CHAPTER 7

ON COINCIDENCE AND COMMON FIXED POINTS OF NONLINEAR HYBRID CONTRACTIONS

‘Knowledge is two fold, and consists not only an affirmation of what is true, but in the negation of that which is false’

(Charles Culson)
CHAPTER 7

On coincidence and common fixed points of nonlinear hybrid contractions

§ 7.1. Introduction

The study of fixed point theorems for multi-valued mappings was initiated by Kakutani [53] in the year 1941. He extended Brouwer’s fixed point theorem for the $n$-cell to upper semi-continuous compact, nonempty, convex set-valued mappings of the $n$-cell. Later in 1946 Eilenberg and Montgomery [20] generalized Kakutani’s result to acyclic absolute neighborhood retracts and upper semi continuous mappings $F$ such that $F(x)$ is nonempty, compact, and acyclic for each $x$. Strother [99] in the year 1953 showed that every continuous multi-valued mapping of the unit interval of $I$ into the nonempty compact subsets of $I$ has a fixed point but that the analogous result for the 2-cell is false.

The development of metric fixed point theory for multifunctions was initiated by Nadler [65] and subsequently pursued by Markin [61], Assad and Kirk [5], Browder [9], Himmelberg [35] and several others. Hybrid fixed point theory for nonlinear single-valued and multi-valued functions is a relatively new development in the domain of contractive type multi-valued theory.

During the last three decades several authors produced a spate of articles employing suitable contractive and weaker versions of commutativity conditions for hybrid pairs. See for instance ([13], [14], [54], [55] and the references given therein).

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In this chapter we investigate the coincidence and common fixed points of non-linear hybrid contractions for multi-valued as well as for single-valued mappings. Our results generalize some earlier results of Cho et al. [13], Fisher [22, 23, 24], Diviccaro et al. [17], Popa [74] and several others. In proving our results we follow the definitions and conventions of Nadler (cf. Chapter 1, Section 1.4) and use some basic needed results established therein. We need not mention here the same to avoid the repetition. However, we use the following definition to prove our main theorem which merely restricts the full force of idempotence to points of coincidence.

**Definition 7.1.1** A pair of self-mappings $(f, I)$ on $X$ is said to be 'coincidently idempotent' if both the partners $f$ and $I$ are idempotent at the points of coincidence of $f$ and $I$.

§ 7.2. Coincidence and common fixed point theorems

In this section we prove some common fixed point theorems for nonlinear hybrid contractions for multi-valued mappings and derive several well known results as corollaries.

In what follows, $S(X)$ and $T(X)$ denote $S(X) = \cup_{x \in X} Sx$ and $T(X) = \cup_{x \in X} Tx$, respectively.

Our objective is to prove the following:

**Theorem 7.2.1** Let $f, g, I$ and $J$ be self-mappings of a complete metric space $(X, d)$ with $fI$ and $gJ$ be $d$-continuous whereas $S, T : X \to CB(X)$ be $H$-continuous multi-valued mappings such that.

(i) $T(X) \subset fI(X)$ and $S(X) \subset gJ(X)$,

(ii) the pairs $(fI, S)$ and $(gJ, T)$ are compatible mappings.
(iii) for all \(x, y \in X\).

\[
H^p(Sx, Ty) \leq \frac{cd(fIx, Sx)d^p(gJy, Ty) + bd(fIx, Ty)d^p(gJy, Sx)}{\delta(fIx, Sx) + \delta(gJy, Ty)}
+ ad^p(gJy, Ty),
\]

(7.2.1.1)

for which \(\delta(fIx, Sx) + \delta(gJy, Ty) \neq 0\), where \(p \geq 1, b \geq 0, c > 0\) and \(1 < (c+2a) < 2\).

Then the following conclusions hold:

(a) There exists a point \(z \in X\) such that \(fIz \in Sz\) and \(gJz \in Tz\), i.e., \(z\) is a coincidence point of the pairs \((fI, S)\) and \((gJ, T)\).

(b) For each \(x \in X\) either (i) \(fIx \neq (fI)^2x \Rightarrow fIx \notin Sx\) (resp. \(gJx \neq (gJ)^2x \Rightarrow gJx \notin Tx\)) or (ii) \(fIx \in Sx \Rightarrow (fI)^nx \to z\) for some \(z \in X\) (resp. \(gJx \in Tx \Rightarrow (gJ)^nx \to z\) for some \(z \in X\)), then \(z\) is a common fixed point of the pair \((fI, S)\) (resp. \((gJ, T)\)).

(c) Moreover, if the pairs of self-mappings \((f, I)\), \((fI, f)\), \((g, J)\) and \((gJ, g)\) commute at the points of coincidence whereas, the pairs \((f, I)\) and \((g, J)\) are coincidentally idempotent then \(z\) is a common fixed point of \(f, I, fI\) and \(S\) and of \(g, J, gJ\) and \(T\).

**Proof.** Choose a real number \(k\) with \(1 < k < (2/c + 2a)^{1/p}\) and let \(x_0\) be an arbitrary point in \(X\). Since \(Sx_0 \subset gJ(X)\), there exists a point \(x_1\) in \(X\) such that \(gJx_1 \in Sx_0\) and so there exists a point \(y \in Tx_1\) such that

\[
d(gJx_1, y) \leq kH(Sx_0, Tx_1),
\]

which is always possible in view of Lemma 1.4.2. Since \(Tx_1 \subset fI(X)\) there exists a point \(x_2 \in X\) such that \(y = fIx_2\) and we obtain

\[
d(gJx_1, fIx_2) \leq kH(Sx_0, Tx_1).
\]
Similarly, there exists a point \( x_3 \in X \) such that \( gJx_3 \in Sx_2 \) and
\[
d(gJx_3, fIx_2) \leq kH(Sx_2, Tx_1).
\]
Inductively, one can obtain a sequence \( \{x_n\} \) in \( X \) such that
\[
\begin{align*}
fIx_{2n} & \in Tx_{2n-1}, \ n \in N, \\
gJx_{2n+1} & \in Sx_{2n}, \ n \in N_0 = N \cup \{0\}, \\
d(gJx_{2n+1}, fIx_{2n}) & \leq kH(Sx_{2n}, Tx_{2n-1}), \ n \in N, \\
\text{and } d(gJx_{2n+1}, fIx_{2n+2}) & \leq kH(Sx_{2n}, Tx_{2n+1}), \ n \in N_0,
\end{align*}
\]
where \( N \) denotes the set of positive integers.

First, suppose that for some \( n \in N \),
\[
\delta(fIx_{2n}, Sx_{2n}) + \delta(gJx_{2n+1}, Tx_{2n+1}) = 0,
\]
then \( fIx_{2n} \in Sx_{2n} \) and \( gJx_{2n+1} \in Tx_{2n+1} \), which show that \( x_{2n} \) is a coincidence point of \( fI \) and \( S \) whereas \( x_{2n+1} \) is a coincidence point of \( gJ \) and \( T \).

Similarly, if \( \delta(fIx_{2n+2}, Sx_{2n+2}) + \delta(gJx_{2n+1}, Tx_{2n+1}) = 0 \), for some \( n \in N \), then \( x_{2n+1} \) is a coincidence point of \( gJ \) and \( T \) whereas \( x_{2n+2} \) is a coincidence point of \( fI \) and \( S \).

Now, suppose that \( \delta(fIx_{2n}, Sx_{2n}) + \delta(gJx_{2n+1}, Tx_{2n+1}) \neq 0 \), for some \( n \in N_0 \). Then, using (7.2.1.1), we obtain
\[
d^p(gJx_{2n+1}, fIx_{2n+2}) \leq k^pH^p(Sx_{2n}, Tx_{2n+1}),
\]
\[
\leq k^p \left[ \frac{cd(fIx_{2n}, Sx_{2n})d^p(gJx_{2n+1}, Tx_{2n+1}) + bd(fIx_{2n}, Tx_{2n+1})d^p(gJx_{2n+1}, Sx_{2n})}{\delta(fIx_{2n}, Sx_{2n}) + \delta(gJx_{2n+1}, Tx_{2n+1})} \right] + ad^p(gJx_{2n+1}, Tx_{2n+1}) 
\leq k^p \left[ \frac{cd(fIx_{2n}, gJx_{2n+1})d^p(gJx_{2n+1}, fIx_{2n+2}) + bd(fIx_{2n}, gJx_{2n+1})d^p(gJx_{2n+1}, gJx_{2n+1})}{d(fIx_{2n}, gJx_{2n+1}) + d(gJx_{2n+1}, fIx_{2n+2})} \right] + ad^p(gJx_{2n+1}, fIx_{2n+2}) \tag{7.2.1.2}
\]
If \( d(gJx_{2n+1}, fIx_{2n+2}) = 0 \) and \( d(fIx_{2n}, gJx_{2n+1}) \neq 0 \) in (7.2.1.2), then \( gJx_{2n+1} = fIx_{2n+2} \in Tx_{2n+1} \) and so \( x_{2n+1} \) is a coincidence point of \( gJ \) and \( T \). But the case of \( d(fIx_{2n}, gJx_{2n+1}) = 0 \) and \( d(gJx_{2n+1}, fIx_{2n+2}) \neq 0 \) in (7.2.1.2) cannot occur. In fact, if \( d(fIx_{2n}, gJx_{2n+1}) = 0 \) and \( d(gJx_{2n+1}, fIx_{2n+2}) \neq 0 \) in (7.2.1.2), then we have \( d^p(gJx_{2n+1}, fIx_{2n+2}) \leq k^p d^p(gJx_{2n+1}, fIx_{2n+2}) \). Since \( k^p a < 1 \) as \( c > 0 \) and \( 1 < k < (2/c + 2a)^{1/p} \), we get \( d(gJx_{2n+1}, fIx_{2n+2}) = 0 \), which is impossible.

Now from (7.2.1.2), we have

\[
d^p(gJx_{2n+1}, fIx_{2n+2}) \leq k^p d^p(gJx_{2n+1}, fIx_{2n+2})
\]

which implies that

\[
d(gJx_{2n+1}, fIx_{2n+2}) = \left( \frac{k^p(c+a)-1}{1-k^p a} \right) d(fIx_{2n}, gJx_{2n+1}), \text{ or}
\]

\[
d(gJx_{2n+1}, fIx_{2n+2}) \leq \beta d(fIx_{2n}, gJx_{2n+1}),
\]

where \( \beta = (k^p(c+a)) - 1/1 - k^p a \).

Again, using (7.2.1.1), we get

\[
d^p(gJx_{2n+3}, fIx_{2n+2}) \leq k^p H^p(Sx_{2n+2}, Tx_{2n+1}),
\]

\[
\leq k^p \left[ \frac{cd(fIx_{2n+2}, Sx_{2n+2})d^p(gJx_{2n+1}, Tx_{2n+1})+bd(fIx_{2n+2}, Sx_{2n+2})d^p(gJx_{2n+2}, Sx_{2n+2})}{\delta(fIx_{2n+2}, Sx_{2n+2})+\delta(gJx_{2n+1}, Tx_{2n+1})} + ad^p(gJx_{2n+1}, Tx_{2n+1}) \right]
\]

\[
\leq k^p \left[ \frac{cd(fIx_{2n+2}, gJx_{2n+3})d^p(gJx_{2n+1}, fIx_{2n+2})+bd(fIx_{2n+2}, fIx_{2n+2})d^p(gJx_{2n+2}, gJx_{2n+3})}{d(fIx_{2n+2}, gJx_{2n+3})+d(gJx_{2n+1}, fIx_{2n+2})} + ad^p(gJx_{2n+1}, fIx_{2n+2}) \right]
\]

\[
d^p(gJx_{2n+3}, fIx_{2n+2}) \leq k^p cd(fIx_{2n+2}, gJx_{2n+3})d^p(gJx_{2n+1}, fIx_{2n+2}) + k^p ad^p(gJx_{2n+1}, fIx_{2n+2})
\]

\[
\leq k^p \left[ d(fIx_{2n+2}, gJx_{2n+3})+d(gJx_{2n+1}, fIx_{2n+2}) \right]
\]

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which implies that, if
\[
\alpha = \frac{d(gJx_{2n+3}, fIx_{2n+2})}{d(fIx_{2n+2}, gJx_{2n+1})},
\]
then \(\alpha^p + \alpha^{p-1} \leq k^p\{c + (1 + \alpha^{-1})a\}\). Thus \(\alpha < 1\) and we have
\[
d(gJx_{2n+3}, fIx_{2n+2}) \leq d(fIx_{2n+2}, gJx_{2n+1}).
\]
Repeating the above argument, since \(0 \leq \beta < 1\), where \(\beta = \left(\frac{kr(a+c)-1}{1-kr^a}\right)\), it follows that \(\{gJx_1, fIx_2, gJx_3, fIx_4, \ldots, gJx_{2n-1}, fIx_{2n}, gJx_{2n+1}, \ldots\}\) is a Cauchy sequence in \(X\). Since \((X, d)\) is a complete metric space, let \(\lim_{n \to \infty} gJx_{2n+1} = \lim_{n \to \infty} fIx_{2n} = z\).

Now, we will prove that \(fIz \in Sz\), that is, \(z\) is a coincidence point of \(fI\) and \(S\).

For every \(n \in N\), we have
\[
d\left( fI(gJx_{2n+1}), Sz \right) \leq d\left( fI(gJx_{2n+1}), S(fIx_{2n}) \right) + H\left( S(fIx_{2n}), Sz \right) \tag{7.2.1.3}
\]
It follows from the \(H\)-continuity of \(S\) that
\[
\lim_{n \to \infty} H\left( S(fIx_{2n}), Sz \right) = 0, \tag{7.2.1.4}
\]
since \(fIx_{2n} \to z\) as \(n \to \infty\).

Since the pair \((fI, S)\) are compatible mappings and \(\lim_{n \to \infty} fIz_n = \lim_{n \to \infty} y_n = z\), where \(y_n = gJx_{2n+1} \in Sx_{2n}\) and \(z_n = x_{2n}\), we have
\[
\lim_{n \to \infty} d(fIy_n, SfIz_n) = \lim_{n \to \infty} d\left( fI(gJx_{2n+1}), SfIx_{2n} \right) = 0. \tag{7.2.1.5}
\]
Thus from (7.2.1.3), (7.2.1.4) and (7.2.1.5), we have \(\lim_{n \to \infty} d(fI(gJx_{2n+1}), Sz) = 0\) and so, from \(d(fIz, Sz) \leq d\left( fIz, fI(gJx_{2n+1}) \right) + d\left( fI(gJx_{2n+1}), Sz \right)\) and the continuity of \(fI\), it follows that \(d(fIz, Sz) = 0\), which implies that \(fIz \in Sz\) as \(Sz\) is a closed subset of \(X\). Similarly, we can prove that \(gJz \in Tz\), that is, \(z\) is a coincidence point of \(gJ\) and \(T\).
For proving (b), assume that \( fIx \neq (fI)^2x \) which implies that \( fIx \notin Sx \), we deduce that \( fIx = (fI)^2x \in fI(Sx) = S(fIx) \), which is always possible in view of Lemma 1.4.1. Assuming that \( fIx \in Sx \) implies that \( (fI)^nx \to z \) for some \( z \) in \( X \), it is straightforward to note that \( fIz = z \) by continuity of \( fI \). We assert that \( (fI)^nx \in S(fI)^{n-1}x \) for each \( n \). To see this let, \( (fI)^2x = fI(fIx) \in fI(Sx) = S(fIx) \). Also \( (fI)^3x = fI((fI)^2x) \in fI(S(fIx)) = S((fI)^2x) \). Repeating this argument, one inductively obtains \( (fI)^nx \in S((fI)^{n-1}x) \) which together with the continuity of \( S \) gives.

\[
d(z, Sz) \leq d(z, (fI)^nx) + d((fI)^nx, Sz) \leq d(z, (fI)^nx) + H(S(fI)^{n-1}x, Sz) \to 0,
\]
i.e., \( z \in Sz \) as \( Sz \) is closed. Hence \( z \) is a common fixed point of the pair \((fI, S)\). Similar argument shows the existence of common fixed point for the pair \((gJ, T)\).

For proving (c), let us write

\[
\begin{align*}
  fz &= f(fIz) = f(I(fz)) = I(fIz) = fIz = z \\
  Ix &= I(fIx) = I(fIz) = fIz = z.
\end{align*}
\]

which show that \( z \) is a common fixed point of \( f, I, fI \) and \( S \). Similarly it can be shown that \( z \) is also a common fixed point of \( g, J, gJ \) and \( T \).

**Corollary 7.2.1.** By restricting \( f, g, I, J, S \) and \( T \) suitably and modifying the conditions (i) and (ii) accordingly, the derived conclusions corresponding to (a), (b) and (c) of Theorem 7.2.1, remain true if we replace contraction condition 7.2.1.1 by any one of the followings:

\[
(A) \quad H^p(Sx, Sy) \leq \frac{cd(fIx, Sx)d^p(gJy, Sy) + bd(fIx, Sy)d^p(gJy, Sx)}{\delta(fIx, Sx) + \delta(gJy, Sy)} + ad^p(gJy, Sy),
\]

where \( \delta(fIx, Sx) + \delta(gJy, Sy) \neq 0 \). (obtained by setting \( S = T \))
where $\delta(f I x, S x) + \delta(f I y, T y) \neq 0$. (obtained by setting $f = g$ and $I = J$)

\[(C) \quad H^p(S x, T y) \leq \frac{c d(f x, S x) d^p(f y, T y)}{\delta(f x, S x) + \delta(f y, T y)} + \alpha d^p(f y, T y),\]

where $\delta(f x, S x) + \delta(g y, T y) \neq 0$. (obtained by setting $I = J = I_x$)

\[(D) \quad H^p(S x, T y) \leq \frac{c d(f I x, S x) d^p(g y, T y) + b d(f I x, T y) d^p(g y, S x)}{\delta(f I x, S x) + \delta(g y, T y)} + \alpha d^p(g y, T y),\]

where $\delta(f I x, S x) + \delta(g y, T y) \neq 0$. (obtained by setting $J = I_x$)

\[(E) \quad H^p(S x, T y) \leq \frac{c d(x, S x) d^p(y, T y) + b d(x, T y) d^p(y, S x)}{\delta(x, S x) + \delta(y, T y)} + \alpha d^p(y, T y),\]

where $\delta(x, S x) + \delta(y, T y) \neq 0$. (obtained by setting $f = g = I = J = I_x$)

\[(F) \quad H^p(S x, T y) \leq \frac{c d(f x, S x) d^p(f y, S y) + b d(f x, S y) d^p(f y, S x)}{\delta(f x, S x) + \delta(f y, S y)} + \alpha d^p(f y, T y),\]

where $\delta(f x, S x) + \delta(f y, S y) \neq 0$. (obtained by setting $f = g, S = T$ and $I = J = I_x$)

**Remark 7.2.1.** (i) Corollary 7.2.1 corresponding to contraction conditions (A) to (F) presents a multitude of known and unknown hybrid fixed point theorems. Particularly, Corollary 7.2.1 corresponding to contraction condition (C) presents an improved version of Theorem 2.1 of Cho et al. [13] which in turn generalizes a result due to Popa [74].
(ii) if we set \( p = 1 \) in Theorem 7.2.1 and in Corollary 7.2.1, we can derive analogous results which are still new and present sharpened versions of some known results especially those contained in Cho et al. [13] and Popa [74]. Apart from known results some unknown results can also be derived.

§ 7.3. Fixed point theorems for single-valued mappings

In this section, using Theorem 7.2.1, we derive some common fixed point theorems for single-valued mappings in a metric space. For this let \( S \) and \( T \) denote the single-valued self-mappings of a metric space \((X,d)\) in Theorem 7.2.1, then we have the following:

**Theorem 7.3.1** Let \( f, g, I, J, S, T \) be continuous self-mappings of a complete metric space \((X,d)\) such that the pairs \((fI, S)\) and \((gJ, T)\) are compatible. If \( S(X) \subset gJ(X), T(X) \subset fI(X) \) and for all \( x, y \in X \), either

\[
\begin{align*}
d^p(Sx, Ty) \leq & \frac{c d(fIx, Sx)d^p(gJy, Ty) + bd(fIx, Ty)d^p(gJy, Sx)}{d(fIx, Sx) + d(gJy, Ty)} \\
& + ad^p(gJy, Ty), \quad (7.3.1.1)
\end{align*}
\]

when \( d(fIx, Sx) + d(gJy, Ty) \neq 0 \), where \( p \geq 1, b \geq 0, c > 0 \) and \( 1 < (c + 2a) < 2 \), or

\( d(Sx, Ty) = 0 \), if \( d(fIx, Sx) + d(gJy, Ty) = 0 \). \quad (7.3.1.2)

Then \( fI, gJ, S \) and \( T \) have a unique common fixed point \( z \) in \( X \). Moreover, \( z \) is the unique common fixed point of both the pairs \((fI, S)\) and \((gJ, T)\) separately.

Further, if the pairs \((f, I), (fI, f), (f, S), (S, I), (g, J), (gJ, g), (g, T) \) and \((T, J)\) commute at the points of coincidence, then \( z \) remains the unique common fixed point of \( f, I, S, T, g \) and \( J \).
Proof. The existence of the point \( v \) with \( fIv = Sv \) and \( gJv = Tv \) is ensured by Theorem 7.2.1. From (7.3.1.2) since \( d(fIv, Sv) + d(gJv, Tv) = 0 \), it follows that \( d(Sv, Tv) = 0 \) and so

\[
Sv = fIv = gJv = Tv
\]

Since the pair \((fI, S)\) are compatible and \( fIv = Sv \), by Lemma 1.5.1, we have

\[
fI(Sv) = SSv = S(fIv) = fI(fIv), \tag{7.3.1.3}
\]

which implies that \( d(fI(Sv), SSv) + d(gJv, Tv) = 0 \), which on using (7.3.1.2) yields to \( d(SSv, Tv) = 0 \), giving \( SSv = Tv \), and we obtain

\[
S(fIv) = SSv = Tv = fIv \tag{7.3.1.4}
\]

Therefore \( fIv = z \) is a fixed point of \( S \). Further, (7.3.1.3) and (7.3.1.4) implies that

\[
Sz = fIz = z.
\]

Similarly, we can show that

\[
Tz = gJz = z.
\]

Using (7.3.1.2), since \( d(fIz, Sz) + d(gJz, Tz) = 0 \), it follows that

\[
d(Sz, Tz) = 0 \quad \text{and so} \quad Sz = Tz.
\]

Therefore, the point \( z \) is a common fixed point of \( fI, gJ, S \) and \( T \). The rest of the proof is straightforward hence it is omitted.

Remark 7.3.1. By choosing \( f, g, I, J, S \) and \( T \) suitably one can derive analogous results for single-valued mappings as those contained in Corollary 7.2.1 some of these are new whereas, rest are partially improved versions of the results contained in Fisher [22, 23, 24], Diviccaro et al. [17], Cho et al. [13], and others.
As a sample we present the following corollary deduced by setting $p = 1$ and $f = g = I = J = S = T$ in Theorem 7.3.1.

**Corollary 7.3.1.** Let $f$ be a self-mapping of a complete metric space $(X, d)$ with $f(X) \subset f^2(X)$ such that either

$$d(fx, fy) \leq \frac{cd(f^2x, fx)d(f^2y, fy) + bd(f^2x, fy)d(f^2y, fx)}{d(f^2x, fx) + d(f^2y, fy)} + ad(f^2y, fy) \tag{7.3.1.1}$$

when $d(f^2x, fx) + d(f^2y, fy) \neq 0, b \geq 0, c > 0, 1 < (c + 2a) < 2$,

or $d(fx, fy) = 0$ if $d(f^2x, fx) + d(f^2y, fy) = 0$. Then $f$ has a unique fixed point.

§ 7.4. Some illustrative examples

In this section, we furnish some examples to discuss the validity of the hypotheses of theorems discussed in the preceding sections which also establish the genuineness of our extensions over the earlier related results in the literature.

**Example 7.4.1.** Let $X = \{1, 2, 3, 4\}$ be a finite set equipped with the metric $d$ defined as

$$d(1, 3) = d(1, 4) = d(2, 3) = d(2, 4) = d(3, 4) = 1,$$

and $d(1, 2) = 2$.

On $X$ define $f, g, I, J, S$ and $T$ as follows:

$$f1 = 1, f2 = 2, f3 = 4, f4 = 3, \quad g = f,$$

$$S1 = S2 = S4 = 2, S3 = 3,$$

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\[ T_1 = T_2 = T_3 = T_4 = 2, \quad I_1 = 1, I_2 = 2, I_3 = 3, I_4 = 4, \]
\[ J_1 = 1, J_2 = 2, J_3 = J_4 = 3. \]

Now from
\[ S(f I(1)) = S(f I) = S(1) = 2 = f_2 = f(I_2) = f I(S_1) \]
\[ S(f I(2)) = S(f_2) = S(2) = 2 = f_2 = f(I_2) = f I(S_2), \]
\[ d(S(f I_3), f I(S_3)) = d(S_4, f_4) = d(2, 3) = 1 = d(4, 3) = d(f I_3, S_3), \]
\[ d(S(f I_4), f I(S_4)) = d(S_3, f_2) = d(3, 2) = 1 = d(f I_4, S_4), \]
\[ f J(T_1) = T(f J_1), \quad f J(T_2) = T(f J_2), \quad f J(T_3) = T(f J_3), \quad f J(T_4) = T(f J_4), \]

It follows that the pairs \((f I, S)\) and \((f J, T)\) are respectively weakly commuting and commuting, hence compatible. Also the maps \(f, I, J, S\) and \(T\) are continuous and
\[ S(X) = \{2, 3\} \subset \{1, 2, 3, 4\} = f I(X), \quad T(X) = \{2\} \subset \{1, 2, 4\} = f J(X). \]

Now adopting Theorem 7.3.1 for \(p = 1\), a routine calculation verifies the contraction condition (7.3.1.1) if we choose \(c = 3/2, b = 2\) and \(a = 1/5\). Clearly, 2 is the unique common fixed point of \(f, I, J, S\) and \(T\).

However, our extension is genuine because on choosing \(x = 3\) and \(y = 4\) in contraction condition \(g\) (cf. Chapter-1, Section 1.2), of Cho et al. [13], we get \(1 \leq c/2\) which is not in lieu of \(1 < c < 2\).

Indeed Theorem 7.3.1 assures that \(f, I, S, T, g\) and \(J\) have a unique common fixed point in \(X\). However, either \(f\), or \(I\) or \(S\) or \(T\) or \(g\) or \(J\) may have other fixed points. One may note that the following component maps have more than one fixed point. For e.g. \(F(f) = \{1, 2\}, F(I) = \{1, 2, 3, 4\}, F(S) = \{2, 3\}, F(g) = \{1, 2\} \) and \(F(J) = \{1, 2, 3\}, \) where \(F(f)\) denotes the fixed point set of the map \(f\) and so on.
The following example exhibits that the condition of compatibility is necessary in Theorem 7.3.1.

**Example 7.4.2.** Let $X = [0,1]$ with the Euclidean metric $d(x,y) = |x - y|$. Set $f = g, S = T, I = J = i_X$ (the identity mapping on $X$) and define $f, S : X \to X$ by

$$Sx = \frac{1}{4} \quad \text{and} \quad fx = \frac{1}{2}x$$

for all $x \in X$. Note that $f$ and $S$ are continuous and $S(X) = \{\frac{1}{4}\} \subset [0, \frac{1}{2}] = f(X)$. Since $d(Sx, Sy) = 0$ for all $x, y \in X$, all the conditions of Theorem 7.3.1 are satisfied except the compatibility of $f$ and $S$. In fact, let $\{x_n\}$ be a sequence in $X$ defined by $x_n = \frac{1}{2}$ for $n = 1, 2, \ldots$. Then we have

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} \frac{1}{2}x_n = \frac{1}{4}, \quad \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} \frac{1}{4} = \frac{1}{4}$$

but

$$\lim_{n \to \infty} d(Sfx_n, fSx_n) = \lim_{n \to \infty} \left| \frac{1}{4} - \frac{1}{8} \right| = \frac{1}{8}.$$ 

Thus $f$ and $S$ are not compatible mappings. But $f$ and $S$ have no common fixed points in $X$.

Finally, one may note that the condition (7.3.1.1) (cf. Corollary 7.3.1) is not always a contraction as it needs $f(X) \subset f^2(X)$. The following simple example illustrates the situation better.

**Example 7.4.3** Consider $X = [0, \infty)$ equipped with usual metric. Define $f(x) = 2x$. Then for any $x, y$ in $X$ ($x > y$), the following holds:

$$|2x - 2y| = d(fx, fy)$$

$$\leq c \frac{|4x - 2x|}{|4x - 2x| + |4y - 2y|} + b \frac{|4x - 2y|}{|4x - 2x| + |4y - 2y|} + a \frac{|2x - 4y|}{|4x - 2x| + |4y - 2y|} + a \frac{|4y - 2y|}{|4x - 2x| + |4y - 2y|}$$

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Since \( x > y \Rightarrow 4x > 4y > 2y \Rightarrow 4y - 2y \leq 4x - 2y, 2x - 4y \leq 2x - 2y \leq 4x - 2y \) and \( 4y - 2y \leq 4y - 2x \) and therefore

\[
|2x - 2y| \leq \frac{c |4x - 2x|}{|4x - 2x|} + \frac{b |4x - 2y|}{|4y - 2y|} + \frac{a |4y - 2x|}{|4y - 2y|}
\]

\[
\leq (a + b + c) |4x - 2y|
\]

\[
\leq (a + b + c) |4x - 4y + 2y|
\]

\[
\leq 2(a + b + c) |2x - 2y| + 2(a + b + c)y,
\]  
(7.3.2.1)

which shows that condition (7.3.2.1) is verified as \( b \geq 0, c > 0 \) and \( 1 < (c + 2a) < 2 \). Also ‘0’ is the unique fixed point of \( f \). Here, one may note that \( f \) is not a contraction in the usual sense.

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