CHAPTER IV
CR-SUBMANIFOLDS OF A NEARLY KAHLER MANIFOLDS

1. Introduction:

In the present chapter we study the CR-submanifolds of yet another class of manifolds viz. nearly Kaehler manifolds. Obviously a nearly Kaehler structure on an almost Hermitian manifold is given by a slightly weaker condition than a Kaehler one. It is therefore interesting to see how far the results of Kaehler setting can be extended to the nearly Kaehler setting, e.g. for the totally umbilical CR-submanifold of a Kaehler manifold, B.Y. Chen [10] showed that they are either totally real or totally geodesic or \( \dim(D^\perp)=1 \). Similar to this, in the present chapter, we have classified the totally umbilical CR-submanifolds of a nearly Kaehler manifold.

2. Some integrability conditions for the distributions on CR-submanifolds of nearly Kaehler manifolds:

In view of the relation,

\[
(\overline{\nabla}_XJ)V = P(X) + Q(X)V,
\]

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the nearly Kaehler condition yields

\[(4.2.1) \quad \mathcal{P}_U V = -\mathcal{P}_V U,\]
\[(4.2.2) \quad Q_U V = -Q_V U,\]

for all \(U, V\) tangent to \(M\), and therefore by proposition (2.2.1) the necessary and sufficient condition for the holomorphic distribution \(D\) on a CR-submanifold of a nearly Kaehler manifold, to be integrable can be written as:

\[(4.2.3) \quad 2Q_X Y = h(X, JY) - h(JX, JY),\]

for all \(X, Y\) in \(D\). Furthermore, it is known that the Neijenhuis tensor for a nearly Kaehler manifold satisfies the relation:

\[(4.2.4) \quad S(U, V) = -4J(\mathcal{P}_U J)V,\]

which in view of property (p5) of \(\mathcal{P}\) and \(Q\) gives

\[(4.2.4) \quad S(X, Y)^{\perp} = 4Q_X JY.\]

The left hand side of the above equation on simplification becomes (i.e., on using eqns. (1.1.4) & (1.3.1))

\[F([PX, Y] + [X, PY]).\]

Hence from equations (4.2.3) & (4.2.4), it follows that the holomorphic distribution \(D\) on a CR-submanifold of a nearly Kaehler manifold \(\mathbb{M}\) is integrable if and only if

\[(4.2.5) \quad Q_X Y = 0 \text{ and } h(X, JY) = h(JX, Y).\]
For the integrability of $D^\perp$, by (2.2.9) and nearly Kaehler character of $\mathcal{R}$, it can be seen that $D^\perp$ is integrable if and only if, for all $Z, W$ in $D^\perp$ and $X$ in $D$,

$$2g(P_ZW, X) = g(A_{JZ}W, X) - g(A_{JW}Z, X),$$

i.e.,

$$2g((\bar{\zeta}J)W, X) = g(A_{JZ}W, X) - g(A_{JW}Z, X).$$

Now, as for nearly Kaehler manifolds $d\Omega(U, V, W) = 3g(\bar{\zeta}UV, V, W)$, the above equation takes the form

$$2/3\ d\Omega(Z, W, X) = g(A_{JZ}W, X) - g(A_{JW}Z, X).$$

Further as $\Omega(D, D^\perp) = \Omega(D^\perp, D^\perp) = 0$, we get

$$2/3 g([Z, W], X) = g(A_{JZ}W, X) - g(A_{JW}Z, X).$$

Hence we conclude that the totally real distribution $D^\perp$ on a CR-submanifold is integrable if and only if,

$$(4.2.6) \begin{cases} g(P_ZW, X) = 0, \quad \text{or} \\ g(A_{JZ}W, X) = g(A_{JW}Z, X). \end{cases}$$

For all $Z, W \in D^\perp$ and $X \in D$.

Remark. The integrability conditions in (4.2.5) and the first one in (4.2.6) have also been established by Urbano (cf., [45]) whereas second relation in (4.2.6) is established by N. Sato (cf., [41]).
These integrability conditions lead to the following characterization of CR-product submanifolds in a nearly Kaehler manifold.

**Theorem (4.2.1).** Let $M$ be a CR-submanifold of a nearly Kaehler manifold $\tilde{M}$ and suppose both the distributions $D$ and $D^\perp$ are integrable then $M$ will be a CR-product if and only if

$$A_{\tilde{D}^\perp} D = 0$$

**Proof.** Making use of equation (4.2.5) and proposition (2.2.3), it follows that the leaves of holomorphic distribution are totally geodesic in $M$ if and only if

$$(4.2.7) \quad h(X,Y) \in \mu$$

for all $X, Y \in D$. Similarly, it follows from equation (4.2.6) and proposition (2.2.4) that leaves of $D^\perp$ are totally geodesic in $M$ if and only if

$$(4.2.8) \quad h(X,Z) \in \mu$$

for all $X \in D$ and $Z \in D^\perp$. The assertion follows immediately on combining (4.2.7) and (4.2.8).

It may be noted that this condition is precisely the same as obtained by B.Y. Chen for the characterization of CR-product in Kaehler manifolds.

Before establishing a classification theorem for the totally umbilical CR-submanifolds of a nearly Kaehler
manifold, we first prove the following preparatory results:

Proposition (4.2.1). Let $M$ be a CR-submanifold of a nearly Kaehler manifold $\mathcal{M}$ with $h(X,JX) = 0$, for each $X$ in $D$. If $D$ is integrable then each of its leaves is totally geodesic in $M$ as well as in $\mathcal{M}$.

Proof. For $X$, $Y \in D$, the Gauss equation gives
\[
h(X,JY) + h(JX,Y) = (\nabla_X J)Y + (\nabla_Y J)X + J(\nabla_X Y + \nabla_Y X) - (\nabla_X JY + \nabla_Y JX).
\]
The left hand side of the above equation is zero as $h(X,JX) = 0$, and the first two terms in the right hand side vanish because of nearly Kaehler character of $\mathcal{M}$. The rest of the terms, on using Gauss equation again, yield
\[(4.2.9) \quad J(\nabla_X JY + \nabla_Y JX) = \nabla_X Y + \nabla_Y X + 2h(X,Y)\]
Moreover, in view of the fact that $h(X,JX) = 0$, equation (2.2.3) gives
\[-F\nabla_X X = fh(X,X)\]
From which on noticing the fact that $FU \in J\mathcal{D}^\perp$ and $fN \in \mu$, $\forall U \in \mathcal{T}$ and $N \in T^\perp M$, one may easily observe that,
\[(4.2.10) \quad \nabla_X X \in D \quad \text{and}\]
\[(4.2.11) \quad h(X,X) \in J\mathcal{D}^\perp.\]
Replacing $X$ by $X+Y$ in (4.2.10), we get that $\nabla_X Y + \nabla_Y X \in D$. 
This observation together with integrability of $D$ implies that

$$\nabla_X Y \in D$$

As Frobenius theorem guarantees the foliation of $M$ by the leaves of $D$, (4.2.12) implies that these leaves are totally geodesic in $M$. Now making use of (4.2.12) and (4.2.9) we get

$$h(X,Y) = 0,$$

proving the assertion completely.

3. Totally Umbilical CR-submanifolds of a nearly Kaehler manifold:

As an immediate consequence of the proposition (4.2.3), we may state the following result which is of some independent interest also.

**Corollary (4.3.1).** Let $M$ be a totally umbilical CR-submanifold of a nearly Kaehler manifold $\mathbb{M}$. If $D$ is integrable then $M$ is totally geodesic in $\mathbb{M}$.

With regard to totally real distribution we establish the following.

**Proposition (4.3.1).** Let $M$ be a totally umbilical CR-submanifold of a nearly Kaehler manifold $\mathbb{M}$. Then the totally real distribution $D^\perp$ is integrable and its leaves are totally geodesic in $M$. 
Proof. For $Z$ in $D^\perp$, equation (2.2.2) gives

$$\mathcal{P}_Z Z = (\nabla_Z \mathcal{P}) Z - A_{FZ} Z - \text{th}(Z, Z).$$

Now as $\bar{M}$ is nearly Kaehler $\mathcal{P}_Z Z = 0$ and as $M$ is totally umbilical in $\bar{M}$, $A_{FZ} Z = g(H, FZ) Z$, and $\text{th}(Z, Z) = g(Z, Z) tH$. With these observations, the above equation can be written as

$$-\mathcal{P} \nabla_Z Z = g(H, FZ) Z + g(Z, Z) tH.$$

Obviously the right hand side of the above equation belongs to $D^\perp$, whereas the left hand side belongs to $D$, implying that

(4.3.1) \quad g(H, FZ) Z + g(Z, Z) tH = 0,

and,

(4.3.2) \quad \nabla_Z Z \in D^\perp.

Equation (4.3.1) has solutions if either

(a) $\dim(D^\perp) = 1$ \quad or \quad (b) $H \in \mu$

If $\dim(D^\perp) = 1$ then from (4.3.2), we conclude that $D^\perp$ is integrable and its leaves are totally geodesic in $M$. Further if $H \in \mu$ then simplifying equation (2.2.2) and noting that in this case $A_{FZ} X = g(H, FZ) = 0$, $\text{th}(Z, X) = g(Z, X) tH = 0$ and $FX = 0$, $\forall X \in D$, we get

$$\mathcal{P}_X Z = (\nabla_X \mathcal{P}) Z, \text{ and, } \mathcal{P}_Z X = (\nabla_Z \mathcal{P}) X,$$

Adding the above two equations with the understanding that $\mathcal{P}_X Z = -\mathcal{P}_Z X$ and $\mathcal{P} Z = 0$, we get

$$\nabla_Z PX = \mathcal{P} (\nabla_X X + \nabla_X Z).$$
implying that $\nabla_z P x \in D$ or equivalently $\nabla_z W x \in D^\perp$. Thus $D^\perp$ is integrable and its leaves are totally geodesic in $M$. This completes the proof.

We are now in a position to establish the following:

**Theorem (4.3.1).** Let $M$ be a totally umbilical CR-submanifold of a nearly Kaehler manifold $\mathcal{M}$. Then at least one of the following is true,

(i) $M$ is totally real,

(ii) $M$ is totally geodesic,

(iii) $\dim(D^\perp) = 1$ & $D$ is not integrable.

**Proof.** If $D=0$ then by definition $M$ is totally real, which is the case (i). If $D \neq 0$ and integrable then by corollary (4.2.1) $M$ is totally geodesic which accounts for case (ii). Suppose now that $D$ is not integrable and $H \notin \mu$ then by virtue of (4.2.11) $M$ is again totally geodesic. If however $H \notin \mu$ then equation (4.3.1) has solutions if and only if $\dim(D^\perp) = 1$ which establishes (iii), this completes the proof.

It is observed in our preceding discussions that the integrability of the holomorphic distribution $D(\neq 0)$ plays an important role in the geometry of CR-submanifolds of a nearly Kaehler manifold as the totally real distribution on a totally umbilical CR-submanifold $M$ is always integrable and
its leaves are totally geodesic in $M$. Thus if we assume that $D$ is also integrable, then by proposition (4.2.2) its leaves will be totally geodesic in $M$, making the CR-submanifold $M$ to be a CR-product.

Hence we may state

Theorem (4.3.2). Let $M$ be a totally umbilical CR-submanifold of a nearly Kaehler manifold $\bar{M}$. Then $M$ is a CR-product if and only if $D$ is integrable.

Since $S^6$, as a special case of nearly Kaehler manifold is known, not to admit a proper CR-product (cf. [42]), the above theorem yields the following.

Corollary (4.3.2). Let $M$ be a totally umbilical proper CR-submanifold of $S^6$, then the holomorphic distribution is not integrable.