CHAPTER I

PRELIMINARIES
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1.1. INTRODUCTION: The object of this chapter is to introduce basic concepts, preliminary notions and some fundamental results, which we shall require for the development of the subject in the present thesis. No attempt will, however, be made to deal with such elementary concepts as those of groups, rings, ideals, fields, modules, vectorspaces, homomorphisms, direct sums and direct products of ideals and rings etc. For most of the material presented in this chapter, we refer to Jacobson [41], Mc Coy([50], [51]), Lambek [46], Divinsky [15], Kurosh [45], Herstein [32], [33] and Rowen [58].

By a ring, until otherwise mentioned, we mean an associative ring (may be without unity) containing at least two elements.

1.2. We begin with the following:

DEFINITION 1.2.1 (NILPOTENT ELEMENT): An element \( a \) of a ring \( R \) is said to be nilpotent if there exists a positive integer \( n \) such that \( a^n = 0 \).

Zero of a ring is trivially nilpotent, moreover, every nilpotent element is necessarily a divisor of zero. For if \( a \neq 0 \) and \( n \) is the smallest positive integer such that \( a^n = 0 \), and \( a(a^{n-1}) = 0 \) with \( a^{n-1} \neq 0 \), because \( n > 1 \).

DEFINITION 1.2.2 (LIE AND JORDAN STRUCTURES): Given an associative ring \( R \) we can induce on \( R \), using its operations, two structures, the Lie structure and the Jordan structure by defining the products
[x, y] = xy - yx and (xoy) = xy + yx respectively.

**DEFINITION 1.2.3 (PRIME IDEAL):** An ideal $P$ in a ring $R$ is said to be a **Prime ideal** if and only if it has the following property:

If $A$ and $B$ are ideals in $R$ such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

**DEFINITION 1.2.4 (MAXIMAL IDEAL):** An ideal $M$ of a ring $R$ is called a **Maximal ideal** if $M \neq R$ and there exists no ideal $A$ in $R$ such that $M \subset A \subset R$.

Thus if $M(\neq R)$ is a maximal ideal, then for any ideal $A$ of $R$, $M \subset A \subset R$ holds only when either $A = M$ or $A = R$.

**DEFINITION 1.2.5 (SEMIPRIME IDEAL):** An ideal $Q$ in a ring $R$ is said to be **Semiprime** if and only if it has the following property:

If $A$ is an ideal in a ring $R$ such that $A^2 \subseteq Q$, then $A \subseteq Q$.

**DEFINITION 1.2.6 (COMMUTATOR IDEAL):** An ideal of a ring $R$ generated by all commutators $[x, y]$, with $x, y$ in $R$ is known as **Commutator ideal** of $R$.

**DEFINITION 1.2.7 (NILPOTENT IDEAL):** An ideal $A$ of a ring $R$ is said to be **nilpotent** if there exists a positive integer $n$ such that $A^n = (0)$.

**DEFINITION 1.2.8 (NIL IDEAL):** An ideal $A$ in a ring $R$ is said to be a **Nil ideal** if every element of $A$ is nilpotent.
Every nilpotent ideal is obviously, a nilideal but not conversely. We for the moment provide an example of a nil ideal which is not nilpotent.

**EXAMPLE 1.2.1:** Let \( \mathbb{Z}_n \) (\( n > 1 \)) be the rings of integers modulo \( p^n \) where \( p \) is a fixed prime. Let \( F = \{ \bar{a}_n \mid \bar{a}_n \in \mathbb{Z}_n \} \) i.e. \( F \) is the family of all sequences \( \{ \bar{a}_n \} \) with \( \bar{a}_n \in \mathbb{Z}_n \). Define addition and multiplication on \( F \) as follows:

\[
\{ \bar{a}_n \} + \{ \bar{b}_n \} = \{ a_n + b_n \} \quad \text{and} \quad \{ \bar{a}_n \} \{ \bar{b}_n \} = \{ a_n b_n \}
\]

Then \( F \) is a ring with sequence \( \{ \bar{0}_n \} \) working as zero, where \( \bar{0}_n \) is zero of \( \mathbb{Z}_n \) and \(-\{ \bar{a}_n \} = \{-\bar{a}_n \} \). Now let \( R \) be the set of all those sequences of \( F \) which have only finitely many non-zero terms. Then \( R \) is an ideal of \( F \).

We consider \( R \) as a ring. Let \( I \) be the ideal of \( R \) consisting of all elements of type \((\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n, \bar{0}, \ldots, -)\), \( \bar{r}_k \in \mathbb{Z}_p \).

Let \( a = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n, \bar{0}, \bar{0}, \ldots) \in I \). Since in \( \mathbb{Z}_p \), \( (\bar{p}_1)^i = \bar{0} \), we get \( (\bar{p}_1)^k = \bar{0} \) for every \( i \leq k \). Thus \( a^k = (\bar{0}, \bar{0}, \bar{0}, \ldots) \). Hence \( I \) is a nilideal. We claim that \( I \) is not nilpotent. Now we show that for any positive integer \( n \) there exists an element \( a \in I \) such that \( a^n \neq 0 \).

Define \( a = (\bar{a}_k) \) by taking \( \bar{a}_{n+1} = \bar{p} \) in \( \mathbb{Z}_{p^{n+1}} \) and \( \bar{a}_k = \bar{0} \) for every \( k \neq n+1 \). Thus \( a = (\bar{0}, \bar{0}, \bar{0}, \ldots, \bar{p}, \bar{0}, \bar{0}, \ldots, -) \).

Since in \( \mathbb{Z}_{p^{n+1}} \), \( \bar{p}_n \neq \bar{0} \), we get

\[
a_n = (\bar{0}, \bar{0}, \bar{0}, \ldots, \bar{p}_n, \bar{0}, \bar{0}, \ldots) \neq 0
\]

Hence \( I^n \neq (0) \), for every integer \( n > 0 \). Consequently \( I \) is not a nilpotent ideal.
DEFINITION 1.2.9 (PRIME RADICAL OF AN IDEAL): The prime radical of an ideal $A$ in a ring $R$ is the intersection of all the prime ideals in $R$ which contain $A$.

DEFINITION 1.2.10 (SUBDIRECT SUM OF RINGS): Let $T$ be a subring of the direct sum $S$ of a family of rings $S_i$, $i \in U$ indexed by the set $U$ and for each $i \in U$, let $\theta_i$ be a homomorphism of $S$ onto $S_i$ defined as follows:

$$a \theta_i = a(i), \ a \in S.$$ 

If $T \theta_i = S_i$ for every $i \in U$, $T$ is said to be a subdirect sum of the rings $S_i$, $i \in U$.

DEFINITION 1.2.11 (IDEMPOTENT ELEMENT): An element $e$ of a ring $R$ is said to be idempotent if $e^2 = e$.

It is obvious that zero is an idempotent element of every ring. Moreover, if $R$ contains unity $1$, then $1$ is also idempotent. Mathematics abounds with examples of rings having one or more idempotent elements other than $0$ and $1$ as well.

DEFINITION 1.2.12 (IRREDUCIBLE R-MODULE): A module $M$ over a ring $R$ is said to be an irreducible $R$-module if $MR \neq (0)$ and if the only submodules of $M$ are $(0)$ and $M$.

DEFINITION 1.2.13 (FAITHFUL R-MODULE): A module $M$ over a ring $R$ is a faithful $R$-module (or that $R$ acts faithfully on $M$) if $Mr = (0)$ forces $r = 0$, $r \in R$. 
DEFINITION 1.2.14 (JACOBSON RADICAL): The Jacobson radical $J(R)$ of $R$ is the set of all those elements of $R$ which annihilate all the irreducible $R$-modules. In case $R$ has no irreducible modules, we say $J(R) = R$.

DEFINITION 1.2.15 (PRIME RING): A ring $R$ is said to be Prime if and only if its zero ideal is a prime ideal in $R$.

Equivalently, a ring $R$ is prime if and only if either of the following conditions holds:

(i) If $A$ and $B$ are ideals in $R$ such that $AB = (0)$, then $A = (0)$ or $B = (0)$.

(ii) If $a, b \in R$ such that $aRb = (0)$, then $a = 0$ or $b = 0$.

DEFINITION 1.2.16 (SEMI PRIME RING): A ring with zero prime radical is called Semi prime. This is equivalent to say that a semi prime ring is one that contains no nonzero nilpotent ideals.

DEFINITION 1.2.17 (SUBDIRECTLY IRREDUCIBLE RING): A ring $R$ is said to be Subdirectly irreducible if the intersection of all its nonzero ideals is not zero.

DEFINITION 1.2.18 (PRIMITIVE RING): A ring $R$ is Primitive if it has a faithful irreducible $R$-module.

DEFINITION 1.2.19 (SEMI SIMPLE RING): A ring $R$ is said to be Semi simple if $J(R) = (0)$.

DEFINITION 1.2.20 (BOOLEAN RING): A ring $R$ is called Boolean if all of its elements are idempotent.
REMARKS

1. A ring $R$ is semi prime if and only if it is a subdirect sum of prime rings.

2. A ring $R$ is semi simple if and only if it is isomorphic to a subdirect sum of primitive rings.

3. Let $R$ be a ring having no nonzero nilideals. Then $R$ is a subdirect sum of prime rings.

4. A subdirectly irreducible ring has no non-trivial representation as a subdirect sum of any rings.

5. A primitive ring is isomorphic to an irreducible ring of endomorphisms of some abelian groups.

6. Every simple ring is subdirectly irreducible.

7. Every ring can be represented as a subdirect sum of subdirectly irreducible rings.

8. A Boolean ring has characteristic 2 and is necessarily commutative.

DEFINITION 1.2.21 (CHARACTERISTIC OF A RING): If there exists a positive integer $n$ such that $na = 0$ for every element $a$ of a ring $R$, the smallest such positive integer is called Characteristic of $R$ (usually written as char $R$). If no such positive integer exists, $R$ is said to have characteristic zero.

Obviously if char $R \neq m$, then $ma = 0$ for some $a \in R$ implies that $a = 0$.

DEFINITION 1.2.22 (CENTER OF A RING): Center $Z(R)$ of a ring $R$ is defined as follows:
Thus a ring \( R \) is commutative if and only if \( Z(R) = R \).

**DEFINITION 1.2.23 (ALGEBRA):** An associative ring \( A \) is called an *algebra* over a field \( F \) if \( A \) is a vectorspace over \( F \) such that for all \( a, b \in A \) and \( \alpha \in F \), \( \alpha(ab) = (\alpha a)b = a(\alpha b) \).

An algebra is said to be *central simple* over a field \( F \) if \( A \) is a simple algebra having \( F \) as its center.

**DEFINITION 1.2.24 (FIELD EXTENSION):** A *field extension* of a field \( F \) is a pair \((K, \sigma)\) where \( K \) is a field and \( \sigma \) is a monomorphism of \( F \) into \( K \).

In case of a field extension \((K, \sigma)\) of a field \( F \), we can identify \( F \) with its isomorphic copy \( \sigma(F) \), a subfield of \( K \) and thus treat \( F \) itself as a subfield of \( K \). Then we simply say that \( K \) is a field extension of \( F \).

**DEFINITION 1.2.25 (SPLITTING FIELD):** Let \( f(x) \) be a polynomial of degree \( n \geq 1 \) over a field \( F \). Then a field extension \( E \) of \( F \) is called *splitting field* of \( f(x) \) if

(a) \( f(x) \) can be factored into \( n \) linear factors over \( E \) and

(b) there does not exist any proper subfield \( E' \) of \( E \) containing \( F \) such that \( f(x) \) can be factored into \( n \) linear factors over \( E' \).

**DEFINITION 1.2.26 (POLYNOMIAL IDENTITY):** A polynomial \( f(x_1, x_2, \ldots, x_t) \) in non-commuting indeterminates \( x_1, x_2, \ldots, x_t \) with integral coefficients is said to be an identity of a ring \( R \) if \( f(r_1, r_2, \ldots, r_t) = 0 \), for every \( r_1, r_2, \ldots, r_t \in R \). We simply say that \( f \) is a *polynomial identity* in \( R \). We also say that \( R \) satisfies \( f \).
DEFINITION 1.2.27 (POLYNOMIAL IDENTITY RING): A Polynomial identity ring (PI-ring) is a ring satisfying a polynomial identity whose coefficients are all ±1.

REMARKS 9. We know that every commutative ring satisfies the identity $f = x_1^2 - x_2^2$, and is thus a trivial example of a PI-ring.

10. Each finite dimensional algebra over a commutative ring is also a PI-ring.

11. All subrings, homomorphic images, and direct products of rings satisfying $f$ also satisfy $f$.

DEFINITION 1.2.28 (TENSOR PRODUCT): Let $A$ be an $R$-module and $B$ be a left -module over the same ring $R$. Then an $R$-module $T$ together with a map $\zeta : A \times B \to T$ is said to be a 'tensor product' of $A$ and $B$ over $R$ if the following conditions are satisfied:

(i) $\zeta(a_1, a_1 + a_2, b) = a_1 \zeta(a_1, b) + a_2 \zeta(a_2, b)$

(ii) $T$ is generated by the element $\zeta(a, b)$, $a \in A$, $b \in B$.

It is easy to verify that all tensor products of a pair of modules $A$ and $B$ over $R$ are unique upto isomorphism and thus we denote essentially the unique tensor product of $A$ and $B$ over $R$ by the symbol

$$A \otimes_R B \text{ or simply } A \otimes B$$

For each $a \in A$ and $b \in B$, the element $\zeta(a, b)$ of $A \otimes_R B$ will be denoted by $a \otimes_R b$ and called the 'tensor product' of elements $a$ and $b$. 
An element \( t \) of the tensor product \( A \otimes B \) is of the form
\[
 t = \sum_{i=1}^{n} (a_i \otimes b_i)
\]
where \( a_i \in A \) and \( b_i \in B \) for \( i = 1, 2, \ldots, n \).

The tensor product \( A \otimes B \) of \( A \) and \( B \) is also referred to as 'Kronecker product'.

1.3. SOME KEY RESULTS: In this section we mention some key results extracted from the above referred texts, which will frequently be used in later chapters.

PROPOSITION 1.3.1 (I. Kaplansky [43]): If a division ring \( D \) satisfies any polynomial identity then it is finite dimensional over its center.

PROPOSITION 1.3.2 (S.A. Amitsur [5]): If \( S \) is a PI-ring without zero divisors then the division ring of quotients \( D \) of \( S \) is a central division algebra of finite order over its center and satisfies the same identities as \( S \).

PROPOSITION 1.3.3 (S.A. Amitsur [5]): A ring \( S \) is a PI-ring without zero divisors if and only if \( S \) is a subring of a central division algebra of finite order.

PROPOSITION 1.3.4 (Cartan-Brauer-Hua-theorem [41, Theorem VII]): The only division subring of a division ring \( D \) which are invariant relative to all inner-automorphisms are \( D \) and the subfields of the center of \( D \).
PROPOSITION 1.3.5 [32, Theorem 2.1.4]: Let \( R \) be a primitive ring. Then for some division ring \( D \) either \( R \) is isomorphic to \( D_n \), the ring of all \( n \times n \) matrices over \( D \) or, given any integer \( m \) there exists a subring \( S_m \) of \( R \) which maps homomorphically onto \( D_m \).

PROPOSITION 1.3.6 [33, Lemma 1.1]: Let \( R \) be a ring and \( 0 \neq A \), a right ideal of \( R \). Suppose that given \( a \in A \), \( a^n = 0 \) for a fixed integer \( n \); then \( R \) has a nonzero nilpotent ideal.

PROPOSITION 1.3.7 (Posner's theorem [57]): Let \( R \) be a prime ring with center \( Z(R) \) and a polynomial identity. Then there exists a simple ring \( S = R\hat{Z}(R) \), where \( \hat{Z}(R) \) is a quotient field of \( Z(R) \), which is a ring of quotient of \( R \) and is finite dimensional over \( \hat{Z}(R) \).

PROPOSITION 1.3.8 (Strengthening of Posner's theorem [57]): With all notions of the Proposition 1.3.7, besides assertions of the Posner's theorem, the center \( Z(S) \) of \( S \) is \( \hat{Z}(R) \).

PROPOSITION 1.3.9 (I.N. Herstein [36]): Let \( R \) be a ring in which given \( a, b \in R \) there exist integers \( m = m(a,b) \geq 1 \), \( n = n(a,b) \geq 1 \) such that \( a^m b^n = b^n a^m \). Then the commutator ideal of \( R \) is nil.

1.4. SOME CLASSICAL COMMUTATIVITY THEOREMS: In this section we give some well known commutativity theorems, which we shall frequently refer to in subsequent chapters.

THEOREM 1.4.1 (Wedderburn [59]): A finite division ring is a field.
THEOREM 1.4.2 (Jacobson [40]): Let $R$ be a ring in which for every $a \in R$ there exists an integer $n(a) > 1$, depending on $a$, such that $a^{n(a)} = a$, then $R$ is commutative.

THEOREM 1.4.3 (Jacobson [40]): An algebraic division algebra over a finite field is commutative.

THEOREM 1.4.4 (Kaplansky [44]): Let $R$ be a ring with center $Z(R)$ and a positive integer $n = n(x) > 1$ such that $x^{n(x)} \in Z(R)$ for every $x \in R$. If $R$ in addition is semi simple, then it is also commutative.

THEOREM 1.4.5 (Faith [16]): Let $D$ be a division ring and $A \neq D$ a subring of $D$. Suppose that for every $x \in D$, $x^{n(x)} \in A$, where $n(x) \geq 1$ depends on $x$. Then $D$ is commutative.

THEOREM 1.4.6 (Herstein [30]): If $R$ is a ring in which the mapping $x \mapsto x^n$ for a fixed integer $n > 1$ is a homomorphism onto, then $R$ is commutative.

THEOREM 1.4.7 (Bell [9]): Let $R$ be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is a polynomial in a finite number of non-commuting indeterminates, its coefficients being integers with highest common factor 1. If there exists no prime $p$ for which the ring of $2 \times 2$ matrices over $GF(p)$ satisfies $q(X) = 0$, then $R$ has a nil commutator ideal. Equivalently, if $R$ has no nonzero nil ideals, then $R$ is commutative.