2.1 INTRODUCTION

Goyal [37], has defined the modified Laguerre polynomial of degree \( m \) as follows:

\[
L_{\alpha, \beta, n, m}(x) = \frac{\beta^m(n)_m}{m!} \, _1F_1[-m; n; ax/\beta] .
\]

Singh and Bala [83] utilized Weisner’s [105] group-theoretic method of obtaining generating functions in the case of modified Laguerre polynomials, by giving suitable interpretation to index \( m \), in order to derive the elements of Lie algebra.

Further Jahan [53] in her Ph.D. Thesis considered more general polynomials \( L_{\alpha, \beta, \gamma, n, m}(x) \) with the name generalized Laguerre-Hermite polynomials, which she defined as

\[
(\gamma - \beta t)^{-n} \exp\left(-\frac{\alpha xt}{\gamma - \beta t}\right) = \sum_{m=0}^{\infty} L_{\alpha, \beta, \gamma, n, m}(x) \, t^m,
\]

where \( m \) is a positive integer and the other parameters are unrestricted in general.

The polynomials \( L_{\alpha, \beta, \gamma, n, m}(x) \) contain the modified Laguerre polynomials \( L_{\alpha, \beta, n, m}(x) \) of Goyal [37], when \( \gamma = 1 \).

Jahan [53] established a fundamental relationship between special linear group \( SL(2) \) and the generalized Laguerre-Hermite polynomials \( L_{\alpha, \beta, \gamma, n, m}(x) \).
Sharma [82] has obtained certain generating functions for modified Laguerre polynomials by Lie-theoretic approach. Group-theoretic origins of generating functions were obtained independently by Das, Sarama [22], Manocha [64], Ghosh, Bandana [36], Chongdar [20], de Oliveira and Capelas [23], Chatterjea and Chakrabarty [18], Chakrabarty [16] and many other research workers. The details of methods and ideas leading to results on special functions are based on the approach of Miller [67] and Weisner [105], [106], [107]. In Sections 2.3 and 2.4, we follow the approach of Miller and obtain generating functions of modified Laguerre polynomials by extending the realizations of $\hat{\uparrow}_\omega, u$ and $\downarrow_\omega, u$ on $V$ to local multiplier representations of $G(0,1)$ defined on $\mathcal{F}$ where $\mathcal{F}$ is the complex vector space of all functions of $x$ and $y$ analytic in some neighbourhood of the point $(0,0)$. Many results of Miller [67], Weisner [105], Jahan [53], Manocha [95], Das, Sarama [22] and Sharma [82] follow as special cases of our results. Generating relations of modified Laguerre polynomials are of interest due to its connections and reductions to a number of other special functions which are well-known. For convenience, we note the following important special cases of $L_{\alpha, \beta, m, n}(x)$.

(a) 
\begin{equation} 
L_{1,1,\alpha+1,n}(x) = L_n^{(\alpha)}(x), 
\end{equation}

where $L_n^{(\alpha)}(x)$ is the associated Laguerre polynomial (1.5.26), [79].

(b) 
\begin{equation} 
L_{1,1,1/2,n}(x^2) = (-1)^n \frac{H_{2n}(x)}{2^{2n} n!}
\end{equation}

and
(2.1.5) \( L_{1,1,3/2,n}(x^2) = (-1)^n \frac{H_{2n+1}(x)}{2^{2n} n!} \),

where \( H_{2n}(x) \) and \( H_{2n+1}(x) \) are the even and odd Hermite polynomials respectively (1.5.23), [79].

(c)

(2.1.6) \( L_{1,1,(\mu+1)/2,n}(x^2) = (-1)^n \frac{H_{2n}^{\mu}(x)}{n!} \),

and

(2.1.7) \( L_{1,1,(\mu+3)/2,n}(x^2) = (-1)^n \frac{H_{2n+1}^{\mu}(x)}{n!} \),

where \( H_{2n}^{\mu}(x) \) and \( H_{2n+1}^{\mu}(x) \) are the generalized even and odd Hermite polynomials [101], [19] defined as

\[
(2.1.8) \quad H_{2n}^{\mu}(x) = \sum_{k=0}^{n} \binom{n+\mu-1/2}{n-k} (-1)^{n-k} \frac{2^{2n} n!}{k!} x^{2k} \]

and

\[
(2.1.9) \quad H_{2n+1}^{\mu}(x) = \sum_{k=0}^{n} \binom{n+\mu+1/2}{n-k} (-1)^{n-k} \frac{2^{2n+1} n!}{k!} x^{2k+1} \]

(d)

(2.1.10) \( L_{-1,4u,\nu+1/2,n}(x^2) = P_{n,\nu}(x,u)/n! \),

where \( P_{n,\nu}(x,u) \) is the generalized Heat polynomial defined by Haimo [45] and Bragg [11]

\[
(2.1.11) \quad P_{n,\nu}(x,u) = \sum_{k=0}^{n} 2^{2k} \left( \binom{n}{k} \frac{\Gamma(\nu+n+1/2)}{\Gamma(\nu+n-k+1/2)} \right) x^{2n-2k} u^k \]
where \( S_n(x) \) is Schultz-Piszachich polynomial (1.5.28), [81] and \( y_n(x) \) is simple Bessel polynomial (1.5.32), [60], [79].

where Shively's pseudo-Laguerre polynomial \( R_n(a,x) \) [79] is defined in terms of \(_1F_1\) as

\[
(2.1.14) \quad R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n} \, _1F_1[-n; a+n; x] .
\]

The recurrence relations for the modified Laguerre polynomials \( L_{\alpha, \beta, n, m}(x) \) are (cf. [83], p. 512(1.1 and 1.2))

\[
(2.1.15) \quad \frac{d}{dx} L_{\alpha, \beta, n, m}(x) \\
= \frac{1}{x} \left\{ \beta (1-n-m) L_{\alpha, \beta, n, m-1}(x) + mL_{\alpha, \beta, n, m}(x) \right\}
\]

\[
(2.1.16) \quad \frac{d}{dx} L_{\alpha, \beta, n, m}(x) \\
= \frac{1}{\beta x} \left\{ \left( \frac{\beta (-n-m)+\alpha x}{\beta} \right) L_{\alpha, \beta, n, m}(x) + \left( m+1 \right) L_{\alpha, \beta, n, m+1}(x) \right\} ,
\]

and the differential equation for \( L_{\alpha, \beta, n, m}(x) \) is (cf. [83], p. 512(1.3))

\[
(2.1.17) \quad \left[ x \frac{d^2}{dx^2} + \left( n - \frac{\alpha x}{\beta} \right) \frac{d}{dx} + \frac{\alpha m}{\beta} \right] L_{\alpha, \beta, n, m}(x) = 0 .
\]
We note the following differential recurrence relations for the polynomials $L_{\alpha, \beta, n-m, m}(x)$

\begin{align*}
& (2.1.18) \quad \frac{d}{dx} L_{\alpha, \beta, n-m, m}(x) = -\beta L_{\alpha, \beta, n-(m-1), m-1}(x), \\
& (2.1.19) \quad \frac{d}{dx} L_{\alpha, \beta, n-m, m}(x) \\
& \quad = \frac{(m+1)}{\alpha x} L_{\alpha, \beta, n-(m+1), m+1}(x) + \left\{ \frac{1+\beta(1+m-n)}{\alpha x} \right\} L_{\alpha, \beta, n-m, m}(x).
\end{align*}

The differential equation for $L_{\alpha, \beta, n-m, m}(x)$ is given by

\begin{equation}
(2.1.20) \quad \left[ \frac{\alpha x}{\beta} \frac{d^2}{dx^2} + \left( n-m-\frac{\alpha x}{\beta} \right) \frac{d}{dx} + m \right] L_{\alpha, \beta, n-m, m}(x) = 0.
\end{equation}

2.2 REALIZATION OF $^{\uparrow} \omega, \mu$ AND GENERATING FUNCTIONS

As we know that $S(0,1)$ essentially coincides with the Lie algebra of the local Lie group $G(0,1)$ given by

$$
G(0,1) = \begin{cases}
\begin{bmatrix}
1 & c & e^\tau & a & \tau \\
0 & e^\tau & b & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, & a, b, c, \tau \in \mathbb{C}.
\end{cases}
$$

The irreducible representation $^{\uparrow} \omega, \mu$ (cf. [67], p. 85) of $S(0,1)$ is defined for each $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum $S$ of $^{\uparrow} \omega, \mu$ is

$$
S = \{-\omega+n:n \text{ a nonnegative integer}\}
$$

and there is a basis $\{f_m, m \in S\}$ for the representation space $V$ with the properties
\[
\begin{align*}
J^3 f_m &= m f_m, \\
E f_m &= \mu f_m, \\
J^+ f_m &= \mu f_{m+1}, \\
J^- f_m &= (m+\omega) f_{m-1}, \\
C_{0,1} f_m &= (J^+ J^- - E J^3) f_m = \mu \omega f_m.
\end{align*}
\]

The commutation relations satisfied by the operators are

\[
\left[ J^3, J^\pm \right] = \pm J^\pm, \quad \left[ J^+, J^- \right] = -E, \quad \left[ J^\pm, E \right] = \left[ J^3, E \right] = 0.
\]

We can extend the realization of \( \tilde{\omega}, \mu \) defined on \( V \) to a local multiplier representation of \( G(0,1) \) defined on \( \mathcal{F} \), where \( \mathcal{F} \) is the complex vector space of all functions of \( x \) and \( y \) analytic in some neighbourhood of the point \( (x^0, y^0) = (0,0) \).

Let us introduce the first order linearly independent differential operators \( J^3, J^+, J^- \) and \( E \) each of the form

\[
A_1(x,y) \frac{\partial}{\partial x} + A_2(x,y) \frac{\partial}{\partial y} + A_3(x,y)
\]

such that

\[
\begin{align*}
J^3 \left[ y^m x^{n-m-1} L_\alpha, \beta, n-m, m(x) \right] &= a_m y^m x^{n-m-1} L_\alpha, \beta, n-m, m(x), \\
J^+ \left[ y^m x^{n-m-1} L_\alpha, \beta, n-m, m(x) \right] &= b_m y^{m+1} x^{n-m-1} L_\alpha, \beta, n-(m+1), m(x), \\
J^- \left[ y^m x^{n-m-1} L_\alpha, \beta, n-m, m(x) \right] &= c_m y^{m-1} x^{n-m-1} L_\alpha, \beta, n-(m-1), m(x), \\
E \left[ y^m x^{n-m-1} L_\alpha, \beta, n-m, m(x) \right] &= y^m x^{n-m-1} L_\alpha, \beta, n-m, m(x),
\end{align*}
\]

where \( a_m, b_m \) and \( c_m \) are expressions in \( m \) which are independent of \( x \) and \( y \), but not necessarily of \( \alpha, \beta \) and \( n \). Each \( A_i(x,y) \),
i=1,2,3, on the other hand, is an expression in x and y which is independent of m but not necessarily of \( \alpha, \beta \) and n.

Using (2.2.3) and recurrence relations (2.1.18) and (2.1.19), we get the following operators

\[
J^3 = \frac{\partial}{\partial y},
\]

\[
J^+ = \alpha y \frac{\partial}{\partial x} + (\alpha - \beta) x^{-1} y^2 \frac{\partial}{\partial y} + (\alpha - \beta) (1 - n) x^{-1} y - \alpha y,
\]

\[
J^- = \frac{-xy^{-1}}{\beta} \frac{\partial}{\partial x} - \frac{1}{\beta} \frac{\partial}{\partial y} + \frac{(n-1)}{\beta} y^{-1},
\]

\[
E = 1.
\]

Commutation relations of these operators are identical with (2.2.2).

To construct a realization of \( \uparrow_{\omega, \mu} \) in terms of the operators (2.2.4), we find nonzero functions \( f_m(x,y) = y^m Z_m(x) \), such that equations (2.2.1) are valid for all \( m \in S \). Following Miller ([67], Section 4.6), the complex constant \( \omega \) is clearly irrelevant as far as the study of the special functions \( Z_m \) is concerned, since we could remove it by relabeling the functions \( Z_m = Z_{m+\omega} \). Hence, without loss of generality we can assume \( \omega = 0 \). Also, there is no loss of generality for special function theory if we set \( \mu = 1 \).

In terms of the functions \( Z_m(x) \) relations (2.2.1) reduce to

\[
\left[ \frac{\alpha}{\beta} + \frac{(\alpha - \beta) (m-n+1) x^{-1}}{\beta} \right] Z_m(x) = Z_{m+1}(x),
\]

\[
\left[ \frac{-x}{\beta} + \frac{(n-m-1)}{\beta} \right] Z_m(x) = m Z_{m-1}(x).
\]
If we choose $Z_m(x) = m! \, x^{n-m-1} \log \alpha \beta n-m \alpha(x)$, $m \in S$, then the functions $f_m(x,y) = y^{m} Z_m(x)$, $m \in S$, form a basis for a realization of the representation $\Upsilon_{0,1}$ of $\mathcal{G}(0,1)$. This realization can be extended to a local multiplier representation of $G(0,1)$ defined on $\mathcal{F}$.

According to Theorem (1.4.1) the operators (2.2.4) generate a Lie algebra, isomorphic to $\mathcal{G}(0,1)$, which is the algebra of generalized Lie derivatives of a local multiplier representation $T$ of $G(0,1)$ acting on $\mathcal{F}$.

Now we proceed to compute the multiplier representation of $G(0,1)$. From the theorem, the action of the one-parameter groups $\exp(b \mathcal{J}^+)$, $\exp(c \mathcal{J}^-)$, $\exp(\tau \mathcal{J}^0)$ and $\exp(a \mathcal{S})$, $a, b, c, \tau \in \mathcal{F}$, are obtained by integrating the following differential equations.

\[
\frac{d}{db} x(b) = \alpha y(b), \quad \frac{d}{db} y(b) = (\alpha - \beta) x^{-1}(b) y^2(b), \quad \frac{d}{db} \nu(b) = \nu(b) \left\{ (\alpha - \beta) (1-n) x^{-1}(b) y(b) - \alpha y(b) \right\},
\]

\[
\frac{d}{dc} x(c) = \frac{-x(c)y^{-1}(c)}{\beta}, \quad \frac{d}{dc} y(c) = \frac{1}{\beta}, \quad \frac{d}{dc} \nu(c) = \frac{\nu(c)(n-1)y^{-1}(c)}{\beta},
\]

\[(2.2.6)\]

\[
\frac{d}{d\tau} x(\tau) = 0, \quad \frac{d}{d\tau} y(\tau) = y(\tau), \quad \frac{d}{d\tau} \nu(\tau) = 0,
\]

\[
\frac{d}{da} x(a) = 0, \quad \frac{d}{da} y(a) = 0, \quad \frac{d}{da} \nu(a) = \nu(a),
\]

subject to conditions $x(0) = x^0$, $y(0) = y^0$, $\nu(0) = 1$, where $\nu$ is multiplier of the representation.

Thus, if $f \in \mathcal{F}$ is analytic in a neighbourhood of $(x^0,y^0)$, then the values of the multiplier representations of $\exp(b \mathcal{J}^+)$, $\exp(c \mathcal{J}^-)$, $\exp(\tau \mathcal{J}^0)$ and $\exp(a \mathcal{S})$ are given by
\[
\begin{align*}
T(\exp(b^+) f)(x^0, y^0) &= \exp\left\{ x^0 \left[ 1 - \left( 1 + \frac{\beta y^0}{x^0} \right) \frac{\alpha}{\beta} \right] \right\} \left[ 1 + \frac{\beta y^0}{x^0} \right] \left( \alpha - \beta \right) (1 - n) \\
&\quad \cdot f \left( x^0 \left[ 1 + \frac{\beta y^0}{x^0} \right] \frac{\alpha}{\beta}, y^0 \left[ 1 + \frac{\beta y^0}{x^0} \right] \left( \alpha - \beta \right) / \beta \right),
\end{align*}
\]

\[
\begin{align*}
T(\exp(c^-) f)(x^0, y^0) &= \left( \frac{1 - c}{\beta y^0} \right)^{1-n} f \left( x^0 \left[ 1 - \frac{c}{\beta y^0} \right], y^0 - \frac{c}{\beta} \right),
\end{align*}
\]

(2.2.7)

\[
\begin{align*}
T(\exp(\tau^0 f)(x^0, y^0) &= f(x^0, y^0 e^{\tau}),
\end{align*}
\]

\[
\begin{align*}
T(\exp(a^s f)(x^0, y^0) &= \exp(a) f(x^0, y^0).
\end{align*}
\]

If \( g \in G(0,1) \) has coordinates \((a,b,c,\tau)\), we have

\[
g = (\exp(b^+)) (\exp(c^+)) (\exp(\tau^0)) (\exp(a^s)),
\]

and the operator \( T(g) \) acting on \( f \in \mathfrak{F} \) is given by

\[
\begin{align*}
T(g) f(x,y) &= T\left[ (\exp(b^+)) (\exp(c^+)) (\exp(\tau^0)) (\exp(a^s)) f \right](x,y) \\
&= \left[ T(\exp(b^+)) T(\exp(c^+)) T(\exp(\tau^0)) T(\exp(a^s)) f \right](x,y) \\
&= \exp \left[ a + x \left( 1 - \frac{\beta y}{x} \right) \frac{\alpha}{\beta} \right] \left[ 1 + \frac{\beta y}{x} \right] (\alpha - \beta) / \beta - \frac{c}{\beta y} \right]^{1-n} \\
&\quad \cdot f \left( x \left( 1 + \frac{\beta y}{x} \right) \left[ 1 + \frac{\beta y}{x} \right] (\alpha - \beta) / \beta - \frac{c}{\beta y} \right), e^{\tau} \left( y \left[ 1 + \frac{\beta y}{x} \right] (\alpha - \beta) / \beta - \frac{c}{\beta y} \right).
\end{align*}
\]

Now the matrix elements of this representation with respect to the analytic basis \( \{ f_m(x,y) = y^m z_m(x) \} \), are the functions \( A_{\ell k}(g) \) defined by
\[ T(g) f_k(x, y) = \sum_{\ell=0}^{\infty} A_{\ell k}(g) f_\ell(x, y), \]
\[ g \in g(0,1), \ k = 0,1,2,\ldots, \]

or, (from (2.2.8))

\[ (2.2.10) \]
\[
\begin{align*}
&= \sum_{\ell=0}^{\infty} A_{\ell k}(g) \ell! x^{k-\ell} L_{\ell+1}^\alpha, \beta, n-k, k(x, y) y^\ell , \\
&= k! \frac{x^{k-\ell} L_{\ell+1}^\alpha, \beta, n-k, k(x, y) y^\ell}{x^{\ell} L_{\ell+1}^\alpha, \beta, n-k, k(x, y) y^\ell}, \\
&= k, \ell = 0,1,2,\ldots.
\end{align*}
\]

The matrix elements \( A_{\ell k}(g) \) are to be determined by expanding the left-hand side of (2.2.10) in a power series in \( y \) and then computing the coefficient of \( y^\ell \). We discuss a very few special cases of the above equation.

For \( \alpha = \beta = 1 \) and \( n = q + 1 \), expression (2.2.10) reduces to

\[
(2.2.11) \]
\[
\begin{align*}
&= \sum_{\ell=0}^{\infty} B_{\ell k}(g) \ell! x^{k-\ell} L_{\ell+1}^1(q-k, k(x, y) y^\ell) , \\
&= k, \ell = 0,1,2,\ldots.
\end{align*}
\]

which gives us (cf. [67], p. 87(4.26)),

\[
(2.2.12) \]
\[
B_{\ell k}(g) = \exp(a+\tau k-b y) c^{k-\ell} L_{\ell}^{1-\ell}(-bc),
\]
\[ k, \ell = 0.\]

Putting this value of \( B_{\ell k}(g) \) in (2.2.11), we get (cf. [67], p. 112(4.94))
\[(2.2.13) \quad k!e^{-by\left(1+\frac{by}{x}\right)}q^{-k}L_k^{(q-k)}\left[x(1+\frac{by}{x})\left(1-\frac{c}{y}\right)\right]y^k\]
\[= \sum_{\ell=0}^{\infty} c^{k-\ell}\ell!\left(k^\ell\right)\left(-bc\right)\ell!x^{k-\ell}L_{\ell}^{(q-\ell)}(x)y^\ell, \quad k = 0,1,2,\ldots .\]

The above equation is valid for all \(b, c, y, x, q \in \mathbb{R}\) such that \(|by/x| < 1\).

Making use of the limits (cf. [67], p. 88(4.29))

\[c^{nL_{\ell}^{(n)}(bc)}|_{c=0} = \begin{cases} 0 & \text{if } n > 0, \\ \frac{(-b)^{-n}}{(-n)!} & \text{if } n \leq 0, \end{cases}\]

\[(2.2.14)\]

\[c^{nL_{\ell}^{(n)}(bc)}|_{b=0} = \begin{cases} \binom{n+\ell}{n}c^n & \text{if } n = 0, \\ 0 & \text{if } n < 0, \end{cases}\]

we can derive some simple consequences of (2.2.13) (cf. [67], pp. 112-113). For \(c = 0, y = x\),

\[(2.2.15) \quad e^{-bx(1+b)}q^{+k}L_k^{(q)}[x(1+b)] = \sum_{\ell=0}^{\infty} b^\ell \binom{\ell+k}{\ell}L_{\ell}^{(q-\ell)}(x), \quad |b| < 1 ,\]

and, setting \(k = 0\) in this expression we obtain a well-known generating function for the associated Laguerre polynomials (cf. [67], p. 87(4.27)).

\[(2.2.16) \quad e^{-bx(1+b)}q = \sum_{\ell=0}^{\infty} b^\ell L_{\ell}^{(q-\ell)}(x), \quad q \in \mathbb{R}, \quad |b| < 1 .\]
When \( b = 0 \), \( y = x \), equation (2.2.13) becomes

\[
(2.2.17) \quad L_k^{(q)}(x-c) = \sum_{\ell=0}^{k} \frac{c^\ell}{\ell!} L_k^{(q+\ell)}(x).
\]

### 2.3 REALIZATION OF \( \downarrow_{\omega,\mu} \) AND GENERATING FUNCTIONS

As was stated in Theorem (1.4.3), the irreducible representation \( \downarrow_{\omega,\mu} \) of \( \mathfrak{g}(0,1) \) is defined for each \( \omega, \mu \in \mathfrak{g} \) such that \( \mu \neq 0 \). The spectrum of this representation is the set

\[ S = \{-\omega-1-n : n \text{ a nonnegative integer}\}, \]

and there is a basis \( \{f_m, m \in S\} \) for the representation space \( V \) such that

\[
\begin{align*}
J^3 f_m &= mf_m, \quad E f_m = -\mu f_m, \\
J^+ f_m &= -(m+\omega+1) f_{m+1}, \quad J^- f_m = \mu f_{m-1}, \\
C_0,1 f_m &= (J^+J^- - EJ^3) f_m = -\mu \omega f_m.
\end{align*}
\]

The commutation relations satisfied by the operators are (2.2.2).

As in Section 2.2, we can extend the realization of \( \downarrow_{\omega,\mu} \) defined on \( V \) to a local multiplier representation of \( G(0,1) \) defined on \( \mathfrak{g} \).

Let us introduce the first order linearly independent differential operators \( J^3, J^+, J^- \) and \( E \) each of the form

\[
A_1(x,y) \frac{\partial}{\partial x} + A_2(x,y) \frac{\partial}{\partial y} + A_3(x,y)
\]

such that
\[ J^3 \left[ y_{m}^{\alpha, \beta, n-m, m(x)} \right] = a_m y_{m}^{\alpha, \beta, n-m, m(x)} , \]

\[ J^+ \left[ y_{m}^{\alpha, \beta, n-m, m(x)} \right] = b_m y_{m+1}^{\alpha, \beta, n-(m+1), m+1(x)} , \]

\[ J^- \left[ y_{m}^{\alpha, \beta, n-m, m(x)} \right] = c_m y_{m-1}^{\alpha, \beta, n-(m-1), m-1(x)} , \]

\[ E \left[ y_{m}^{\alpha, \beta, n-m, m(x)} \right] = y_{m}^{\alpha, \beta, n-m, m(x)} , \]

where \( a_m, b_m \) and \( c_m \) are expressions in \( m \) which are independent of \( x \) and \( y \), but not necessarily of \( \alpha, \beta \) and \( n \). Each \( A_i(x, y), i=1,2,3 \), on the other hand, is an expression in \( x \) and \( y \) which is independent of \( m \) but not necessarily of \( \alpha, \beta \) and \( n \).

Using (2.3.2) and recurrence relations (2.1.18) and (2.1.19), we get the following operators

\[ J^3 = \frac{\partial}{\partial y} , \]

\[ J^+ = -\alpha xy \frac{\partial}{\partial x} + \beta y^2 \frac{\partial}{\partial y} - (\beta n - \alpha x - \beta) y , \]

\[ J^- = \frac{y^{-1}}{\beta} \frac{\partial}{\partial x} , \]

\[ E = 1 . \]

These operators satisfy the commutation relations (2.2.2).

To obtain a realization of \( J^\omega, J^\mu \) in terms of the operators (2.3.3), we find nonzero functions \( f_m(x,y) = y^m z_m(x) \), such that equations (2.3.1) are valid for all \( m \in S \). Just as in the previous section, it is easy to show that without loss of generality for special function theory we can assume \( \omega=0, \mu=1 \). Then, expressed in terms of the functions \( z_m(x) \), equations
(2.3.1) become the recursion relations

\[
\left\{ -\alpha x \frac{d}{dx} + \alpha x + \beta (m-n+1) \right\} Z_m(x) = -(m+1) Z_{m+1}(x),
\]

(2.3.4)

\[
\left( \frac{1}{\beta} \frac{d}{dx} \right) Z_m(x) = -Z_{m-1}(x).
\]

Now if we take \( Z_m(x) = L_{\alpha, \beta, n-m, m}(x) \), \( m \in S \), then the functions \( f_m(x,y) = y^m Z_m(x) \), \( m \in S \), form a basis for a realization of the representation \( \mathfrak{g}_{0,-1} \) of \( \mathfrak{g}(0,1) \). This realization of \( \mathfrak{g}(0,1) \) can be extended to a local multiplier representation of \( G(0,1) \). Now, proceeding exactly as in the previous section we compute the multiplier representation of \( G(0,1) \). The action of the one-parameter groups \( \exp(b\beta^+), \exp(c\beta^-), \exp(\tau\beta^3) \) and \( \exp(a\epsilon) \), \( a, b, c, \tau \in \mathbb{C}, \) are obtained by integrating the following differential equations

\[
\frac{d}{db} x(b) = -\alpha x(b) y(b), \quad \frac{d}{db} y(b) = \beta y^2(b), \quad \frac{d}{db} \nu(b) = \nu(b) \{\alpha x(b) + \beta - \beta n\} y(b),
\]

(2.3.5)

\[
\frac{d}{dc} x(c) = \frac{y^{-1}(c)}{\beta}, \quad \frac{d}{dc} y(c) = 0, \quad \frac{d}{dc} \nu(c) = 0,
\]

\[
\frac{d}{d\tau} x(\tau) = 0, \quad \frac{d}{d\tau} y(\tau) = y(\tau), \quad \frac{d}{d\tau} \nu(\tau) = 0,
\]

\[
\frac{d}{da} x(a) = 0, \quad \frac{d}{da} y(a) = 0, \quad \frac{d}{da} \nu(a) = \nu(a),
\]

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with initial conditions \( x(0) = x^0, \ y(0) = y^0, \ v(0) = 1, \) where \( v \) is multiplier of the representation.

Thus, if \( f \in \mathcal{F} \) is analytic in a neighbourhood of \((x^0,y^0)\) then the values of the multiplier representations of \( \exp(b_j^+), \ \exp(c_j^-), \ \exp(\tau_j^3) \) and \( \exp(a\theta) \) are given by

\[
\begin{align*}
[T'_{\exp(b_j^+)}f](x^0,y^0) &= (1-\beta y^0)^{n-1} \exp\left(\frac{x^0-x^0(1-\beta y^0)\alpha/\beta}{1-\beta y^0}\right) f\left(x^0(1-\beta y^0)\alpha/\beta, \frac{y^0}{1-\beta y^0}\right), \\
[T'_{\exp(c_j^-)}f](x^0,y^0) &= f(x^0, y^0 e^\tau), \\
[T'_{\exp(\tau_j^3)}f](x^0,y^0) &= \exp(a) f(x^0, y^0).
\end{align*}
\] (2.3.6)

As mentioned in Section 2.2, if \( g \in G(0,1) \) has parameters \((a,b,c,\tau)\) then

\[g = \left[\exp(b_j^+)\right]\left[\exp(c_j^-)\right]\left[\exp(\tau_j^3)\right]\left[\exp(a\theta)\right].\]

Thus,

\[T'(g)f = T'_{\exp(b_j^+)} T'_{\exp(c_j^-)} T'_{\exp(\tau_j^3)} T'_{\exp(a\theta)} f,\]

for all \( f \in \mathcal{F}. \) Direct computation gives

\[
[T'_{g}f](x,y) = (1-\beta y)^{n-1} \exp\left(x - x (1-\beta y) \frac{\alpha/\beta + a}{\beta y}\right) f\left(x (1-\beta y) \alpha/\beta + \frac{c (1-\beta y)}{\beta y}, \frac{y e \tau}{1-\beta y}\right). \]

As in Section 2.2, the matrix elements of the operators
$T'(g)$ with respect to the basis vectors $\{f_m(x,y) = y^m z_m(x)\}$, are the functions $A_{\ell k}(g)$ defined by

$$
(2.3.8) \quad \left[ T'(g) f_k \right](x,y) = \sum_{\ell=0}^{\infty} A_{\ell k}(g) f_\ell(x,y), \\
k = 0,1,2,\ldots,
$$

valid for all $g \in G(0,1)$, or, (from (2.3.7))

$$
(2.3.9) \quad (1-\beta y)^{n-k-1} \exp \left( x - x(1-\beta y) \alpha/\beta + a + \tau k \right) y^k \\
\cdot {}_1 F_1 \left[ \begin{array}{c} -k; \\
1+\ell-k; \\
\end{array} \frac{-abc}{\beta} \right] \\
= \sum_{\ell=0}^{\infty} A_{\ell k}(g) y^\ell {}_1 F_1 \left[ \begin{array}{c} x(1-\beta y) \alpha/\beta + \frac{c(1-\beta y)}{\beta y}; \\
\ell-k; \\
\end{array} \right], \\
|\beta y| < 1.
$$

To find $A_{\ell k}(g)$, we set $x = 0$, this enables us to arrive at

$$
(2.3.10) \quad A_{\ell k}(g) = \frac{\exp(a+\tau k)(-b)^{\ell-k}}{k!} \cdot {}_1 F_1 \left[ \begin{array}{c} -k; \\
1+\ell-k; \\
\end{array} \frac{-abc}{\beta} \right],
$$

or, alternatively

$$
A_{\ell k}(g) = \exp(a+\tau k)(-b)^{\ell-k} {}_1 F_1 \left[ \begin{array}{c} x(1-\beta y) \alpha/\beta + \frac{c(1-\beta y)}{\beta y}; \\
\ell-k; \\
\end{array} \right].
$$

Thus the generating function (2.3.9) becomes

$$
(2.3.11) \quad (1-\beta y)^{n-k-1} \exp \left( x - x(1-\beta y) \alpha/\beta \right) \\
\cdot {}_1 F_1 \left[ \begin{array}{c} x(1-\beta y) \alpha/\beta + \frac{c(1-\beta y)}{\beta y}; \\
\ell-k; \\
\end{array} \right],
$$

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The generating function (2.3.11) was obtained under the assumption that $k$ is a nonnegative integer.

Now assuming that $k$ and $n-k$ are not integers, say $k = \nu$ and $n - k = n - \nu$, the generating function emanating from (2.3.7) is

\[
\exp\left( x - x(1 - \beta y) \frac{\alpha}{\beta} \right) \left( 1 - \beta y \right)^{n - \nu - 1} 
= \sum_{m = -\infty}^{\infty} \frac{\Gamma(1 + \nu + m)}{\Gamma(1 + \nu)} L_{\alpha, \beta, n - \nu, \nu} \left( x (1 - \beta y) \frac{\alpha}{\beta} + \frac{c(1 - \beta y)}{\beta y} \right) (-\beta y)^m 
\leq \sum_{m = -\infty}^{\infty} \frac{\Gamma(1 + \nu + m)}{\Gamma(1 + \nu)} L_{\alpha, \beta, n - (\nu + m), \nu + m} (x) \{\Gamma(1 + m)\}^{-1} 
\left( -\beta y \right)^m,
\]

where

\[0 < |\beta| < |\beta b|^{-1}.
\]

**Special cases**

I. For $\alpha, \beta = 1, a = d, b = a, c = b, \tau = c, \ell = n$ and $n = \alpha + 1$, the generating function (2.3.11) reduces to (cf. [95], p. 343(23))

\[
(2.3.13) \quad \exp(ax) (a - bx) \frac{\alpha - k}{\beta} \left( (1 - ay) \left( x + \frac{b}{y} \right) \right) 
= \sum_{n=0}^{\infty} \frac{n!}{k!} L_n(a - n) (x) \{\Gamma(1 + n - k)\}^{-1} \left( -\beta y \right)^n
\leq \sum_{n=0}^{\infty} \frac{n!}{k!} L_n(a - n) (x) \{\Gamma(1 + n - k)\}^{-1} \left( -\beta y \right)^n,
\]

where

\[0 < |\beta| < |\alpha|^{-1}.
\]
II. Taking $\alpha, \beta = 1, \ a = d, \ b = a, \ c = b, \ \tau = c, \ m = n$ and $n = \alpha+1$, the generating function (2.3.12) gives us (cf. [95], p. 342(20))

\[
(2.3.14) \quad \exp(axy) (1-ay)^{\alpha-\nu} L^\nu_\nu(\alpha-\nu) \left[ (1-ay) \left( x + \frac{b}{y} \right) \right]
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{\Gamma(1+\nu+n)}{\Gamma(1+\nu)} L^\nu_\nu(\alpha-\nu-n)(x) \{ \Gamma(1+n) \}^{-1} \cdot \ _1F_1[-\nu;1+n;-ab](\operatorname{ay})^n,
\]

\[0 < |y| < |a|^{-1} .\]

III. Taking $c = 0$ and $y = -1$ in (2.3.12) we shall arrive at

\[
(2.3.15) \quad \exp\left( x-x(1+\beta b)\frac{\alpha}{\beta} \right) (1+\beta b)^{n-\nu-1} L_\alpha,\beta, n-\nu, \nu (x(1+\beta b)\frac{\alpha}{\beta})
\]

\[
= \sum_{m=0}^{\infty} \frac{(1+\nu)_m}{m!} L_\alpha,\beta, n-(\nu+m), \nu+m (x) b^m ,
\]

\[|\beta b| < 1 ,\]

which for $\alpha = \beta = 1, \ b = a, \ m = n$ and $n = \alpha+1$ gives us (cf. [95], p. 342(21))

\[
(2.3.16) \quad \exp(-ax) (1+a)^{\alpha-\nu} L^\nu_\nu(\alpha-\nu) (x(1+a))
\]

\[
= \sum_{n=0}^{\infty} \frac{(1+\nu)_n}{n!} L^\nu_\nu(\alpha-\nu-n)(x) a^n , \ |a| < 1 .
\]

IV. Taking $b = 0, \ y = -1$, and using the limit (cf. [67], p. 84(4.16))
where $n$ is an integer, equation (2.3.12) becomes

\begin{equation}
(2.3.18) \quad L_{\alpha, \beta, n-\nu, \nu}^{(n)} \left( x - \frac{c}{\beta} \right) = \sum_{m=0}^{\infty} \frac{(\alpha c / \beta)^m}{m!} L_{\alpha, \beta, n-\nu+m, \nu-m}(x)
\end{equation}

Further for $\alpha, \beta = 1$, $c = b$, $m = n$ and $n = \alpha + 1$, the above equation reduces to (cf. [95], p. 342(22))

\begin{equation}
(2.3.19) \quad L_{\nu}^{(\alpha-\nu)}(x-b) = \sum_{n=0}^{\infty} \frac{b^n}{n!} L_{\nu-n}^{(\alpha-\nu+n)}(x).
\end{equation}