cdot)$ respectively.
The pairing between $H^*$ and $H$ is denoted by $|\cdot|$ and $(\cdot,\cdot)$ respectively. The pairing between $H^*$ and $H$ is denoted by $<\cdot,\cdot>$. Let $\Lambda$ be canonical isomorphism from $H^*$ onto $H$.

Let a multivalued mapping $K:H \rightarrow 2^H$ which associates a closed convex subset $K(u)$ of $H$ with energy element $u$ of $H$. Let $T,A:H \rightarrow 2^{H^*}$ be two multivalued mapping $g:H \rightarrow H$ be a nonlinear operator. Then strongly nonlinear quasi-variational inequality problem (GSNQVIP) is as follows:

Find $u \in H$, $x \in T(u)$ and $y \in A(u)$ such that $g(u) \in K(u)$ and

$$<x,v-g(u)> \geq <y,v-g(u)>, \text{ for all } v \in K(u),$$

where

$$K(u) = K + m(u),$$

$m$ is a single valued operator from $H$ into $H$ and $K$ is a nonempty, closed and convex subset of $H$.

**Special Cases:**

(i) If $T$ and $A$ are single valued mappings then Problem 3.2.1 is equivalent to finding $u \in H$, such that $g(u) \in K(u)$ and

$$<T(u),v-g(u)> \geq <A(u),v-g(u)>, \text{ for all } v \in K(u)$$

which is introduced by Luchuan [145] and Siddiqi and Ansari [126].

(ii) If $K$ is independent of the solution $u$, $T$ is single valued and $A(u)=0$, $\forall u \in K$ then Problem 3.2.1 reduces to the problem of finding $u \in K$, such that

$$<T(u),v-g(u)> \geq 0 \text{ for all } v \in K,$$

which is introduced and studied by Noor [88] and Isac [48].
(iii) If $m(u) = 0$, $\forall u \in K$, $T$ is single valued, $g = I$, the identity map and $A(u) = 0$, $\forall u \in K$, then Problem 3.2.1 is equivalent to Problem 1.1.1.

(iv) If $m(u) = 0$, $\forall u \in K$, $g = I$, the identity mapping and $A(u) = 0$, $\forall u \in K$, then Problem 3.2.1 is equivalent to the problem of finding $u \in K(u)$, $x \in T(u)$ such that

$$ <x, v-u> \geq 0 \text{ for all } v \in K(u), $$

which is introduced and studied by Saigal [114], Fang and Peterson [38].

We need the following lemma to suggest the following new unified iterative algorithm for GSNQVIP.

Lemma 3.2.1 [35]: Let $K$ be a nonempty closed convex subset of $H$, $K(u)$ has the form (3.2.2). Then the general strongly nonlinear quasi-variational inequality problem GSNQVIP has a solution $u \in H$, $x \in T(u)$ and $y \in A(u)$ if and only if $u \in H$ is a fixed point of the mapping $F : H \rightarrow 2^H$ defined by

$$ F(u) = \bigcup_{x \in T(u)} \bigcup_{y \in A(u)} \left[ u - g(u) + m(u) + p_K(g(u) - \rho \Lambda(x-y) - m(u)) \right] $$

for each $u \in H$, $\rho$ is some positive constant.

Perturbed Ishikawa Iterative Algorithm (PIIA):

Let $K$ be a nonempty closed convex subset of $H$ and $K(u)$ has the form 3.2.2 where $m : H \rightarrow H$. Let $g : H \rightarrow H$ and $T, A : H \rightarrow 2^{H^*}$.

Given $u_0 \in H$ and choose $x_0 \in T(u_0)$ and $y_0 \in A(u_0)$, the iterative sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ are defined by

$$ u_{n+1} = (1 - \alpha_n) u_n + \alpha_n \left[ v_n - g(v_n) + m(v_n) + p_K(g(v_n) - \rho \Lambda(x_n-y_n) - m(v_n)) \right] + e_n, $$

$$ x_{n+1} = \sigma_n x_n + \beta_n y_n, $$

$$ y_{n+1} = \zeta_n y_n + \gamma_n x_n, $$

where $\alpha_n$, $\beta_n$ and $\gamma_n$ are sequences in $(0, 1)$ and $\rho > 0$ is some positive constant.
\[ v_n = (1-\beta_n)u_n + \beta_n[u_n - g(u_n) + m(u_n) + p_k(g(u_n) - \rho \lambda (x_n - y_n) - m(u_n))] + r_n, \]

where \( e_n \) and \( r_n \) are the error terms which are taken into account the possible inexact computation of the projection points; \( x_n \in T(u_n), \bar{x}_n \in T(v_n), y_n \in A(u_n), \bar{y}_n \in A(v_n); \)

\( \rho > 0 \) is a constant, and \( \{a_n\} \) and \( \{\beta_n\} \) are the sequences in \([0,1]\) satisfy the following conditions:

(i) \( a_0 = 1 \)

(ii) \( 0 \leq a_n, \beta_n \leq 1, n \geq 0 \)

(iii) \( \sum_{n=0}^{\infty} a_n \) diverges and \( \sum_{n=0}^{\infty} \frac{1}{n} \) converges, where \( 0 \leq c < 1 \).

(iv) \( \sum_{j=0}^{\infty} \prod_{i=j+1}^{n} (1-a_i(1-c)) \) converges, where \( 0 \leq c < 1 \).

The following are the special cases of (PIIA):

(i) If \( m(u) = 0 \) for all \( u \in H \) and \( e_n = r_n = 0 \) for all \( n \geq 0 \), then PIIA reduces to Algorithm 3.1 of Ding and Deng [33].

(ii) If \( m(u) = 0, \forall u \in H \) and \( g(u) = u \) for all \( u \in H \) and \( \beta_n = 0, e_n = r_n = 0 \) for all \( n \geq 0 \), then PIIA reduces to the algorithm in Theorem 3.5 of Ding [32].

(iii) If \( T \) and \( A \) are single valued mapping and \( \beta_n = 0, \alpha_n = 1, e_n = r_n = 0 \) for all \( n \geq 0 \), then PIIA reduces to the Algorithm 3.1 of Siddiqi and Ansari [126].

(iv) If \( m(u) = 0, \forall u \in H \), \( T \) and \( A \) are single valued mapping and \( \alpha_n = 1, \beta_n = 0, e_n = r_n = 0 \) for all \( n \geq 0 \), then PIIA reduces to Algorithm 3.1 of Noor [88].
3.3 EXISTENCE THEORY

In this section, we shall establish some results for existence of solution to GSNQVIP and show that the approximate solution obtained by PIIA strongly converges to the exact solution.

Theorem 3.3.1: Let $K$ be a nonempty closed convex subset of $H$, $K(u)$ has the form (3.2.2), $T:H \rightarrow 2^{H^*}$ be $\alpha$-strongly monotone and $\beta$-Lipschitz continuous, $A:H \rightarrow 2^{H^*}$ be $\nu$-Lipschitz continuous, $g:H \rightarrow H$ be $\lambda$-strongly monotone and $\sigma$-Lipschitz continuous and let $m:H \rightarrow H$ be $\mu$-Lipschitz continuous. Assume that

$$\Re(m(u)-m(v), u-v-(g(u)-g(v))) \leq \eta \|u-v\|^2, \text{ for all } u, v \in H$$

for some constant $\eta$ such that $\eta_0 \leq \eta \leq \mu(1-2\lambda+\sigma^2)$, where,

$$\eta_0 = \inf \left\{ M: \Re(m(u)-m(v), u-v-(g(u)-g(v))) \leq M\|u-v\|^2, \forall \ u, v \in H \right\}$$

If there exists a constant $\rho > 0$ such that

$$\left| \rho - \frac{\sqrt{2(\alpha+\nu(k-1))}}{\beta^2-\nu^2} \right| < \frac{\sqrt{2(\alpha+\nu(k-1))^2 - (\beta^2-\nu^2)k(2-k)}}{\beta^2-\nu^2}$$

$$\alpha > \nu(1-k) + \frac{1}{\sqrt{2}} \sqrt{(\beta^2-k^2)k(2-k)}, \nu(1-k) < \beta \text{ and } k < 1,$$

where $k = \frac{2+1-2\lambda+\sigma^2+\mu^2+2\eta}{2\sqrt{2}}$.

Then GSNQVIP has a solution $u^* \in H$, $x^* \in T(u^*)$, $y^* \in A(u^*)$ and the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ generated by PIIA with $\lim \|e_n\| = \lim \|r_n\| = 0$, strongly converge to $u^*$, $x^*$ and $y^*$ respectively.
Proof: We first prove that GSNQVIP has a solution $u^* \in H$, $x^* \in T(u^*)$, $y^* \in A(u^*)$. By Lemma 3.2.1, it is sufficient to prove that the mapping $F$ defined in Lemma 3.2.1, has a fixed point $u^* \in H$. For any $u, v \in H$, $a \in F(u)$ and $b \in F(v)$, there exists $x_1 \in T(u)$, $y_1 \in A(u)$, $x_2 \in T(v)$ and $y_2 \in A(v)$ such that

$$a = u - g(u) + m(u) + \rho \Lambda (x_1 - y_1) - m(u) ,$$

$$b = v - g(v) + m(v) + \rho \Lambda (x_2 - y_2) - m(v) ,$$

We have

$$|a - b| \leq |u - v - (g(u) - g(v)) + m(u) - m(v)| +$$

$$+ |g(u) - \rho \Lambda (x_1 - y_1) - m(u) - g(v) + \rho \Lambda (x_2 - y_2) + m(v)|$$

(since $F$ is nonexpansive).

$$\leq |u - v - (g(u) - g(v)) + m(u) - m(v)| +$$

$$+ |g(u) - g(v) - (m(u) - m(v)) - \rho \Lambda (x_1 - x_2)| + \rho \|y_1 - y_2\|$$

$$\leq 2|u - v - (g(u) - g(v)) + m(u) - m(v)| +$$

$$+ |u - v - \rho \Lambda (x_1 - x_2)| + \rho \delta (A(u), A(v)) \quad (3.3.3)$$

Since $g$ is $\lambda$-strongly monotone and $\sigma$-Lipschitz continuous, and $m$ is $\mu$-Lipschitz continuous, it can be obtained that, using (3.3.1),

$$|u - v - (g(u) - g(v)) + m(u) - m(v)|^2 = |u - v - (g(u) - g(v))|^2 +$$

$$+ |m(u) - m(v)|^2 + 2 \Re (m(u) - m(v), u - v - (g(u) - g(v)))$$

$$\leq \{(1 - 2 \lambda + \sigma^2) + \mu^2 + 2 \eta\} |u - v|^2 \quad (3.3.4)$$

Since $T$ is strongly monotone and $\beta$-Lipschitz continuous, it can be obtained that

$$|u - v - \rho \Lambda (x_1 - x_2)|^2 \leq (1 - 2 \rho \alpha + \rho^2 \beta^2) |u - v|^2 \quad (3.3.5)$$
On combining (3.3.3), (3.3.4) and (3.3.5), and using \( \nu \)-Lipschitz continuity of \( \Lambda \), we have

\[
|a-b| \leq \left[ 2(1-2\lambda+\sigma^2+\mu^2+2\eta)^{1/2} + (1-2\rho\alpha+\rho^2\beta^2)^{1/2} + \rho \nu \right] \| u-v \|
= \theta \| u-v \| ,
\]

(3.3.6)

where \( \theta = 2(1-2\lambda+\sigma^2+2\eta)^{1/2} + (1-2\rho\alpha+\rho^2\beta^2)^{1/2} + \rho \nu \).

By condition (3.3.2), we have \( 0 < \theta < 1 \). It follows from (3.3.6) and Theorem 3.1 of Siddiqi and Ansari [124] that \( F \) has a fixed point \( u^* \in H \). By Lemma 3.2.1, there exist \( x^* \in T(u^*) \) and \( y^* \in A(u^*) \) such that \( u^* \in H, x^* \in T(u^*) \) and \( y^* \in A(u^*) \) is a solution of GSNQVIP.

Next, we prove that the iterative sequences \( \{ u_n \}, \{ x_n \} \) and \( \{ y_n \} \) strongly converge to \( u^*, x^* \) and \( y^* \) respectively. Since \( u^* \in H, x^* \in T(u^*) \), \( y^* \in A(u^*) \) is a solution of the GSNQVIP, we have

\[
u^* = u^* - g(u^*) + m(u^*) + P_K[g(u^*) - \rho \Lambda(x^*-y^*) - m(u^*)]
\]

By making the similar arguments as above, we obtain

\[
\begin{align*}
|u_n - u^* - (g(u_n) - g(u^*)) + m(u_n) - m(u^*)| & \leq (1-2\lambda+\sigma^2+2\eta)^{1/2} \| u_n - u^* \| , \\
|u_n - u^* - \rho \Lambda(x_n - x^*)| & \leq (1-2\rho\alpha+\rho^2\beta^2)^{1/2} \| u_n - u^* \| , \\
|v_n - u^* - (g(v_n) - g(u^*)) + m(v_n) - m(u^*)| & \leq (1-2\lambda+\sigma^2+\mu^2+2\eta)^{1/2} \| v_n - u^* \| , \\
|v_n - u^* - \rho \Lambda(x_n - x^*)| & \leq (1-2\rho\alpha+\rho^2\beta^2)^{1/2} \| v_n - u^* \| .
\end{align*}
\]

Thus, by the \( \nu \)-Lipschitz continuity of \( \Lambda \), we obtain

\[
\begin{align*}
|u_{n+1} - u^*| &= |(1-\alpha_n)u_n + \alpha_n[v_n - g(v_n) + m(v_n) + \\
&+ P_K[g(v_n) - \rho \Lambda(x_n - y_n) - m(v_n)] - \\
&- (1-\alpha_n)u^* - \alpha_n[u^* - g(u^*) + m(u^*) + \\
&+ P_K[g(u^*) - \rho \Lambda(x^*-y^*) - m(u^*)]] + e_n| ,
\end{align*}
\]
\[\|v_n - u^*\| \leq (1 - \alpha_n) \|u_n - u^*\| + \alpha_n \|v_n - u^* - (g(v_n) - g(u^*)) + m(v_n) - m(u^*)\| + \\
+ \|v_n - u^* - \rho \Lambda (\bar{x}_n - \bar{y}_n)\| + \rho \|\bar{y}_n - y^*\| + \|e_n\|,\]
(since \(P_K\) is nonexpansive)
\[\leq (1 - \alpha_n) \|u_n - u^*\| + \alpha_n \theta \|v_n - u^*\| + \|e_n\| \tag{3.3.7}\]

Similarly, we have
\[\|v_n - u^*\| = \|(1 - \beta_n)u_n + \beta_n [u_n - g(u_n) + m(u_n) + \\
+ P_K [g(u_n) - \rho \Lambda (\bar{x}_n - \bar{y}_n) - m(u_n))]\| + r_n - \\
- (1 - \beta_n)u^* - \beta_n [u^* - g(u^*) + m(u^*) + \\
+ P_K [g(u^*) - \rho \Lambda (x^* - y^*) - m(u^*))]\| |
\[\leq (1 - \beta_n) \|u_n - u^*\| + \beta_n \theta \|u_n - u^*\| + \|r_n\|
\leq \|u_n - u^*\| + \|r_n\| \tag{3.3.8}\]
(since \((1 - \beta_n(l - \theta)) \leq 1\).

It follows from (3.3.7) and (3.3.8) that
\[\|u_{n+1} - u^*\| \leq (1 - \alpha_n) \|u_n - u^*\| + \alpha_n \theta \|u_n - u^*\| + \alpha_n \theta \|r_n\| + \|e_n\|
= (1 - \alpha_n(1 - \theta)) \|u_n - u^*\| + \alpha_n \theta \|r_n\| + \|e_n\|
\leq \prod_{i=0}^{n} (1 - \alpha_1(1 - \theta)) \|u_0 - u^*\| + \theta \sum_{j=0}^{n} \prod_{i=j+1}^{n} (1 - \alpha_1(1 - \theta)) \|r_j\|
+ \sum_{j=0}^{n} \prod_{i=j+1}^{n} (1 - \alpha_1(1 - \theta)) \|e_j\|, \tag{3.3.9}\]

where \(\prod_{i=j+1}^{n} (1 - \alpha_1(1 - \theta)) = 1\) when \(j = n\).

Now, let \(B\) denotes lower triangular matrix with entries
\[b_{nj} = \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_1(1 - \theta)).\]

Then \(B\) is multiplicative, see Rhoades [112], so that
Let $D$ be the lower triangular matrix with entries
\[ d_{nj} = \prod_{i=j+1}^{n} (1-\alpha_i(1-\theta)) . \]
Condition (iv) of PIIA implies that $D$ is multiplicative, and hence
\[ \lim_{n \to \infty} \sum_{j=0}^{n} \prod_{i=j+1}^{n} (1-\alpha_i(1-\theta)) \|e_j\| = 0 , \]
since $\lim_{n \to \infty} \|e_n\| = 0$. Also condition (iii) of PIIA implies
\[ \lim_{n \to \infty} \prod_{i=0}^{n} (1-\alpha_i(1-\theta)) = 0 . \]
Hence, it follows from (3.3.9) that $\lim_{n \to \infty} \|u_{n+1}-u^*\| = 0$, i.e. the sequence $\{u_n\}$ strongly converges to $u^*$ in $H$. The inequality (3.3.8) implies that $v_n$ strongly converges to $u^*$. Since $x_n \in T(v_n)$, $x^* \in T(u^*)$ and $T$ is $\beta$-Lipschitz continuous, we have
\[ \|x_n - x^*\| \leq \delta(T(v_n), T(u^*)) \leq \beta \|v_n - u^*\| \to 0 \text{ as } n \to \infty \]
and hence the sequence $\{x_n\}$ strongly converges to $x^*$. Similarly we can show that the sequence $\{\bar{y}_n\}$ strongly converges to $y^*$.

**Theorem 3.3.2:** Let $K$ be a nonempty closed and convex subset of $H$; $K(u)$ be of the form (3.2.2), $T:H \to 2^H$ be $\alpha$-strongly...
monotone and $\beta$-Lipschitz continuous; $A: H \to 2^H$ be $\nu$-Lipschitz continuous; $(g-m): H \to H$ be $\lambda$-strongly monotone; $g: H \to H$ be $\sigma$-Lipschitz continuous and let $m: H \to H$ be $\mu$-Lipschitz continuous. Assume that

$$\text{Re}(m(v) - m(u), g(u) - g(v)) \leq \eta \|u - v\|^2, \text{ for all } u, v \in H \tag{3.3.10}$$

for some constant $\eta$ such that $\eta_0 \leq \eta \leq \mu \sigma$, where

$$\eta_0 = \inf \left\{ M: \text{Re}(m(v) - m(u), g(u) - g(v)) \leq M\|u - v\|^2, \forall u, v \in H \right\}.$$

If there exist a constant $\rho > 0$ such that

$$\rho - \frac{\sqrt{2(\alpha + \nu(k-1))}}{\beta^2 - \nu^2} < \frac{\sqrt{2(\alpha + \nu(k-1))^2 - (\beta^2 - \nu^2)k(2-k)}}{\beta^2 - \nu^2},$$

$$\alpha > \nu(1-k) + \frac{1}{\sqrt{2}} \sqrt{(\beta^2 - \nu^2)k(2-k)}, \nu(1-k) < \beta \text{ and } k < 1, \tag{3.3.11}$$

where $k = 2\lambda + \sigma^2 + \mu^2 + 2\eta$.

Then GSNQVIP has a solution $u^* \in H$, $x^* \in T(u^*)$, $y^* \in A(u^*)$ and the iterative sequences $\{u_n\}, \{x_n\}, \{y_n\}$ generated by PIIA with

$$\lim\|e_n\| = \lim\|r_n\| = 0$$

strongly converge to $u^*$, $x^*$ and $y^*$ respectively.

**Proof:** We first prove that GSNQVIP has a solution $u^* \in H$, $x^* \in T(u^*)$, $y^* \in A(u^*)$. In the light of Lemma 3.2.1, it is enough to show that the mapping $F$ defined in Lemma 3.2.1 has a fixed point $u^* \in H$.

For any $u, v \in H$, $a \in F(u)$ and $b \in F(v)$, there exist $x_1 \in T(u)$, $y_1 \in A(u)$, $x_2 \in T(v)$ and $y_2 \in A(v)$ such that...
a = u - g(u) + m(u) + P_\mathcal{K}(g(u) - \rho \Lambda(x_1 - y_1) - m(u)) ,

b = v - g(v) + m(v) + P_\mathcal{K}(g(v) - \rho \Lambda(x_2 - y_2) - m(v)) .

We have

\[ |a - b| \leq 2|u - v - (g(u) - g(v)) + m(u) - m(v)| + \\
+ |u - v - \rho \Lambda(x_1 - x_2)| + \rho |y_1 - y_2| \]

\[ \leq 2|u - v - (g(u) - g(v)) + m(u) - m(v)| + \\
+ |u - v - \rho \Lambda(x_1 - x_2)| + \rho \delta(A(u), A(v)) \quad (3.3.12) \]

Since \((g - m)\) is \(\lambda\)-strongly monotone, \(g\) is \(\sigma\)-Lipschitz continuous and \(m\) is \(\mu\)-Lipschitz continuous, it can be obtained that, using \((3.3.10)\),

\[ |u - v - (g(u) - g(v)) + m(u) - m(v)|^2 = |u - v|^2 - 2Re(u - v, (g(u) - m(u)) - \\
- (g(v) - m(v))) + |m(u) - m(v)|^2 + \\
+ |g(u) - g(v)|^2 + 2Re(m(v) - m(u), (g(u) - g(v))) \]

\[ \leq (1 - 2\lambda + \sigma^2 + \mu^2 + 2\eta) |u - v|^2 \quad (3.3.13) \]

From \((3.3.5)\), \((3.3.12)\) and \((3.3.13)\) and the \(\nu\)-Lipschitz continuity of \(A\), we have

\[ |a - b| \leq [2(1 - 2\lambda + \sigma^2 + \mu^2 + 2\eta)^{1/2} + (1 - 2\rho \alpha + \rho^2 \beta^2)^{1/2} + \rho \nu] |u - v| \]

\[ \leq \theta |u - v| , \quad (3.3.14) \]

where \(\theta = [2(1 - 2\lambda + \sigma^2 + \mu^2 + 2\eta)^{1/2} + (1 - 2\rho \alpha + \rho^2 \beta^2)^{1/2} + \rho \nu] \).

By condition \((3.3.11)\), we have \(0 < \theta < 1\). It follows from \((3.3.13)\) and Theorem 3.1 of Siddiqi and Ansari [124] that \(F\) has a fixed point \(u^* \in \mathcal{H}\). Hence, by Lemma 3.2.1, there exist \(x^* \in \mathcal{T}(u^*)\) and \(y^* \in \mathcal{A}(u^*)\) such that \(u^* \in \mathcal{H}, x^* \in \mathcal{T}(u^*)\) and \(y^* \in \mathcal{A}(u^*)\) is a solution of GSNQVIP. By using the similar arguments as
above and the technique used in Theorem 3.3.1, we can easily prove that the iterative sequences \{u_n\}, \{x_n\} and \{y_n\} strongly converges to \(u^*\), \(x^*\) and \(y^*\) respectively.

3.4 EXISTENCE THEORY OF COMPLETELY GENERALIZED STRONGLY NONLINEAR QUASI-VARIATIONAL INEQUALITY

Let \(H\) be a real Hilbert space with norm \(|\cdot|\) and inner product \(<\cdot,\cdot>\) respectively. Let \(K:H\rightarrow\mathbb{C}(H)\), where \(\mathbb{C}(H)\) denotes the family of all nonempty closed and convex subset of \(H\).

Let \(\mathcal{F}(H)\) be a collection of all fuzzy sets on \(H\). A mapping \(F:H\rightarrow\mathcal{F}(H)\) is called a fuzzy mapping on \(H\). If \(F\) is a fuzzy mapping on \(H\), \(F(x), x\in H\) (denoted by \(F_x\)) is a fuzzy set in \(\mathcal{F}(H)\) and \(F_x(s), s\in H\) is the degree of membership of \(s\) in \(F_x\). Let \(S\in\mathcal{F}(H)\) and \(m\in[0,1]\) then the set \((S)^m = \{x\in H: S(x)\geq m\}\) is said to be an \(m\)-cut set of \(S\).

Further, let \(F, G:H\rightarrow\mathcal{F}(H)\) be two fuzzy mappings such that there exists two real numbers \(p, q\in(0,1]\) such that for all \(x\in H\), the sets \((F_x)^p\) and \((G_x)^q\) belong to \(\mathbb{C}(H)\), where \(\mathbb{C}(H)\) denotes the family of all nonempty bounded closed subsets of \(H\).

Now we define two multivalued mappings \(\tilde{F}, \tilde{G}:H\rightarrow\mathcal{F}(H)\) by \(\tilde{F}(x) = (F_x)^p\) and \(\tilde{G}(x) = (G_x)^q\) respectively for any \(x\in H\).

Given two nonlinear operators \(A, T:H\rightarrow H\), the real numbers \(p, q\in(0,1]\) and fuzzy mappings \(F, G:H\rightarrow\mathcal{F}(H)\), the completely generalized strongly nonlinear quasi-variational inequality problem for fuzzy mappings (in short, CGSNQVIPF) is to find
u,x,y \in H \text{ such that } F_u(x) > p, \ G_u(y) > q, \ u \in K(u) \text{ and }

\langle Ax + Ty, v - u \rangle \geq 0, \text{ for all } v \in K(u). \quad (3.4.1)

In 1989, Chang and Zhu [17] first introduced and studied a class of variational inequalities for fuzzy mappings. Recently, several kinds of variational and quasi-variational inequality problems for fuzzy mappings were considered by Chang [20], Lee et al [66,67] and Noor [94], etc.

In particular, if \( F,G:H \rightarrow 2^H \) are classical multivalued mappings then Problem (3.4.1) is equivalent to find \( u,x,y \in H \) such that \( x \in F(u), \ y \in G(u), \ u \in K(u) \) and

\[ <Ax+Ty, v-u> \geq 0, \text{ for all } v \in K(u). \]  

(3.4.2)

First, we prove the following technical lemma:

**Lemma 3.4.1:** Let \( K:H \rightarrow \mathcal{C}(H) \) then \( u \in H, \ x \in F(u), \ y \in G(u) \) satisfy (3.4.1) if and only if \( u \in H, \ x \in F(u), \ y \in G(u) \) satisfy \( u \in K(u) \) and the relation

\[ u = P_K(u) \left( u - \rho (Ax+Ty) \right) \]  

(3.4.3)

for some \( \rho > 0 \).

**Proof:** Let \( u \in H, \ x \in F(u), \ y \in G(u) \) are solution of GQVIPFM then \( u \in K(u) \) and

\[ <Ax+Ty, v-u> \geq 0, \text{ for all } v \in K(u) \]

\[ <\rho (Ax+Ty), v-u> \geq 0, \text{ for some } \rho > 0 . \]

\[ <u-(u-\rho (Ax+Ty)), v-u> \geq 0 \]

By Lemma 1.2.3, we have

\[ u = P_K(u) \left( u - \rho (Ax+Ty) \right). \]
Conversely, suppose that \( u \in H, \ x \in F(u), \ y \in G(u) \) satisfy \( u \in K(u) \) and the relation
\[
u = P_K(u) \left( u - \rho(Ax + Ty) \right), \text{ for some } \rho > 0.\]
Again, by Lemma 1.2.3, we have
\[
\langle u - (u - \rho(Ax + Ty)), v - u \rangle \geq 0,
\]
\[
\langle \rho(Ax + Ty), v - u \rangle \geq 0, \text{ for all } v \in K(u).
\]
\[
\langle Ax + Ty, v - u \rangle = 0, \text{ where } \rho > 0.
\]
Hence, \( u \in H, \ x \in F(u), \ y \in G(u) \) are solution of (3.4.1).

Based on Lemma (3.4.1), we propose the following general and unified iterative algorithm for CGSNQVIPF:

**Iterative Algorithm 3.4.1:** Suppose that \( K \in \text{CC}(H) \) and \( A, T : H \rightarrow H \). Let \( F, G : H \rightarrow \text{CB}(H) \) be multivalued mappings induced by the fuzzy mappings \( F, G : H \rightarrow \mathcal{F}(H) \).

Now, for given \( u_0 \in H \), we take \( x_0 \in F(u_0) \) and \( y_0 \in G(u_0) \) and let, for \( 0 < \lambda < 1 \),
\[
u_1 = (1 - \lambda)u_0 + \lambda P_K(u_0) \left( u_0 - \rho(Ax_0 + Ty_0) \right).
\]

Since \( x_0 \in F(u_0) \in \text{CB}(H) \), \( y_0 \in G(u_0) \in \text{CB}(H) \), by Nadler [76] it follows that for each \( \varepsilon > 0 \), there exists \( x_1 \in F(u_1) \) and \( y_1 \in G(u_1) \) such that
\[
\| x_0 - x_1 \| \leq (1 + \varepsilon) H(Fu_0, Fu_1) \text{ and }
\]
\[
\| y_0 - y_1 \| \leq (1 + \varepsilon) H(Gu_0, Gu_1),
\]
where \( H(\cdot, \cdot) \) is a Hausdorff metric on \( \text{CB}(H) \). Let
\[
u_2 = (1 - \lambda)u_1 + \lambda P_K(u_1) \left( u_1 - \rho(Ax_1 + Ty_1) \right).
\]
By induction, we can obtain sequences \( \{u_n\} \), \( \{x_n\} \), and \( \{y_n\} \) such that for each \( n = 1, 2, 3, \ldots \),

\[
\begin{align*}
&x_n \in \mathcal{F}u_n, \quad ||x_n - x_{n+1}|| \leq (1+\epsilon/2^n) \text{ } H(\mathcal{F}u_n, \mathcal{F}u_{n+1}) \\
y_n \in \mathcal{G}u_n, \quad ||y_n - y_{n+1}|| \leq (1+\epsilon/2^n) \text{ } H(\mathcal{G}u_n, \mathcal{G}u_{n+1}) \\
u_{n+1} = (1-\lambda)u_n + \lambda \text{ } P_{K}(u_n)(u_n - \rho(Ax_n + Ty_n)),
\end{align*}
\]

where \( 0 < \lambda < 1 \) and \( \rho > 0 \) are both constants.

If \( F,G:H\rightarrow \text{CB}(H) \) are two classical multivalued mappings then from Iterative Algorithm 3.4.1 we can obtain the following:

**Iterative Algorithm 3.4.2:** For given \( u_0 \in H, \ x_0 \in \mathcal{F}u_0, \ y_0 \in \mathcal{G}u_0, \) we can obtain the sequences \( \{x_n\}, \{y_n\}, \) and \( \{u_n\} \) as follows:

\[
\begin{align*}
&x_n \in \mathcal{F}u_n, \quad ||x_n - x_{n+1}|| \leq (1+\epsilon/2^n) \text{ } H(\mathcal{F}u_n, \mathcal{F}u_{n+1}) \\
y_n \in \mathcal{G}u_n, \quad ||y_n - y_{n+1}|| \leq (1+\epsilon/2^n) \text{ } H(\mathcal{G}u_n, \mathcal{G}u_{n+1}) \\
u_{n+1} = (1-\lambda)u_n + \lambda \text{ } P_{K}(u_n)(u_n - \rho(Ax_n + Ty_n)),\quad n = 0, 1, 2, \ldots
\end{align*}
\]

By using the technique of Kazmi [56], we study the existence of solutions of CGSNQVIPF and discuss the convergence criteria for the iterative sequences generated by Iterative Algorithm (3.4.1).

**Theorem 3.4.1:** Let \( K \in \text{CC}(H) \) and \( \mathcal{F}, \mathcal{G}:H\rightarrow \text{CB}(H) \) be multivalued mappings induced by the fuzzy mapping \( F,G:H\rightarrow \mathcal{F}(H) \) such that there exist \( p,q \in (0,1] \) such that \( \mathcal{F}(x) = (Fx)^p \) and \( \mathcal{G}(x) = (Gx)^q \) respectively. Let \( A:H\rightarrow H \) be \( \beta \)-Lipschitz continuous, \( \mathcal{F}, \mathcal{G} \) be \( \sigma \)-H-Lipschitz continuous and \( \eta \)-H-Lipschitz continuous.
respectively, \( F \) be \( \alpha \)-strongly monotone with respect to \( A \) and let \( T:H\rightarrow H \) be \( \xi \)-Lipschitz continuous. If there exists a constant \( c > 0 \) such that
\[
\|P_K(u)^+ - P_K(v)^+\| \leq c \|u - v\|, \quad \text{for all } u, v, w \in H, \quad (3.4.6)
\]
and if
\[
0 < \rho < \frac{2(\alpha - \xi \eta)}{\beta \sigma^2 - \xi^2 \eta^2}, \quad \beta \sigma > \alpha > \xi \eta, \quad \alpha \xi \eta < 1,
\]
\[
c < 1 - \sqrt{1 - 2\rho \alpha + \rho \beta \sigma^2 - \rho ^2 \xi \eta}.
\]

Then there exist \( u, x, y \in H \) which are the solutions of Problem 3.4.1. Furthermore, the sequences \( \{u_n\}, \{x_n\} \) and \( \{y_n\} \) generated by Iterative Algorithm (3.4.1) strongly converges to \( u, x, y \) in \( H \) respectively.

**Proof:** From the Iterative Algorithm 3.4.1, we have
\[
\|u_{n+1} - u_n\| \leq (1-\lambda) \|u_n - u_{n-1}\| + \lambda \|P_K(u_n)(u_n - \rho (Ax_n + Ty_n)) - P_K(u_{n-1})(u_{n-1} - \rho (Ax_{n-1} + Ty_{n-1}))\|
\]
\[
\leq (1-\lambda) \|u_n - u_{n-1}\| + \lambda \|P_K(u_n)(u_n - \rho (Ax_n + Ty_n)) - P_K(u_{n-1})(u_{n-1} - \rho (Ax_{n-1} + Ty_{n-1}))\|
\]
\[
\leq (1-\lambda) \|u_n - u_{n-1}\| + \lambda c \|u_n - u_{n-1}\| + \lambda \|u_n - u_{n-1} - \rho (Ax_n - Ax_{n-1})\| + \lambda \rho \|Ty_n - Ty_{n-1}\| \quad (3.4.8)
\]

Since $F$ is $\alpha$-strongly monotone with respect to $A$ and $\sigma$-$H$-Lipschitz continuous, and $A$ is $\beta$-Lipschitz continuous, we have
\[
\|u_n - u_{n-1} - \rho(Ax_n - Ax_{n-1})\|^2 \leq (1-2\rho\alpha + \rho^2\beta^2(1+\varepsilon/2^n)^2\sigma^2)\|u_n - u_{n-1}\|^2.
\] (3.4.9)

Further, since $G$ is $\eta$-$H$-Lipschitz continuous and $T$ is $\xi$-Lipschitz continuous, we get
\[
\|Ty_n - Ty_{n-1}\| \leq \xi\|y_n - y_{n-1}\| = \xi\eta(1+\varepsilon/2^n)\|u_n - u_{n-1}\|. \tag{3.4.10}
\]

From (3.4.8)-(3.4.10), we have
\[
\|u_{n+1} - u_n\| \leq \theta_n\|u_n - u_{n-1}\|, \tag{3.4.11}
\]
where $\theta_n = \lambda c + (1-\lambda) + \lambda(1-2\rho\alpha + \rho^2\beta^2(1+\varepsilon/2^n)^2\sigma^2 + \lambda\rho\varepsilon\eta(1+\varepsilon/2^n))$.

Denoted by $\theta := \lim_{n \to \infty} \theta_n$, it is easy to see that
\[
\theta = \lambda c + (1-\lambda) + \lambda(1 - 2\rho\alpha + \rho^2\beta^2\sigma^2 + \lambda\rho\varepsilon\eta).
\]

But condition (3.4.7) gives $0 < \theta < 1$. Hence $\theta_n < 1$, for $n$ sufficiently large. Therefore (3.4.11) implies that $\{u_n\}$ is a Cauchy sequence in $H$. Letting $u_n \to u$ as $n \to \infty$, by Iterative Algorithm (4.3.1) we have
\[
\|x_n - x_{n-1}\| \leq (1+\varepsilon/2^{n-1}) \sigma\|u_n - u_{n-1}\|
\]
\[
\|y_n - y_{n-1}\| \leq (1+\varepsilon/2^{n-1}) \eta\|u_n - u_{n-1}\|.
\]

These imply that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $H$. Letting $x_n \to x$, $y_n \to y$ as $n \to \infty$. By using the continuity of $P_K(u)$, $A$, $T$, and Iterative Algorithm (4.3.1), we have
\[ u = (1 - \lambda)u + \lambda P_K(u)(u - \rho(Ax + Ty)) \]
\[ u = P_K(u)(u - \rho(Ax + Ty)) \in K(u). \]

Further we have
\[
d(x, Fu) = \inf \left\{ \|x - z\| : z \in Fu \right\}
\leq \|x - x_n\| + d(x_n, Fu)
\leq \|x - x_n\| + \sigma \|u_n - u\|
\rightarrow 0 \text{ as } n \rightarrow \infty
\]

Hence \( x \in Fu \). Similarly \( y \in Gu \).

By Lemma 3.4.1, it follows that \( u \in H, x \in Fu, y \in Gu \) are solution of Problem (3.4.1) and the sequences \( \{u_n\}, \{x_n\} \) and \( \{y_n\} \) generated by Iterative Algorithm (3.4.1), strongly converges to \( u, x, y \) in \( H \) respectively.

**Application:** Let \( K \) be a nonempty closed convex cone in \( H \) then the set
\[
K^* := \{x \in H : \langle y, x \rangle \geq 0, \forall y \in K\}
\]
is called the polar cone of \( K \).

Now we prove the following Lemma:

**Lemma 3.4.2:** Let \( K(u) := u - g(u) + K \) for each \( u \in H \), where \( g : H \rightarrow H \) is a nonlinear mapping, then Problem (3.4.1) and the implicit complementarity problem for fuzzy mappings of finding \( u, x, y \in H \) such that \( F_u(x) \geq p, \ G_u(y) \geq q, \ g(u) \in K \) and
\[
Ax + Ty \in K^* \text{ and } \langle g(u), Ax + Ty \rangle = 0
\]
have the same set of solutions.

**Proof:** Let \( u, x, y \in H \) such that \( F_u(x) \geq p, \ G_u(y) \geq q, \ u \in K(u) \) are solution of Problem (3.4.1). Since \( u \in K(u) := u - g(u) + K, \)
g(u) ∈ K and hence 2g(u) ∈ K. Since 0 ∈ K, u - g(u) ∈ K(u). Taking 
\( v := u + g(u) \) ∈ K(u), we get

\[
0 \leq \langle Ax + Ty, v - u \rangle = \langle Ax + Ty, u + g(u) - u \rangle \\
0 \leq \langle Ax + Ty, g(u) \rangle \\
0 \leq \langle Ax + Ty, u - g(u) - u \rangle = \langle Ax + Ty, -g(u) \rangle \\
\langle Ax + Ty, g(u) \rangle \leq 0,
\]

\[\Rightarrow \langle Ax + Ty, g(u) \rangle = 0.\]

Conversely, let \( u \in H, x \in F(u) \) and \( y \in G(u) \) be a solution of 
Problem (3.4.1). Since \( K(u) := u - g(u) + K \), if \( v \in K(u) \) then 
\( v = u - g(u) + z \) for some \( z \in K \). Using the assumption we have

\[
\langle Ax + Ty, v - u \rangle = \langle Ax + Ty, u - g(u) + z - u \rangle \\
= \langle Ax + Ty, z \rangle - \langle Ax + Ty, g(u) \rangle \\
= \langle Ax + Ty, z \rangle \geq 0.
\]

We remark that Theorem 3.4.1 with \( K(u) := u - g(u) + K \) gives 
also the suitable conditions for the existence of solution 
to complementarity problem for fuzzy mappings. In this 
\textit{case}, condition 3.4.1 is satisfied if \( u - g(u) \) is \( c/2 \)- 
Lipschitz continuous. It is followed by a well known 
property of projection mapping i.e.

\[\| P_{a + K}(w) - P_K(w) \| \leq \| a \|, \text{ for all } w \in K,\]

where \( a \in H \) is fixed.