CHAPTER 2

ON COMMON FIXED POINTS OF NONCONTINUOUS NONCOMMUTING MAPPINGS

‘No idea is so antiquated that it was not once modern. No idea is so modern that it will not some day antiquated’

(Elfen Glasgow)
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§ 2.1. Introduction

Every metrical common fixed point theorem generally involves conditions on commutativity and continuity of the involved maps besides a suitable contraction condition. Researchers in this domain are aimed at weakening one or more of these conditions.

Since the appearance of weak commutativity of Sessa [88], researchers started utilizing weak conditions of commutativity. Recent literature has witnessed the evolution of several such conditions of commutativity such as: Compatible mappings of type (A) [48], Compatible mappings of type (B) [62], Compatible mappings of type (P) [68], Compatible mappings of type (C) [70], Biased maps [43] and several others. The details of such weak conditions of commutativity are included in Chapter 1 (see Section 1.3.). In our subsequent work we choose to utilize the most natural of these weak conditions namely ‘coincidently commuting property’ (cf. [16] also see [51]).

It has been known since the paper of Kannan [56] that there exist maps possessing discontinuities in their domain but still admitting fixed points. However, in every case the maps involved were continuous at the fixed point. Recently, some authors endeavoured to relax continuity requirements in such results and in this regard the work of Singh-Mishra [93] and Pant [72, 73] deserves special mention.

In this chapter combining these ideas we demonstrate the effectiveness of the coincidence commutativity concept under quite tight conditions. Here we notice that an appreciable number of fixed point theorems can be improved by muting the continuity requirements of the maps completely besides reducing the commutativity requirement of the maps to merely coincidence points. Also, the completeness requirement of the space $X$ is weakened to a set of four alternative natural conditions. In process results of Fisher [22, 24], Diviccaro et al. [17], Aqel et al. [2], Imdad-Ahmad [38], Jeong-Rhoades [43], Kannan [56], Hardy-Rogers [32], Ahmad-Imdad [2] and others are generalized and improved.

§ 2.2. Fixed point theorems via certain rational inequalities

Here, we prove two general common fixed point theorems satisfying unified rational inequalities for pairwise coincidently commuting mappings. We utilize our main theorems to demonstrate how several fixed point theorems can be improved by muting the continuity requirements and improving commutativity requirement upto the extent of coincidently commuting property. The main results of this section are Theorem 2.2.1 and Theorem 2.2.2.

**Theorem 2.2.1.** Let $A, B, S, T, I$ and $J$ be self-mappings of a metric space $(X, d)$ with $AB(X) \subseteq J(X)$ and $ST(X) \subseteq I(X)$. If one of $AB(X), ST(X), I(X)$ or $J(X)$ is a complete subspace of $X$ and for any $x, y$ in $X$, either

$$d(ABx, STy) \leq \frac{a_0d(Ix, ABx)d(Iy, STy) + b_0d(Ix, STy)d(Iy, ABx)}{d(Ix, ABx) + d(Iy, STy)}$$

$$+ d_0d(Ix, Iy),$$

(2.2.1.1)

whenever $d(Ix, ABx) + d(Iy, STy) \neq 0$, provided $a_0, b_0, d_0 \geq 0$, so that at least one of these is non zero and $a_0 + 2d_0 < 2$, or
Then the following conclusions hold:

(a) \((AB, I)\) has a point of coincidence,

(b) \((ST, J)\) has a point of coincidence.

(c) Further, if the pairs \((AB, I)\) and \((ST, J)\) are coincidently commuting, then \(AB, ST, I\) and \(J\) have a unique common fixed point \(z\).

(d) Moreover, if the pairs \((A, B), (BA, B), (S, T), (TS, T), (A, I), (B, I), (S, J)\) and \((T, J)\) commute at \(z\), then \(z\) also remains the unique common fixed point of \(A, B, S, T, I\) and \(J\) separately.

**Proof.** Let \(x_0 \in X\) be an arbitrary point of \(X\). Since \(AB(X) \subset J(X)\), we can find a point \(x_1\) in \(X\) such that \(ABx_0 = Jx_1\). Also, since \(ST(X) \subset I(X)\), we can choose a point \(x_2\) with \(STx_1 = Ix_2\). Thus in general for the point \(x_{2n}\) one can find a point \(x_{2n+1}\) such that \(ABx_{2n} = Jx_{2n+1}\) and then a point \(x_{2n+2}\) with \(STx_{2n+1} = Ix_{2n+2}\) for \(n = 0, 1, 2, \ldots\). Let us put \(u_{2n} = d(ABx_{2n}, STx_{2n+1})\) and \(u_{2n+1} = d(STx_{2n+1}, ABx_{2n+2})\). Now, we distinguish two cases:

**Case (i)**: Suppose that \(u_{2n} + u_{2n+1} \neq 0\) for \(n = 0, 1, 2, \ldots\). Then on using the inequality (2.2.1.1), we have.

\[
u_{2n+1} \leq \frac{a_0 u_{2n} \cdot u_{2n+1}}{u_{2n} + u_{2n+1}} + d_0 u_{2n},\]

so that \(u_{2n+1} + (1-a_0-d_0)u_{2n} \cdot u_{2n+1} - d_0 u_{2n}^2 \leq 0\). The positive root \(k\) of the quadratic equation \(t^2 + (1-a_0-d_0)t - d_0 = 0\) is \([((1-a_0-d_0)^2 + 4d_0)]^{1/2} - (1-a_0-d_0)/2\) and since \(a_0 + 2d_0 < 2\), it follows that \(k < 1\). Thus \(u_{2n+1} \leq k u_{2n} \leq k^{2n+1} u_0\), for \(n = 0, 1, 2, \ldots\). It follows that the sequence \(\{y_n\}\) defined as \(y_{2n} = Jx_{2n+1} = ABx_{2n}\) and \(y_{2n+1} = Ix_{2n+2} = STx_{2n+1}\) for \(n \in N_0 = N \cup \{0\}\), is a Cauchy sequence.

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Now suppose that $I(X)$ is complete. Then by observing that the subsequence \( \{y_{2n+1}\} \) which is contained in $I(X)$ must gets a limit $z$ in $I(X)$. Let $u \in I^{-1}z$, then $Ju = z$. Now one needs to note that the subsequence $\{y_{2n}\}$ converges to $z$. Otherwise suppose that $\{y_{2n}\}$ converges to $z' \neq z$, then we have

$$d(y_{2n}, y_{2n+1}) = d(ABx_{2n}, STx_{2n+1})$$

$$\leq a_0 d(Ix_{2n}, ABx_{2n})d(Jx_{2n+1}, STx_{2n+1}) + b_0 d(Ix_{2n}, STx_{2n+1})d(Jx_{2n+1}, ABx_{2n})$$

$$+ d_0 d(Ix_{2n}, Jx_{2n+1})$$

which on letting $n \to \infty$, reduces to

$$d(z, z') \leq \left( \frac{a_0 + 2b_0}{2} \right) d(z, z') < d(z, z'),$$

which is a contradiction giving thereby $z = z'$.

To prove $ABu = z$, set $x = u$ and $y = x_{2n+1}$ in (2.2.1.1), then

$$d(ABu, STx_{2n+1})$$

$$\leq a_0 d(Iu, ABu) \cdot d(Jx_{2n+1}, STx_{2n+1}) + b_0 d(Iu, STx_{2n+1}) \cdot d(Jx_{2n+1}, ABu)$$

$$+ d_0 d(Iu, Jx_{2n+1})$$

which on letting $n \to \infty$, reduces to

$$d(ABu, z) \leq 0,$$

giving thereby $ABu = z$. Thus we have $ABu = Iu = z$, which shows that $(AB, I)$ has a point of coincidence. This proves (a).

Since $AB(X) \subset J(X)$, $ABu = z$ implies that $z \in J(X)$. Let $v \in J^{-1}z$, then $Jv = z$. Now using the earlier arguments it can be easily shown that $STv = z$, giving thereby $Jv = STv = z$, so that $(ST, J)$ has a point of coincidence which establishes (b).
If we assume that $J(X)$ is complete, then arguments analogous to the previous one can be produced to establish (a) and (b). The remaining two cases pertain essentially to the previous cases. Indeed, if $ST(X)$ is complete, then $z \in ST(X) \subseteq I(X)$. Similarly, if $AB(X)$ is complete, then $z \in AB(X) \subseteq J(X)$. Thus in each case (a) and (b) are completely established.

To prove (c), note that $(AB, I)$ and $(ST, J)$ are coincidently commuting at $u$ and $v$ respectively, then

\[ z = ABu = Iu = Jv = STv, \]  
\[ ABz = AB(Iu) = I(ABu) = Iz, \]
\[ \text{and } STz = ST(Jv) = J(STv) = Jz. \]

To prove $ABz = z$, we note that

\[ d(Iz, ABz) + d(Jv, STv) = 0, \]

which, due to (2.2.1.2), amounts to say that

\[ d(ABz, STv) = d(ABz, z) = 0, \]

yielding thereby $ABz = z$. Similarly, one can also show that $STz = z$. Now, in view of (2.2.1.4) and (2.2.1.5), $z$ is a common fixed point of $AB, ST, I$ and $J$. The uniqueness of the common fixed point of $AB, ST, I$ and $J$ is an easy consequence of contraction condition (2.2.1.1).

If the pairs $(A, I), (BA, B), (B, I), (A, B), (S, T), (TS, T), (S, J)$ and $(T, J)$ commute at $z$, then

\[ Az = A(ABz) = A(BAz) = AB(Az), \quad Az = A(Iz) = I(Az), \]
\[ Bz = B(ABz) = BA(Bz) = AB(Bz), \quad Bz = B(Iz) = I(Bz), \]
\[ Sz = S(STz) = S(TSz) = ST(Sz), \quad Sz = S(Jz) = J(Sz), \]
\[ Tz = T(STz) = TS(Tz) = ST(Tz), \quad Tz = T(Jz) = J(Tz), \]
which establish that $Az$ and $Bz$ are common fixed point of the pair $(AB, I)$, whereas $Sz$ and $Tz$ are the common fixed point of the pair $(ST, J)$. Now due to the uniqueness of common fixed point of both the pairs, one gets

$$z = Az = Bz = Sz = Tz = I_z = J_z.$$  

**Case (ii)**: If $d(ABx, Ix) + d(STy, Jy) = 0$ implies $d(ABx, STy) = 0$, then we argue as follows:

Let $u_n + u_{n+1} = 0$ for some $n$. Then $y_n = y_{n+1} = y_{n+2}$. If $n = 2k$, we have $y_{2k+2} = ABx_{2k+2} = Ix_{2k+2}$, so there exist $v_1, w_1$ such that $v_1 = ABw_1 = Iw_1$. Similarly, there exist $v_2, w_2$ such that $v_2 = STw_2 = Jw_2$. Since $d(ABw_1, Iw_1) + d(STw_2, Jw_2) = 0$, from (2.2.1.2) $d(ABw_1, STw_2) = 0$, implies $v_1 = ABw_1 = STw_2 = v_2$. Note also that $Iv_1 = I(ABw_1) = AB(Iw_1) = ABv_1$. Similarly $STv_2 = Jv_2$. Define $y_1 = ABv_1$, $y_2 = STv_2$. Since $d(ABv_1, Iv_1) + d(STv_2, Jv_2) = 0$, it follows from (2.2.1.2) that $d(ABv_1, STv_2) = 0$. i.e., $y_1 = y_2$. Thus $ABv_1 = Iv_1 = STv_2 = Jv_2$. But $v_1 = v_2$, therefore $AB, I, ST$ and $J$ have a common coincidence point. Define $w = ABv_1$, it then follows that $w$ is also a common coincidence point of $AB, ST, I$ and $J$. If $ABw \neq ABv_1 = STv_1$, then $d(ABw, STv_1) > 0$. But since $d(ABw, Iv_1) + d(STv_1, Jv_1) = 0$. It follows from (2.2.1.2) that $d(ABw, STv_1) = 0$. i.e., $ABw = STv_1$, which is a contradiction. Therefore, $ABw = ABv_1 = w$ and $w$ is a common fixed point of $AB, ST, I$ and $J$. The rest of the proof is identical to that of Case (i), hence it is omitted. This evidently completes the proof.

**Corollary 2.2.1.** The conclusions (a), (b), (c) and (d) of Theorem 2.2.1 remain valid if we replace contraction condition 2.2.1.1 by any one of the followings (modifying the rest of the hypotheses accordingly): For all $x, y$ in $X$.

$$d(Ax, Sy) \leq a_0d(Ix, Ax)d(Jy, Sy) + b_0d(Ix, Sy)d(Jy, Ax) \leq d(Ix, Ax) + d(Jy, Sy)$$

(deduced by restricting $B = T = I_X$)
\[ (B) \ d(Ax, Ay) \leq \frac{a_0 d(Ix, Ax)d(Jy, Ay) + b_0 d(Ix, Ay)d(Jy, Ax)}{d(Ix, Ax) + d(Jy, Ay)} + d_0 d(Ix, Jy) \]

(deduced by restricting \( A = S \) and \( B = T = I_x \))

\[ (C) \ d(Ax, Ay) \leq \frac{a_0 d(Ix, Ax)d(Iy, Ay) + b_0 d(Ix, Ay)d(Iy, Ax)}{d(Ix, Ax) + d(Iy, Ay)} + d_0 d(Ix, Iy) \]

(deduced by restricting \( A = S, I = J \) and \( B = T = I_x \))

\[ (D) \ d(Ax, Sy) \leq \frac{a_0 d(x, Ax)d(y, Sy) + b_0 d(x, Sy)d(y, Ax)}{d(x, Ax) + d(y, Sy)} + d_0 d(x, y) \]

(deduced by restricting \( I = J = B = T = I_x \))

\[ (E) \ d(Ax, Sy) \leq d_0 d(Ix, Jy) \]

(deduced by restricting \( B = T = I_x, a_0 = b_0 = 0 \))

**Remark 2.2.1.** By choosing \( a_0, b_0, d_0 \) and \( A, B, S, T, I \) and \( J \) suitably in Corollary 2.2.1, one can derive improved and generalized versions of certain results contained in Fisher [22, 24], Diviccaro et al. [17], Aqeel-Imdad-khan [2], Jeong-Rhoades [43] and Imdad-Ahmad [38]. Note that most of the results in [2, 17, 22, 24, 38, 43] involves three or four maps out of which atleast one is required to be continuous whereas, our Theorem 2.2.1 never needs continuity requirement besides increasing the number of involved maps from four to six and improving 'commutativity' requirement upto the extent of 'coincidently commuting property'. Also, completeness requirement of the space is weakened to a set of four alternative conditions which is also situationally useful.
While proving our next theorem, we adopt a more natural way than employed earlier. In fact, first we prove a common fixed point theorem for two pairs of coincidentally non-commuting mappings which is then utilized to prove a common fixed point theorem for four finite families of mappings. In doing so, we are motivated by the observations that any common fixed point theorem for four mappings can be used to prove common fixed point theorems for six, eight or any finite number of mappings as pointed out in Imdad [36].

**Theorem 2.2.2.** Let $A, S, I$ and $J$ be self-mappings of a metric space $(X, d)$ with $A(X) \subseteq J(X)$ and $S(X) \subseteq I(X)$ satisfying the conditions

$$d(Ax, Sy) \leq \alpha_1 \frac{[d(Ax, Ix)]^2 + [d(Sy, Jy)]^2}{[d(Ax, Ix)] + [d(Sy, Jy)]} + \alpha_2[d(Ax, Jy) + d(Sy, Ix)] + \alpha_3 d(Ix, Jy)$$

(2.2.2.1)

if $d(Ax, Ix) + d(Sy, Jy) \neq 0$, $\alpha_i \geq 0$ (with at least one $\alpha_i \neq 0$) and $2\alpha_1 + 2\alpha_2 + \alpha_3 < 1$,

or , $d(Ax, Sy) = 0$, whenever $d(Ax, Ix) + d(Sy, Jy) = 0$,  \hspace{1cm} (2.2.2.2)

for all $x, y$ in $X$. If one of $A(X), S(X), I(X)$ or $J(X)$ is a complete subspace of $X$, then

(e) $(A, I)$ has a point of coincidence.

(f) $(S, J)$ has a point of coincidence.

Moreover, if the pairs $(A, I)$ and $(S, J)$ are coincidently commuting, then $A, S, I$ and $J$ have a unique common fixed point $z$. Also, $z$ remains the unique common fixed point of both the pairs separately.

**Proof.** Following the proof of Theorem 2.2.1 one can construct a sequence $\{z_n\}$ such that $z_{2n} = Ax_{2n} = Jx_{2n+1}$, $z_{2n+1} = Sx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$. 

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For the sake of brevity, we write $u_{2n} = d(Ax_{2n}, Sx_{2n+1})$ and $u_{2n+1} = d(Sx_{2n+1}, Ax_{2n+2})$. Now we distinguish two cases.

Case (i): Suppose $u_{2n} + u_{2n+1} \neq 0$, for $n = 0, 1, 2, \ldots$. Then from inequality (2.2.2.1), we have

$$
\begin{align*}
d(z_{2n+1}, z_{2n+2}) &= d(Sx_{2n+1}, Ax_{2n+2}) \\
&\leq \alpha_1 \frac{[d(Ax_{2n+2}, Ix_{2n+2})]^2 + [d(Sx_{2n+1}, Jx_{2n+1})]^2}{[d(Ax_{2n+2}, Ix_{2n+2})] + [d(Sx_{2n+1}, Jx_{2n+1})]} \\
&+ \alpha_2 [d(Ax_{2n+2}, Jx_{2n+1}) + d(Sx_{2n+1}, Jx_{2n+2})] \\
&+ \alpha_3 d(Ix_{2n+2}, Jx_{2n+1})
\end{align*}
$$

which yields to

$$
d(z_{2n+1}, z_{2n+2}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_1 - \alpha_2} d(z_{2n+1}, z_{2n}).
$$

Similarly, one can show that

$$
d(z_{2n}, z_{2n+1}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_1 - \alpha_2} d(z_{2n}, z_{2n-1}).
$$

Thus for every $n$ we have

$$
d(z_n, z_{n+1}) \leq kd(z_{n-1}, z_n) \leq \leq k^n d(z_0, z_1),
$$

which shows that $\{z_n\}$ is a Cauchy sequence in $X$. Now suppose that $I(X)$ is a complete subspace of $X$, then note that the subsequence $\{z_{2n+1}\}$ which is contained in $I(X)$ must gets a limit $z$ in $X$. Let $u \in (I)^{-1}(z)$, then $Iu = z$. Now one needs to note that the subsequence $\{z_{2n}\}$ also converges to $z$. Otherwise suppose that $\{z_{2n}\}$ converges to some $z' \neq z$, then we have

$$
d(z_{2n}, z_{2n+1}) = d(Ax_{2n}, Sx_{2n+1})
$$

$$
\leq \alpha_1 \frac{[d(Ax_{2n}, Ix_{2n})]^2 + [d(Sx_{2n+1}, Jx_{2n+1})]^2}{d(Ax_{2n}, Ix_{2n}) + d(Sx_{2n+1}, Jx_{2n+1})} \\
+ \alpha_2 [d(Ax_{2n}, Jx_{2n+1}) + d(Sx_{2n+1}, Ix_{2n})] + \alpha_3 d(Ix_{2n}, Jx_{2n+1})
$$

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which on letting \( n \to \infty \), and using the fact that
\[
\frac{a^2 + b^2}{a + b} \leq \frac{(a + b)^2}{a + b} \leq a + b,
\]
reduces to
\[
d(z, z') \leq (2\alpha_1 + \alpha_3)d(z, z') < d(z, z'),
\]
which is a contradiction, giving thereby \( z = z' \).

To prove \( Au = z \), set \( x = u \) and \( y = x_{2n+1} \) in (2.2.2.1), then
\[
d(Au, Sx_{2n+1}) \leq \alpha_1 \left[ \frac{d(Au, Iu)^2}{d(Au, Iu)} + \frac{d(Sx_{2n+1}, Jx_{2n+1})^2}{d(Sx_{2n+1}, Jx_{2n+1})} \right]
+ \alpha_2[d(Au, Jx_{2n+1}) + d(Sx_{2n+1}, Iu)]
+ \alpha_3d(Iu, Jx_{2n+1}),
\]
which on letting \( n \to \infty \), reduces to
\[
d(Au, z) \leq (\alpha_1 + \alpha_2)d(Au, z),
\]
implies thereby \( Au = z \). Thus one gets \( Au = Iu = z \), which shows that \( (A, I) \) has a point of coincidence. This proves (e).

Since \( A(X) \subset J(X) \), \( Au = z \) implies that \( z \in J(X) \). Let \( v \in J^{-1}(z) \) then \( Jv = z \). Again using the earlier arguments, it can be easily shown that \( Sv = z \), giving thereby \( Jv = Sv = z \) which establishes (f). If one assumes \( J(X) \) to be a complete subspace of \( X \), then analogous arguments can be produced to establish (e) and (f). The remaining two cases pertain essentially to the previous one. Indeed, if \( S(X) \) is complete, then \( z \in S(X) \subset I(X) \). Similarly, if \( A(X) \) is complete, then \( z \in A(X) \subset J(X) \). Thus in each case (e) and (f) are completely established.

Moreover, if the pairs \( (A, I) \) and \( (S, J) \) are coincidently commuting at \( u \) and \( v \) respectively, then
\[
z = Au = Iu = Sv = Jv \tag{2.2.2.3}
\]
Now in order to prove $Az = z$, we note that

$$d(Az, Iz) + d(Sv, Jv) = 0$$

which due to (2.2.2.2), amounts to say that

$$d(Az, Sv) = d(Az, z) = 0,$$

yielding thereby $Az = z$. Similarly one can show that $z = Sz$. Thus $z$ is a common fixed point of $A, S, I$ and $J$. The uniqueness of common fixed point follows easily.

**Case (ii) :** Suppose $d(Ax, Ix) + d(Sy, Jy) = 0$, implies $d(Ax, Sy) = 0$, then we argue as follows:

Let $u_n + u_{n+1} = 0$ for some $n$. Then $z_n = z_{n+1} = z_{n+2}$. If $n = 2k$, we have $z_{2k+2} = Ax_{2k+2} = Ix_{2k+2}$. It then follows that there exist two points $w_1$ and $w_2$ such that $v_1 = Av_1 = Iw_1$ and $v_2 = Sw_2 = Jw_2$. Since $d(Aw_1, Iw_1) + d(Sw_2, Jw_2) = 0$, from (2.2.2.2) $d(Aw_1, Sw_2) = 0$, implies $v_1 = Aw_1 = Sw_2 = v_2$. Note also that $Iv_1 = I(Aw_1) = A(Iw_1) = Av_1$. Similarly $Sv_2 = Jv_2$. Define $y_1 = Av_1, y_2 = Sv_2$. Since $d(Av_1, Iv_1) + d(Sv_2, Jv_2) = 0$, it follows from (2.2.2.2) that $d(Av_1, Sv_2) = 0$, i.e., $y_1 = y_2$. Thus $Av_1 = Iv_1 = Sv_2 = Jv_2$. But $v_1 = v_2$. Therefore, $A, B, I$ and $J$ have a common coincidence point. Define $w = Av_1$, it then follows that $w$ is also a common coincidence point of $A, S, I$ and $J$. If $Aw \neq Av_1 = Sv_1$, then $d(Aw, Sv_1) > 0$. But since $d(Aw, Iw) + d(Sv_1, Jv_1) = 0$, it again follows from (2.2.2.2) that $d(Aw, Sv_1) = 0$ i.e., $Aw = Sv_1$, a contradiction. Therefore, $Aw = Av_1 = w$ and $w$ is a common fixed point of $A, S, I$ and $J$. The rest of the proof is identical to the Case (i), hence it is omitted.

This completes the proof

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Corollary 2.2.2. The conclusions of Theorem 2.2.2 remain true if conditions (2.2.2.1) and (2.2.2.2) are replaced by any one of the following contractions:

\((A)'\) Either
\[
d(A, S) \leq \frac{\alpha_1 [d(A, I) + d(S, J)]^2 + [d(S, J)]^2}{d(A, I) + d(S, J)} + \alpha_2 [d(A, J) + d(S, I)],
\]
provided \(d(A, I) + d(S, J) \neq 0\), \(\alpha_1, \alpha_2 \geq 0\), \(2\alpha_1 + 2\alpha_2 < 1\), or
\[
d(A, S) = 0, \text{ if } d(A, I) + d(S, J) = 0.
\]

\((B)'\) Either
\[
d(A, S) \leq \alpha_1 \frac{[d(A, I)]^2 + [d(S, J)]^2}{d(A, I) + d(S, J)} + \alpha_3 d(I, J),
\]
provided \(d(A, I) + d(S, J) 
eq 0\), one of \(\alpha_1\) and \(\alpha_3\) is non-zero with \(2\alpha_1 + \alpha_3 < 1\), or
\[
d(A, S) = 0, \text{ whenever } d(A, I) + d(S, J) = 0.
\]

\((C)'\) Either
\[
d(A, S) \leq \alpha_1 \frac{[d(A, I)]^2 + [d(S, J)]^2}{d(A, I) + d(S, J)},
\]
provided \(d(A, I) + d(S, J) \neq 0\), \(\alpha_1 \geq 0\) with \(\alpha_1 < 1/2\), or
\[
d(A, S) = 0, \text{ whenever } d(A, I) + d(S, J) = 0.
\]

\((D)'\) \(d(A, S) \leq \alpha_1 [d(A, I) + d(S, J)] + \alpha_2 [d(A, J) + d(S, I)] + \alpha_3 d(I, J),\)
provided \(2\alpha_1 + 2\alpha_2 + \alpha_3 < 1\), or
\[
\quad \text{with } \alpha_1 < 1/2, \text{ or}
\]
\[
\quad \text{with } \alpha_2 < 1/2, \text{ or}
\]
\[
\quad \text{with } \alpha_3 < 1.
\]
Proof. Corollaries corresponding to the contraction conditions \((A)'\), \((B)'\) and \((C)'\) are immediate from Theorem 2.2.2 by setting \(\alpha_3 = 0, \alpha_2 = 0\) and \(\alpha_2 = \alpha_3 = 0\), respectively. The corollary corresponding to the contraction condition \((D)'\) follows from Theorem 2.2.2 after noting the fact
\[
\frac{(d(AX, Ix))^2 + (d(SY, Jy))^2}{d(AX, Ix) + d(SY, Jy)} \leq \frac{(d(AX, Ix) + d(SY, Jy))^2}{d(AX, Ix) + d(SY, Jy)} \leq [d(AX, Ix) + d(SY, Jy)]
\]
Finally one notes that contractions \((E)'\), \((F)'\) and \((G)'\) are special cases to the contraction \((D)'\).

Remark 2.2.2. Corollary 2.2.2 corresponding to the condition \((D)'\) is an extension of a theorem of Hardy and Rogers [32] to four discontinuous pairwise coincidently commuting mappings. Corollary 2.2.2 corresponding to the condition \((A)'\) unifies the result of Fisher [24] and Kannan [56, 57] whereas, Corollary 2.2.2 corresponding to the condition \((B)'\) extends the result of Ahmad and Imdad [2]. Corollary 2.2.2 corresponding to the condition \((C)'\) extends a theorem of Fisher [24] to four discontinuous coincidently commuting mappings.

§ 2.3. Fixed point theorems for families of mappings

In this section, as an application of Theorem 2.2.2, we prove a common fixed point theorem for four finite families of mappings which runs as follow:

Theorem 2.3.1. Let \(\{A_1, A_2, \ldots, A_m\}, \{S_1, S_2, \ldots, S_n\}, \{I_1, I_2, \ldots, I_p\}\) and \(\{J_1, J_2, \ldots, J_q\}\) be four finite families of self-mappings of a metric space \((X, d)\) with \(A = A_1 A_2 \ldots A_m, S = S_1 S_2 \ldots S_n, I = I_1 I_2 \ldots I_p\) and \(J = J_1 J_2 \ldots J_q\) satisfying conditions (2.2.2.1) and (2.2.2.2) with \(A(X) \subseteq J(X), S(X) \subseteq I(X)\). If one of \(A(X), S(X), I(X)\) or \(J(X)\) is a complete subspace of \(X\), then

\[(g) (A, I)\] has a point of coincidence.

\[(h) (S, J)\] has a point of coincidence.
Moreover, if \( A_iA_j = A_jA_i, \) \( I_kI_l = I_lI_k, \) \( S_rS_s = S_sS_r, \) \( J_tJ_u = J_uJ_t, \) \( A_iI_k = I_kA_i \) and \( S_rJ_t = J_tS_r \) for all \( i, j \in I_1 = \{1, 2, \ldots, m\}, \) \( k, l \in I_2 = \{1, 2, \ldots, p\}, \) \( r, s \in I_3 = \{1, 2, \ldots, n\} \) and \( t, u \in I_4 = \{1, 2, \ldots, q\}, \) then (for all \( i \in I_1, k \in I_2, r \in I_3 \) and \( t \in I_4 \)) \( A_i, I_k, S_r \) and \( J_t \) have a common fixed point.

**Proof.** The conclusions (g) and (h) are immediate as \( A, S, I \) and \( J \) satisfy all the conditions of Theorem 2.2.2. Now appealing to componentwise commutativity of various pairs, one can immediately prove that \( AI = IA \) and \( SJ = JS \) and hence, obviously both the pairs \( (A, I) \) and \( (S, J) \) are coincidently commuting. Note that all the conditions of Theorem 2.2.2 (for mappings \( A, S, I \) and \( J \)) are satisfied ensuring the existence of unique common fixed point \( z. \) Now one needs to show that \( z \) remains the fixed point of all the component maps. For this consider

\[
A(A_i z) = ((A_1, A_2, \ldots, A_m) A_i) z = (A_1 A_2 \ldots A_{m-1})(A_m A_i)(z) = (A_1 \ldots A_{m-1})(A_i A_m z)
\]

\[
= (A_1 \ldots A_{m-2})(A_{m-1} A_i(A_m z)) = (A_1 \ldots A_{m-2})(A_i A_{m-1}(A_m z)) = \ldots
\]

\[
= \ldots = \ldots = \ldots = \ldots = \ldots = \ldots = \ldots = \ldots = \ldots = \ldots = \ldots
\]

\[
= A_1 A_i(A_2 A_3 A_4 \ldots A_m z) = A_i A_1(A_2 A_3 \ldots A_m z) = A_i(A z) = A_i z.
\]

Similarly, one can show that,

\[
A(I_k z) = I_k(A z) = I_k z, \quad I(I_k z) = I_k(I z) = I_k z,
\]

\[
J(A_i z) = A_i(I z) = A_i z, \quad S(S_r z) = S_r(S z) = S_r z,
\]

\[
S(J_t z) = J_t(S z) = J_t z, \quad J(J_t z) = J_t(J z) = J_t z,
\]

\[
\text{and } J(S_r z) = S_r(J z) = S_r z,
\]

which show that (for all \( i, r, k \) and \( t \)) \( A_i z, I_k z \) are other fixed points of the pair \( (A, I) \) whereas \( S_r z \) and \( J_t z \) are other fixed points of the pair \( (S, J) \). Now appealing to the uniqueness of common fixed points of both the pairs separately, we get

\[
z = A_i z = S_r z = I_k z = J_t z,
\]
which shows that \( z \) is a common fixed point of \( A_i, S_r, I_k \) and \( J_t \) for all \( i, r, k \) and \( t \).

This completes the proof.

By setting \( A_1 = A_2 = \ldots = A_m = F, S_1 = S_2 = \ldots = S_n = G, I_1 = I_2 = \ldots = I_p = B \) and \( J_1 = J_2 = \ldots = J_q = T \) in Theorem 2.3.1, we deduce the following:

**Corollary 2.3.1.** Let \( F, G, B \) and \( T \) be self-mappings of a metric space \((X, d)\) with \( F^m(X) \subseteq T^q(X) \) and \( G^n(X) \subseteq B^p(X) \) satisfying the condition

\[
d(F^m x, G^n y) \leq \alpha_1 \frac{[d(F^m x, B^p x)]^2 + [d(G^n y, T^q y)]^2}{d(F^m x, B^p x) + d(G^n y, T^q y)}
+ \alpha_2 [d(F^m x, T^q y) + d(G^n y, B^p x)] + \alpha_3 d(B^p x, T^q y),
\]

if \( d(F^m x, B^p x) + d(G^n y, T^q y) \neq 0, \alpha_i \geq 0 (i = 1, 2, 3) \) and at least one \( \alpha_i \neq 0 \) and \( 2\alpha_1 + 2\alpha_2 + \alpha_3 < 1 \), or \( d(F^m x, G^n y) = 0 \), whenever \( d(F^m x, B^p x) + d(G^n y, T^q y) = 0 \), for all \( x, y \in X \). If one of \( F^m(X), G^n(X), B^p(X) \) or \( T^q(X) \) is a complete subspace of \( X \), then \( F, G, B \) and \( T \) have a unique common fixed point provided \( FB = BF \) and \( GT = TG \).

**Remark 2.3.1.** By restricting four families as \( \{A_1, A_2\}, \{S_1, S_2\}, \{I_1\} \) and \( \{J_1\} \) in Theorem 2.3.1, we deduce a substantial but partial generalization of the main result of Imdad and Khan [40] as such a result will deduce stronger commutativity condition besides relaxing continuity requirements and improving completeness requirement of the space to four alternative natural conditions.

**Remark 2.3.2.** Corollary 2.3.1 is a slight but partial generalization of Theorem 2.3.1 as the commutativity requirements (i.e. \( FB = BF \) and \( GT = TG \)) in this corollary are stronger as compared to Theorem 2.3.1.
Remark 2.3.3. A result similar to Corollary 2.2.2 can be derived from Corollary 2.3.1 for iterates of maps. For the sake of brevity, we have not included the entire details.

§ 2.4. Illustrative examples

Our first example illustrates the hypotheses of Theorem 2.2.1 besides establishing its utility over related results contained in [2, 17, 22, 24, 25, 38 and 43].

Example 2.4.1. Consider $X = [0,6]$ equipped with usual metric. On $X$ define self maps $A, B, S, T, I$ and $J$ as follows:

\[
A(x) = \begin{cases} 
0 & \text{if } x = 0, \\
1 & \text{if } 0 < x \leq 6,
\end{cases}
\]
\[
B(x) = \begin{cases} 
0 & \text{if } x = 0, \\
2 & \text{if } 0 < x \leq 6,
\end{cases}
\]
\[
S(x) = \begin{cases} 
0 & \text{if } x = 0, \\
3 & \text{if } 0 < x < 6,
\end{cases}
\]
\[
I(x) = \begin{cases} 
0 & \text{if } x = 0, \\
5 & \text{if } 0 < x < 6, \\
3 & \text{if } x = 6,
\end{cases}
\]
\[
J(x) = \begin{cases} 
0 & \text{if } x = 0, \\
6 & \text{if } 0 < x < 6, \\
1 & \text{if } x = 6,
\end{cases}
\]
\[
T(x) = \begin{cases} 
4 & \text{if } 0 < x < 6, \\
6 & \text{if } x = 6,
\end{cases}
\]

so that

\[
AB(x) = \begin{cases} 
0 & \text{if } x = 0, \\
1 & \text{if } 0 < x \leq 6,
\end{cases}
\]
\[
ST(x) = \begin{cases} 
0 & \text{if } x = 0 \text{ or } x = 6, \\
3 & \text{if } 0 < x < 6.
\end{cases}
\]

It is worth noting that all the six maps are discontinuous even at their unique common fixed point ‘0’. Note that $AB(X) = \{0,1\} \subset \{0,1,6\} = J(X)$, and $ST(X) = \{0,3\} \subset \{0,3,5\} = I(X)$. Also, the pairs $(AB, I)$ and $(ST, J)$ commute at ‘0’ which is their common coincidence point. Also all needed pairwise commutativity is immediate at coincidence point ‘0’. By a routine calculation one can verify the contraction condition (2.2.1.1) for control constants $a_0 = 11/10, b_0 = 1$ and $d_0 = 1/5$. Clearly, all the conditions of Theorem 2.2.1 are satisfied and ‘0’ is the unique common fixed point of $A, B, S, T, I$ and $J$. 

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However, the unification of contraction condition (2.2.1.1) remains genuine because for $0 < x < 6$ and $0 < y < 6$ the contraction condition (2.2.1.1) with $a_0 = b_0 = 0$ implies $2 \leq d_0$ which is a contradiction to the fact $0 \leq d_0 < 1$ (cf. [2, Theorem 1]).

Our next example discusses the validity of the hypotheses and degree of generality of Theorem 2.2.2 over relevant results especially those contained in [23, 40].

**Example 2.4.2.** Consider $X = [0, 6]$ with usual metric. Set $A = S$ and $I = J$ and define self-mappings $A$ and $I$ as follows:

$$A(x) = \begin{cases} 
0 & \text{if } x = 0 \text{ or } x = 6 \\
1 & \text{if } 0 < x \leq 6,
\end{cases}$$

$$I(x) = \begin{cases} 
0 & \text{if } x = 0, \\
3 & \text{if } 0 < x < 6, \\
1 & \text{if } x = 6,
\end{cases}$$

Again all the maps in this example are discontinuous even at their unique common fixed point '0' which is their common coincidence point as well. Clearly, $A(X) = \{0, 1\} \subset I(X) = \{0, 1, 3\}$. Also the pair $(A, I)$ is commutative at coincidence point '0'. Now one can easily verify the contraction conditions (2.2.2.1) (resp. 2.2.2.2) for $\alpha_1 = \alpha_2 = 1/12$ and $\alpha_3 = 1/2$, in turn satisfying all the hypotheses of Theorem 2.2.2 with '0' as the unique common fixed point of the involved maps.

**Remark 2.4.1.** Example 2.4.1 and 2.4.2 exhibit that the main theorem of Imdad and Khan [40] and other related results (cf. [16, 24, 40, 44]) for two or more maps cannot be used in this context as all the involved maps in our results are discontinuous whereas, all the earlier known theorems require the continuity of at least one of the involved maps. Also, Example 2.4.2 shows that the pair $(A, I)$ is not compatible as there exists no sequence $\{x_n\} \subset [0, 6]$ such that $\lim_{n \to \infty} Ix_n = \lim_{n \to \infty} Ax_n$. Thus all the known theorems (cf. [46, 47, 48]), in the literature with compatibility requirements cannot be used in the context of this example.
Finally, for the verification of the hypotheses involved in Theorem 2.3.1, we give the following example. Here, we notice that requirement of commutativity at the coincidence points is necessary in Theorem 2.3.1.

**Example 2.4.3.** Let \( X = \{0, 1, 1/2, 1/2^2, 1/2^3, \ldots\} \) be a metric space with the usual metric \( d(x, y) = |x - y| \) for all \( x, y \in X \). Define mappings \( A, I : X \to X \) by \( A(0) = 1/2^2, A(1/2^n) = 1/2^{n+2}, I(0) = 1/2, I(1/2^n) = 1/2^{n+1} \) for \( n = 0, 1, 2, \ldots \) respectively. Also set \( A = S \) and \( I = J \). Clearly

\[
A(X) = \{1/2^2, 1/2^3, \ldots\} \subset \{1/2, 1/2^2, 1/2^3, \ldots\} = I(X).
\]

By a routine calculation one can verify that the contraction condition (2.2.2.1) or (2.2.2.2) is satisfied for \( \alpha_1 = 1/8, \alpha_2 = 1/6, \alpha_3 = 1/4 \). Thus all the conditions of Theorem 2.3.1 are satisfied except the completeness of the subspace \( A(X) \) and \( S(X) \). Note that \( A \) and \( I \) have no point of coincidence. Here it is fascinating to note that in the set up of Theorem 2.3.1 even the completeness of the space cannot ensure the existence of coincidence point as the space \( X \) is complete in the present example. Also note that \( A \) and \( S \) are not continuous at the origin.

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