The purpose of this chapter is to introduce basic concepts, preliminary notions and some fundamental results which we require for the development of the subject in the present thesis. Thus we have given a brief resume of some of the results in the geometry of almost Hermitian manifolds and their allied structures, and the geometry of submanifolds of these manifolds. Much though all these results are readily available in review articles and some in the standard books e.g. Nomizu & Kobayashi [29], Blair [7], B.Y. Chen [9], nevertheless, we have collected them here for ready references and to fix-up our terminology.

1. Structure on $C^\infty$-manifolds:

Basically the geometry of a differentiable manifold is revealed by knowing a Riemannian structure on it i.e., a positive definite inner product in the tangent bundle of the manifold. Further refined information can be had by knowing additional structures on the manifold, for example almost complex, almost Kaehler, nearly Kaehler etc. [29]. In this section we briefly discuss some of these structures.
In what follows, we shall always take a differentiable manifold which is connected and paracompact, so that it can always be endowed with a Riemannian metric $g$ and a Riemannian connexion $\nabla$.

An almost complex structure on a real differentiable manifold $\mathcal{M}$ is a tensor field $J$ which is at every point $p \in \mathcal{M}$, an endomorphism of the tangent space $T_p \mathcal{M}$ such that $J^2 = -I$, where $I$ denotes the identity transformation. A manifold with a fixed almost complex structure is called an almost complex manifold. On almost complex manifold, there always exist a Riemannian metric $g$ consistent with the almost complex structure $J$ i.e., satisfying $g(JU,JV) = g(U,V)$, $\forall U,V \in T\mathcal{M}$, by virtue of which $g$ is called a Hermitian metric. An almost complex manifold (resp. a complex manifold) with a Hermitian metric is called an almost Hermitian manifold (resp. a Hermitian manifold).

Analogous to the almost complex structure $J$, there is defined another important $(0,2)$ tensor (infact, a two form) which plays an important role in the geometry and mechanics on the manifold [31] & [14]. We describe it as follows:

Definition (1.1.1). A symplectic form on a real vector space $V$ of dimension $n$ is a non-degenerate exterior 2-form $\Omega$ i.e.,
a 2-form of rank $n$. We, then say that $\Omega$ defines a symplectic structure on $V$ or that $(V, \Omega)$ is a symplectic vector space. The idea of orthogonality with respect to $\Omega$ is defined similar to the usual orthogonality with respect to the metric tensor (or inner product) of $V$, i.e., Two elements $x$ and $y$ of $V$ are said to be $\Omega$-orthogonal if $\Omega(x, y) = 0$. If $W$ is a subspace of $V$, then we define

$$\text{Orth}_{\Omega}(W) = \{x \in V / \Omega(x, y) = 0, \forall y \in W\}.$$ 

Now we have the following definitions:

Definitions (1.1.2) [31]. A vector sub space $W$ of $V$ is said to be

(i) Isotropic if $\Omega_{W} = 0$ i.e., if $W \subset \text{orth}(W)$,
(ii) Coisotropic if $\Omega_{\text{orth}(W)} = 0$ i.e., if $\text{orth}(W) \subset W$,
(iii) Lagrangian if $W$ is both isotropic & coisotropic i.e., if $W = \text{orth}(W)$,
(iv) Symplectic if $\Omega_{W}$ defines a symplectic structure on $W$, i.e., $W \cap \text{orth}(W) = \{0\}$.

For manifolds, we have

Definition (1.1.3). A symplectic structure on a manifold $\bar{M}$ is defined by the choice of a differential 2-form $\Omega$ satisfying the following two conditions:

(1) For all $p \in \bar{M}, \Omega_{p}$ is non degenerate.
(2) $\Omega$ is closed i.e., $d\Omega = 0$.

Isotropic, Coisotropic and Lagrangian manifolds are defined similarly. At this point it is also easy to realize that one can define the Isotropic, Coisotropic, Lagrangian and Symplectic distributions and bundles.

Now, we will introduce Kaehlerian manifolds using the fundamental tensors $\Omega$, $J$ and $g$.

Let $\bar{\Omega}$ be a fundamental 2-form associated to the Hermitian metric $g$ on $\mathbb{R}$, i.e.,

$$\bar{\Omega}(U,V) = g(JU,V),$$

for all vector fields $U$ & $V$. Since $g$ is invariant under $J$, so is $\bar{\Omega}$, i.e.,

$$\bar{\Omega}(JU,JV) = \bar{\Omega}(U,V).$$

The almost complex structure $J$ is not in general parallel with respect to the Riemannian Connexion $\bar{\nabla}$ defined by the Hermitian metric $g$. In fact we have the following formula [21].

$$2g(\bar{\nabla}_J V,W) = d\omega(U,V,W) - d\omega(U,JV,JW) - g(U S(V,JW)),$$

where $S$ is the Nijenhuis tensor of $J$ defined by


It is easy to verify that $S$ satisfies

$$S(JU,V) = S(U,JV) = -JS(U,V).$$
It is well known that vanishing of the tensor $S(U,V)$ is the necessary and sufficient condition for an almost complex manifold to be a complex manifold [29].

If we extend the Riemannian connexion $\nabla$ of $\mathbb{M}$ to be a derivative on the tensor algebra of $\mathbb{M}$, then we have the following formulae:

(1.1.6) \( (\bar{\nabla}_U J)V = \bar{\nabla}_U JV - J\bar{\nabla}_U V \),

(1.1.7) \( (\bar{\nabla}_U \Omega)(V,W) = g(\bar{\nabla}_U J)V, W) \).

At this stage, one can appreciate the following definitions:

**Definition (1.1.4).** A Hermitian metric on an almost complex manifold is called Kaehler metric if the fundamental 2-form $\Omega$ is closed. An almost complex manifold (resp. a complex manifold) with a Kaehler metric is called an almost Kaehler manifold (resp. a Kaehler manifold). A manifold with a 2-form (resp. a closed 2-form) $\Omega$ which is nondegenerate at each point $p$ of $\mathbb{M}$ is called an almost symplectic (resp. symplectic) manifold [29].

We may now, write down the following as the defining conditions for the pre-Kaehler structures.
Definition (1.1.5). An almost Hermitian manifold $\bar{M}$ is said to be a
(a) Kaehler manifold if
\[ (\bar{\nabla}_U J)V=0, \]
(b) nearly Kaehler manifold if
\[ (\bar{\nabla}_U J)V+(\bar{\nabla}_V J)U=0, \]
(e) almost Kaehler if
\[ g((\bar{\nabla}_U J)V,W)+g((\bar{\nabla}_V J)W,U)+g(\bar{\nabla}_W J)U,V)=0, \]
or equivalently
\[ d\Omega(U,V,W)=0, \]
(d) quasi-Kaehler if
\[ (\bar{\nabla}_U J)V+(\bar{\nabla}_V J)JU=0, \]
for all $U, V, W$ in $T\bar{M}$.

For the relation among these classes, let us denote by $K, AK, NK, QK$ & $H$ the classes of Kaehler, almost Kaehler, nearly Kaehler, quasi-Kaehler and Hermitian manifolds respectively. Then it is known that [21]
\[
\begin{align*}
K & \subseteq AK \\
& \subseteq QK \quad \& \quad K \subseteq H. \\
& \subseteq NK
\end{align*}
\]
Further,
\[ K = H \cap QK = AK \cap NK. \]
There is another important class of manifolds which are not Kaehlerian but they are locally conformal to a Kaehler manifold, we now discuss them in the following paragraph.

Definition (1.1.6). Let \((\mathbb{M},\Omega)\) be an almost symplectic manifold. \(\mathbb{M}\) is said to be a locally conformal symplectic (l.c.s.) manifold if every point \(p \in \mathbb{M}\) has an open neighbourhood \(U\) such that

\[
d(\omega^{-\sigma}_U) = 0
\]

for some function \(\sigma: U \to \mathbb{R}\). If \(U = \mathbb{R}\), then \((\mathbb{M},\Omega)\) is said to be globally conformal symplectic and if \(\sigma\) is constant, then \((\mathbb{M},\Omega)\) is obviously a symplectic manifold. If the 2-form on a Hermitian manifold is, in particular, the fundamental 2-form i.e., if \(\Omega(U,V) = g(JU,V)\), then the locally conformal symplectic manifolds are said to be locally conformal Kaehler (l.c.k.) manifolds i.e., in this case every point of \(\mathbb{M}\) has a neighbourhood \(U\) endowed with a Kaehler metric of the form

\[
g' = e^{-\sigma}(g|_U)
\]

We refer to [46] for the main properties of such manifolds. Particularly for such manifolds, the forms \(d\sigma\) glue up to a global pfaffian form \(w\), called the Lee-form, and for which

\[
d\omega = \omega \wedge w.
\]

Its corresponding Lee-vector field \(\lambda\) is obtained as
(1.1.16) \( g(\lambda, U) = w(U) \).

\( \mathbb{M} \) is globally conformal Kaehler if and only if \( w \) is exact.
And it is Kaehler if and only if \( w = 0 \) or equivalently \( \lambda = 0 \).

The following is an important relation [47] and is used subsequently in the thesis.

(1.1.17) \( 2(\nabla_U \Omega)(V, W) = g(J\lambda, W)g(U, V) + g(\lambda, W)g(JU, V) \)
\[-g(J\lambda, V)g(U, W) - g(\lambda, V)g(JU, W). \]

On the other hand, on odd dimensional manifolds we define contact structures as follows:

Definition (1.1.7) [7]. A \((2n+1)\)-dimensional differentiable manifold \( \mathbb{M} \) is called an almost contact manifold if it is equipped with a triplet \((\mathbb{O}, \xi, \eta)\) where \( \mathbb{O} \) is a \((1,1)\) tensor field; \( \xi \), a vector field and \( \eta \) a 1-form on \( \mathbb{M} \) satisfying.

(1.1.18) \( \mathbb{O}^2 = -I + \eta \otimes \xi; \mathbb{O} \xi = 0; \eta(\xi) = 1, \eta \circ \mathbb{O} = 0 \)
Moreover if \( \mathbb{M} \) is endowed with a Riemannian metric \( g \) satisfying

(1.1.19) \( g(\mathbb{O}U, \mathbb{O}V) = g(U, V) - \eta(U)\eta(V); g(U, \xi) = \eta(U), \)
for all vector field \( U, V \) on \( \mathbb{M} \), then \( \mathbb{M} \) is said to be an almost contact metric manifold.

Definition (1.1.8). An almost contact manifold \( \mathbb{M} \) is said to
be a

(a) Cosymplectic manifold if
\begin{equation}
(\tilde{\nabla}_U \phi)V = 0 \text{ and } (\tilde{\nabla}_U \eta)V = 0,
\end{equation}
(b) nearly cosymplectic manifold if
\begin{equation}
(\tilde{\nabla}_U \phi)U = 0 \text{ and } (\tilde{\nabla}_U \eta)U = 0,
\end{equation}
(c) Sasakian manifold if
\begin{equation}
(\tilde{\nabla}_U \phi)V = \eta(V)U - g(U,V)\phi,
\end{equation}
(d) nearly Sasakian manifold if
\begin{equation}
(\tilde{\nabla}_U \phi)U = \eta(U)U - g(U,U)\phi,
\end{equation}
(e) quasi-Sasakian manifold if
\begin{equation}
g((\tilde{\nabla}_U \phi)V,W) + g(\tilde{\nabla}_V \phi)W,U) + g(\tilde{\nabla}_U \phi)U,V) = 0,
\end{equation}
for all \(U,V,W \in TM\). Obviously for the definitions (c), (d) & (e), \(M\) is taken to be an almost contact metric manifold.

2. Submanifold Theory:

If an \(n\)-dimensional differentiable manifold \(M\) admits an immersion \(f: M \rightarrow \bar{M}\) into an \(m\)-dimensional differentiable manifold \(\bar{M}\), then \(M\) is said to be a submanifold of \(\bar{M}\). Naturally \(n \leq m\). If \(M\) and \(\bar{M}\) are Riemannian manifolds, then \(f\) is said to be an isometric immersion if the differential map \(f_\bullet : TM \rightarrow T\bar{M}\) preserves the Riemannian metric, that is for \(U,V \in TM\),
\begin{equation}
g(f_\bullet U, f_\bullet V) = g(U,V),
\end{equation}
where we use $g$ to denote Riemannian metric on both $M$ and $\mathbb{R}$. Consequently $f_*$ becomes an isomorphism. When only local questions are involved, we shall identify $TM$ with $f_*(TM)$ through this isomorphism. Hence a tangent vector in $T\mathbb{R}$ tangent to $M$, shall mean a tangent vector which is the image of an element in $TM$ under $f_*$. More generally, a $C^\infty$-cross section of the restriction of $T\mathbb{R}$ on $M$ shall be called a vector field of $\mathbb{R}$ on $M$. Those tangent vectors of $T\mathbb{R}$ which are normal to $TM$ form the normal bundle $T^\perp M$ of $M$. Hence for every point $p \in M$, the tangent space $T_f(p)\mathbb{R}$ of $\mathbb{R}$ admits the following decomposition.

$$T_f(p)\mathbb{R}=T_pM \oplus T^\perp_pM.$$  

The Riemannian connexion $\nabla$ of $\mathbb{R}$ induces canonically the connexions $\nabla$ and $\nabla^\perp$ on $TM$ and on the normal bundle $T^\perp M$ respectively governed by the Gauss & Weingarten formulae viz:

\begin{align*}
(1.2.2) \quad \nabla_UV &= \nabla_UV + h(U,V), \\
(1.2.3) \quad \nabla_UN &= -A_NU + \nabla_U^\perp N.
\end{align*}

Where $U,V$ are vector fields on $M$ and $N \in T^\perp M$, $h$ and $A_N$ are second fundamental forms and are related by

$$g(h(U,V),N) = g(A_NU,V).$$

Looking into the Guass formula, we observe that we can
classify the submanifolds, putting conditions on h.

Definition (1.2.1) [9]. A submanifold for which the second fundamental form $h$ is identically zero is called totally geodesic submanifold.

Definition (1.2.2) [9]. The submanifold is called totally umbilical if its second fundamental form $h$ satisfies

\[(1.2.5) \quad h(U,V) = g(U,V)H,\]

where $H = 1/n(\text{trace of } h)$, is called the mean curvature vector field.

Definition (1.2.3) [9]. The submanifold $M$, is called minimal if the mean curvature vector $H$ vanishes identically i.e., $H = 0$.

3. CR-submanifold:

On an almost Hermitian manifold $\mathbb{H}$, $g(JU, JV) = g(U, V)$ for all vector fields $U, V$ on $\mathbb{H}$. In other words, $g(JU, U) = 0$, i.e., $JU \perp U$ for each $U$ on $\mathbb{H}$. Hence for a submanifold $M$ of $\mathbb{H}$ if $U \in T_pM$, $JU$ may or may not belong to $T_pM$. Thus the action of the almost complex structure $J$ on the tangent vectors of the submanifold of an almost Hermitian manifold gives rise to its classification into invariant and anti-invariant submanifolds. These submanifolds therefore are defined as follows:

Definition (1.3.1) [53]. A submanifold $M$ of an almost
Hermitian manifold $\mathcal{R}$ is said to be invariant (or holomorphic) if $J(T_p\mathcal{R}) = T_p\mathcal{R}, \forall \ p \in \mathcal{R}$.

Definition (1.3.2) [53]. A submanifold $\mathcal{M}$ of an almost Hermitian manifold $\mathcal{R}$ is said to be anti-invariant (or totally real) if $J(T_p\mathcal{M}) \subseteq T_p^\perp \mathcal{M}, \forall \ p \in \mathcal{M}$.

In 1978, A. Bejancu considered a new class of submanifolds of an almost Hermitian manifold of which the above classes are particular cases and he named this class of submanifolds as CR-submanifolds. That is, a CR-submanifold provides a single setting to study the invariant and anti-invariant submanifolds of an almost Hermitian manifold. Since the present thesis basically deals with the CR-submanifolds, we enlist here, in this section, some of the basic notions and results about these submanifolds which are relevant for the subsequent chapters.

Let $\mathcal{R}$ be an almost Hermitian manifold with almost complex structure $J$ and Hermitian metric $g$ and $\mathcal{M}$, a Riemannian submanifold immersed in $\mathcal{R}$. At each point $p \in \mathcal{M}$, let $D_p$ be the maximal holomorphic subspace of the tangent space $T_p\mathcal{M}$ i.e., $JD_p = D_p$. If the dimension of $D_p$ is same for all $p \in \mathcal{M}$, we have a holomorphic distribution $D$ on $\mathcal{M}$.

Definition (1.3.3). $\mathcal{M}$ is said to be a CR-submanifold of an
almost Hermitian manifold $\mathbb{R}$ if there exists on $M$ a $C^\infty$-holomorphic distribution $D$ such that its orthogonal complement $D^\perp$ is totally real in $M$

i.e., $JD_p^\perp \subseteq T_p^\perp M$, $\forall p \in M$. [2]

Clearly every real hypersurface $M$ of an almost Hermitian manifold is a CR-submanifold if $\dim M > 1$.

Remark: We observe from the above definition that the dimension of $D$ is always even, and that $JD^\perp$ being a sub-bundle of $T^\perp M$ splits as

$$T^\perp M = JD^\perp \oplus \mu$$

where $\mu$ is complement of $JD^\perp$ in $T^\perp M$ and that $\mu$ is invariant under $J$.

Definition (1.3.4). A CR-submanifold $M$ is said to be proper if neither $D=0$ nor $D^\perp=0$.

Obviously, if $D=0$ then $M$ is a totally real submanifold and if $D^\perp=0$ then $M$ is holomorphic.

Remark: Throughout the thesis we will denote by $M$, a CR-submanifold of the ambient space $\mathbb{R}$, unless mentioned otherwise.

For any vector field $U$ tangent to $M$, we put

(1.3.1) $JU=PU+FU$,

where $PU$ & $FU$ are the tangential and normal components of $JU$. 
respectively. Then \( P \) is an endomorphism of the tangent bundle \( TM \) and \( F \) is a normal bundle valued one form on \( TM \). It is easy to observe that \( PU \in D \& FU \in JD^\perp \). Infact \( P \) and \( F \) are annihilators on \( D^\perp \) and \( D \) respectively. Similarly for any vector \( N \), normal to \( M \), if we put

\[
(1.3.2) \quad JN = tN + fN,
\]

with \( tN \) and \( fN \) as tangential and normal components of \( JN \) respectively then \( f \) can be treated as an endomorphism of the normal bundle \( T^\perp M \) and \( t \), a tangent bundle valued 1-form on \( T^\perp M \) with kernel as \( JD^\perp \& \mu \) respectively.

The covariant differentiation of the operators \( P, F, t \) & \( f \) are defined respectively as:

\[
(1.3.3) \quad (\nabla_U P)V = \nabla_U PV - PV^U V,
\]

\[
(1.3.4) \quad (\nabla_U F)V = \nabla_U^1 FV - FV^U V,
\]

\[
(1.3.5) \quad (\nabla_U t)N = \nabla_U tN - tV^U N,
\]

\[
(1.3.6) \quad (\nabla_U f)N = \nabla_U^1 fN - fV^U N,
\]

A. Bejancu obtained the following integrability conditions for the distributions \( D \) and \( D^\perp \) on a CR-submanifold \( M \) of an almost Hermitian manifold \( \mathcal{M} \).

Proposition (1.3.1) [4]. The distribution \( D \) is integrable if and only if any one of the following is satisfied.

(i) \( S(X,Y)_T = S_p(X,Y) \),
(ii) $S(X,Y)^⊥ = 0$ and $CS_p(X,Y) = 0$

For any $X, Y$ in $D$, where $S_p$ stands for the Nijenhuis tensor of $P$, viz. $S_p(U,V) = P[X, PY] + P[PX, Y] - P^2[X, Y] - [PX, PY]$, $T$ and $⊥$ denote the tangential and normal components respectively, and $C$ denotes the projection operator onto $D^⊥$ respectively.

Proposition (1.3.2) [4]. The totally real distribution $D^⊥$ is integrable if and only if the Nijenhuis tensor of $P$ vanishes identically on $D^⊥$.

If the ambient space $\mathbb{R}$ is Kaehler, the integrability conditions for the distribution $D$ and $D^⊥$ are given by B.Y. Chen as:

Proposition (1.3.3) [11]. The holomorphic distribution $D$ on a CR-submanifold $M$ of a Kaehler manifold $\mathbb{R}$ is integrable if and only if

$$g(h(X,JY),JZ) = g(h(JX,Y),JZ)$$

for any vector fields $X, Y$ in $D$ and $Z$ in $D^⊥$.

Proposition (1.3.4) [11]. The totally real distribution $D^⊥$ on a CR-submanifold in a Kaehler manifold is integrable.

This theorem has been generalized by Blair & Chen [8] to a CR-submanifold of a locally conformal Kaehler manifold.
With regard to the geometry of leaves of $D$ and those of $D^\perp$ we have

**Proposition (1.3.5) [11].** Let $M$ be a CR-submanifold of a Kaehler manifold $\mathcal{M}$. Then

1) The leaves of $D$ are totally geodesic in $M$, if and only if
   \[ g(h(D, D), JD^\perp) = 0, \]

2) The leaves of $D^\perp$ are totally geodesic in $M$ if and only if
   \[ g(h(D, D^\perp), JD^\perp) = 0. \]

Under what conditions the CR-submanifold is a Riemannian product of holomorphic submanifold and totally real submanifold? This situation has important geometric significance. The problem was studied by B.Y. Chen.

**Definition (1.3.5) [11].** A CR-submanifold $M$ is called a CR-product if it is locally a Riemannian product of a holomorphic submanifold $M$ and a totally real submanifold $M$.

For a CR-product submanifold, the leaves of $D$ and $D^\perp$ are totally geodesic in $M$ and vice-versa. Thus we have

**Theorem (1.3.1) [11].** A CR-submanifold $M$ in a Kaehler manifold $\mathcal{M}$ is a CR-product if and only if

\[ AJD^\perp D = 0 \]

Another equivalent condition for CR-product is obtained in [13] and is given by
Theorem (1.3.2) [11]. A CR-submanifold of a Kaehler manifold $\mathcal{M}$ is a CR-product if and only if
$$\nabla P = 0$$

The condition $\nabla P = 0$ was further extended by Chen [13] for a submanifold of almost Hermitian manifolds. He, in fact proved the following:

Theorem (1.3.3) [13]. Let $M$ be a submanifold of an almost Hermitian manifold $\mathcal{N}$ then $\nabla P = 0$ if and only if $M$ is locally the Riemannian product $M_1 \times M_2 \cdots \times M_k$, where each $M_i$ is either a Kaehler submanifold, a totally real submanifold or a Kaehlerian slant submanifold.

Here the Kaehlerian slant submanifolds are defined as follows:

Definition (1.3.6) [13]. For each non zero vector $U$ tangent to $M$ at $p$, the angle $\theta(U)$ between $jU$ and $T_p M$ is called Wirtinger angle of $U$. The immersion $f: M \to \mathcal{M}$ is said to be a general slant immersion if the Wirtinger angle $\theta(U)$ is constant (i.e., independent of the choice of $p \in M$ & $U \in T_p M$). Holomorphic and totally real immersions are general slant immersions with Wirtinger angle equal to $0$ and $\pi/2$ respectively. A general slant immersion which is not holomorphic is simply called a slant immersion. A slant
submanifold is said to be proper if it is not totally real. Finally, a proper slant submanifold is said to be Kaehlerian slant if the endomorphism P is parallel i.e., \( \nabla P = 0 \).