CHAPTER 4

SOME COMMON FIXED POINT THEOREMS

‘Whenever science makes a discovery, the devil grabs it while the angels are debating the best way to use it’

(Alan Valentine)
CHAPTER 4

Some common fixed point theorems

§ 4.1. Introduction

The present chapter comprises of three sections: Section 4.1 offers a brief introduction to the contents of this chapter. In Section 4.2, we introduce the notion of compositely asymptotically regular (*abbreviated as c.a.r.*) maps and utilize it to prove a common fixed point theorem satisfying a relatively more general contraction condition. Our work generalizes some earlier results of Nesic [66], Guay and Singh [29], Sharma and Yuel [91] and several others. Section 4.3 opens with some basic definitions and a brief introduction on results on best approximation theory. Also, some important results, like the one by Sahab, Khan and Sessa [88], have been delineated there. Among the main features of this section is the well known result of Jungck [47](Theorem 4.3.1). An extension and unification of this important result is given in the form of Theorem 4.3.2. Finally, using Theorem 4.3.2, we prove our main result employing the notion of best approximation. Our results are in fact the generalizations and extensions of earlier known results of Brosowski [8], Singh [95], Hicks-Humphries [34], Sahab et al. [85] and others.

§ 4.2. Composite asymptotic regularity and common fixed points

In this section, we propose a generalization to the concept of asymptotically regular (*abbreviated as a.r.*) mapping (cf. Definition 1.5.1) by introducing the notion of compositely asymptotically regular (*abbreviated as c.a.r.*) mappings.

In doing so, we are motivated by those functions which are not a.r. but their composition is a.r. To substantiate this, let us consider the following example.

**Example 4.2.1.** Let $X = R$ be the set of reals equipped with usual metric. On $X$ define the pair of maps $(S, T)$ by

$$S(x) = x - 1 \text{ and } T(x) = x + 1,$$

for all $x \in R$. Then

$$\lim_{n \to \infty} d(S^n x, S^{n+1} x) = \lim_{n \to \infty} |x - n - x + (n + 1)| = 1,$$

whereas

$$\lim_{n \to \infty} d(T^n x, T^{n+1} x) = \lim_{n \to \infty} |x + n - x - (n + 1)| = 1,$$

which show that both the maps $S$ and $T$ are not a.r. But on taking their composition, we get $STx = x$ and hence, we deduce

$$\lim_{n \to \infty} d((ST)^n x, (ST)^{n+1} x) = \lim_{n \to \infty} |x - x| = 0,$$

which shows that the pair $(S, T)$ is c.a.r.

Thus it seems worthwhile to introduce the following:

**Definition 4.2.1.** A pair of self-mappings $(S, T)$ of a metric space $(X, d)$ is said to be c.a.r. at a point $x$ if their composition $SoT$ is a.r. at $x$.

It is immediate to note that if we choose $T = I_x$ (or $S = I_x$), where $I_x$ is the identity mapping on $X$, then the notion of c.a.r. mappings reduces to that of a.r. mapping.

Let $R^+$ denotes the set of non-negative real numbers, and let $F : R^+ \to R^+$ be a mapping such that $F(0) = 0$ and $F$ is continuous at 0.
Employing the notion of c.a.r. mappings, we first prove the following:

**Theorem 4.2.1.** Let $S$ and $T$ be self-mappings of a complete metric space $(X, d)$ satisfying

\[
d(STx, STy) \leq \alpha \max\{d(x, y), d(x, STx), d(y, STy), d(x, STy), d(y, STx)\} + F\left( \min\{d^2(x, y), d(x, STx) \cdot d(y, STy), d(x, STx) \cdot d(x, STy), d(y, STx) \cdot d(y, STy), d(x, STy) \cdot d(y, STx)\} \right),
\]

for all $x, y$ in $X$, where $0 < \alpha < 1$. Then $ST$ has a unique fixed point $z$ provided the pair $(S, T)$ is c.a.r. at some point of $X$. Moreover, if the pair $(S, T)$ commutes at $z$ and $Tz$, then the fixed point of $ST$ also remains the fixed point of $S$ and $T$ separately.

**Proof.** Let the pair $(S, T)$ be c.a.r. at $x_0$ in $X$, then using (4.2.1.1), we get

\[
d\left((ST)^m x_0, (ST)^n x_0\right) \leq \alpha \max\{d\left((ST)^{m-1} x_0, (ST)^{n-1} x_0\right), d\left((ST)^{m-1} x_0, (ST)^m x_0\right), d\left((ST)^n x_0, (ST)^m x_0\right)\}
\]

\[
+ F\left( \min\{d^2\left((ST)^{m-1} x_0, (ST)^{n-1} x_0\right), d\left((ST)^{m-1} x_0, (ST)^m x_0\right)\}\right)\cdot d\left((ST)^m x_0, (ST)^n x_0\right).
\]

Substituting

\[
d\left((ST)^{m-1} x_0, (ST)^{n-1} x_0\right) \leq d\left((ST)^{m-1} x_0, (ST)^m x_0\right) + d\left((ST)^m x_0, (ST)^n x_0\right)
\]

\[+ d\left((ST)^n x_0, (ST)^{m-1} x_0\right),
\]

\[
d\left((ST)^{m-1} x_0, (ST)^n x_0\right) \leq d\left((ST)^{m-1} x_0, (ST)^m x_0\right) + d\left((ST)^m x_0, (ST)^n x_0\right),
\]

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\[
d((ST)^{n-1}x_0, (ST)^{m}x_0) \leq d((ST)^{n-1}x_0, (ST)^{m}x_0) + d((ST)^n x_0, (ST)^m x_0),
\]
in (4.2.1.2) and using the composite asymptotic regularity of the pair \((S,T)\) at \(x_0\), we get as \(n,m \to \infty\)
\[
d((ST)^{m}x_0, (ST)^{n}x_0) \leq
\]
\[
\leq \alpha \max \{d((ST)^{m}x_0, (ST)^{n}x_0), 0, 0, d((ST)^{m}x_0, (ST)^{n}x_0), d((ST)^{m}x_0, (ST)^{n}x_0)\} + F\left(\min \left\{d^2((ST)^{m}x_0, (ST)^{n}x_0), 0, 0, d^2((ST)^{m}x_0, (ST)^{n}x_0)\right\}\right),
\]
which yields to \(d((ST)^{m}x_0, (ST)^{n}x_0) \leq \alpha d((ST)^{m}x_0, (ST)^{n}x_0) + F(0)\). Since \(\alpha < 1\) and \(F(0) = 0\), we get \(d((ST)^{m}x_0, (ST)^{n}x_0) \to 0\) as \(n,m \to \infty\). Hence \(\{(ST)^{n}x_0\}\) is a Cauchy sequence in \(X\) and so, since \((X,d)\) is complete, it converges to a point \(z\) in \(X\). Now using (4.2.1.1), we obtain
\[
d(z, STz) \leq
\]
\[
d\left(z, (ST)^{m}x_0\right) + d\left((ST)^{n}x_0, STz\right) \leq
\]
\[
\leq d\left(z, (ST)^{m}x_0\right) + \alpha \max \left\{d\left((ST)^{n-1}x_0, z\right), d\left((ST)^{n-1}x_0, (ST)^{m}x_0\right)\right\},
\]
d\((z, STz), d\left((ST)^{n-1}x_0, STz\right), d\left(z, (ST)^{n}x_0\right)\} + F\left(\min \left\{d^2\left((ST)^{n-1}x_0, z\right),
\right.\right.
\]
\[
d\left((ST)^{n-1}x_0, (ST)^{n}x_0\right) \cdot d(z, STz), d\left((ST)^{n-1}x_0, (ST)^{n}x_0\right) \cdot d\left((ST)^{n-1}x_0, STz\right)
\]
\[
d\left(z, (ST)^{m}x_0\right) \cdot d(z, STz), d\left((ST)^{n-1}x_0, STz\right) \cdot d\left(z, (ST)^{m}x_0\right)\} \right\}
\]
\[
\leq d\left(z, (ST)^{m}x_0\right) + \alpha \max \left\{d\left((ST)^{n-1}x_0, z\right), d\left((ST)^{n-1}x_0, (ST)^{m}x_0\right), d(z, STz)\right\}
\]
\[
\left[d\left((ST)^{n-1}x_0, z\right) + d(z, STz)\right], d\left(z, (ST)^{m}x_0\right)\} + F\left(\min \left\{d^2\left((ST)^{n-1}x_0, z\right),
\right.\right.
\]
\[
d\left((ST)^{n-1}x_0, (ST)^{m}x_0\right) \cdot d(z, STz), d\left((ST)^{n-1}x_0, (ST)^{m}x_0\right) \cdot d\left((ST)^{n-1}x_0, STz\right)
\]
\[
d\left(z, (ST)^{m}x_0\right) \cdot d(z, STz), \left[d\left((ST)^{n-1}x_0, z\right) + d(z, STz)\right], d\left(z, (ST)^{m}x_0\right)\} \right\}
\]
which on letting \(n,m \to \infty\), reduces to
\[
d(z, STz) \leq \alpha d(z, STz) < d(z, STz),
\]
a contradiction giving thereby \(z = STz\).
The uniqueness assertion follows immediately from contraction condition (4.2.1.1) of the hypotheses.

Now, it remains to show that \( z \) is also a common fixed point of \( S \) and \( T \) separately. For this let us write

\[
Sz = S(STz) = S(TSz) = ST(Sz),
\]

\[
Tz = T(STz) = TS(Tz) = ST(Tz),
\]

which show that \( Sz \) and \( Tz \) are other fixed points of \( ST \). Therefore, in view of the uniqueness of the fixed point of \( ST \), one can write

\[
Sz = Tz = STz = z
\]

which shows that \( z \) is the common fixed point of \( S, T \) and \( ST \).

**Remark 4.2.1.** By choosing \( S, T \) and \( F \) suitably, one can deduce earlier results from Nesic [66], Guay and Singh [29] and others.

**Theorem 4.2.2.** Let \((X, d)\) be a metric space and \( S \) and \( T \) be mappings of \( X \) into itself satisfying (4.2.1.1), where \( 0 < \alpha < 1 \). If the pair \((S, T)\) is c.a.r. at a point \( x \) in \( X \) and the sequence of iterates \( \{(ST)^nx\} \) has a subsequence converging to a point \( u \) in \( X \), then \( u \) is the unique fixed point of \( ST \) and \( \{(ST)^nx\} \) also converges to \( u \).

Moreover, if the pair \((S, T)\) commutes at \( u \) and \( Tu \), then the fixed point of \( ST \) also remains the fixed point of \( S \) and \( T \) separately.

**Proof.** Let the pair \((S, T)\) be c.a.r. at some point \( x \) of \( X \) and consider the sequence \( \{(ST)^nx\} \). Suppose that \( \lim k(\{ST\}^nx) = u \) and \( STu \neq u \).
By (4.2.1.1), we obtain
\[ d(u, STu) \leq \]
\[ \leq d\left( u, (ST)^n x \right) + d\left( (ST)^n x, (ST)^{n+1} x \right) + d\left( (ST)^{n+1} x, STu \right) \leq \]
\[ \leq d\left( u, (ST)^n x \right) + d\left( (ST)^n x, (ST)^{n+1} x \right) + \alpha \max \left\{ d\left( (ST)^n x, u \right), \right. \]
\[ d\left( (ST)^n x, (ST)^{n+1} x \right), d(u, STu), d\left( (ST)^n x, STu \right), d\left( u, (ST)^{n+1} x \right) \left\} \right., \]
\[ + F \left( \min \left\{ d^2 \left( (ST)^n x, u \right), d\left( (ST)^n x, (ST)^{n+1} x \right) \cdot d(u, STu), \right. \right. \]
\[ d\left( (ST)^n x, (ST)^{n+1} x \right) \cdot d\left( (ST)^n x, STu \right), d\left( u, (ST)^{n+1} x \right) \cdot d(u, STu), \]
\[ d\left( (ST)^n x, STu \right) \cdot d\left( u, (ST)^{n+1} x \right) \left\} \right. \]
which on letting \( k \to \infty \), reduces to
\[ d(u, STu) \leq \alpha \max \{ 0, 0, d(u, STu), d(u, STu), 0 \} + F(0), \]
since \( \alpha \in [0,1) \) and \( F(0) = 0 \), we get \( d(u, STu) \leq \alpha d(u, STu) < d(u, STu) \), a contradiction, giving thereby \( STu = u \).

Now
\[ d\left( u, (ST)^n x \right) = d\left( STu, (ST)^n x \right) \leq d\left( STu, (ST)^{n+1} x \right) + d\left( (ST)^{n+1} x, (ST)^n x \right). \]
Since the pair \((S, T)\) is c.a.r. using (4.2.1.1), \( STu = u \) and letting \( n \to \infty \), we obtain
\[ d\left( u, (ST)^n x \right) \leq \alpha \max \{ d\left( u, (ST)^n x \right), 0, 0, d\left( u, (ST)^n x \right), d\left( u, (ST)^n x \right) \} + F(0). \]
\[ d\left( u, (ST)^n x \right) \leq \alpha d\left( u, (ST)^n x \right) + F(0). \]
Since \( \alpha < 1 \) and \( F(0) = 0 \), we get \( d\left( u, (ST)^n x \right) \to 0 \) as \( n \to \infty \). Consequently, \( \{(ST)^n x\} \) converges to \( u \). The remaining part follows from Theorem 4.2.1.
The following theorem regarding $ST$ is predictable.

**Theorem 4.2.3.** Let $S$ and $T$ be self-mappings of a metric space $(X, d)$ such that $ST$ is continuous. Then the following conclusions hold:

(a) If a sequence $\{x_n\}$ in $X$ converges to a fixed point $z$ of $ST$, then $\{x_n\}$ is asymptotically $ST$-regular.

(b) If $\{x_n\}$ be a sequence in $X$ admitting a subsequence $\{x_{n_k}\}$ with $\lim x_{n_k} = z$ and $\lim d(STx_{n_k}, x_{n_k}) = 0$, then $z$ is a fixed point of $ST$. If the pair $(S, T)$ commutes at $z$ and $Tz$, then $Sz$ and $Tz$ also remains the fixed point of $ST$.

**Proof.** The proof is straightforward, hence it is omitted.

§ 4.3. Best approximation and common fixed points

In what follows, we consider a selfmap $T$ of a normed linear space $X$, a subset $C$ of $X$, and $\bar{x} \in X$. Let $F(I, T)$ denotes the set of common fixed point of $I$ and $T$ whereas $F(A, B, S, T, I, J)$ denotes the set of common fixed point of the mappings $A, B, S, T, I, J$, let $D$ be the set of $C$-approximants to $\bar{x}$ and $\partial C$ denotes the boundary of $C$ in $X$.

Brosowski [8] proved that if $T$ is nonexpansive with $\bar{x} \in F(T), T(C) \subset C$ and $D$ is nonempty, compact convex, then $T$ has a fixed point in $D = B_C(\bar{x})$. Subrahmanyam [100] substituted the nonempty requirement of $B_C(\bar{x})$ with the finite dimensionality of $C$ (as a subspace of $X$). Singh [95], relaxing the linearity of the operator $T$ and the convexity of $D$ in the result of Brosowski [8], proved the following:

**Theorem A.** Let $T : X \to X$ be a contractive operator on $X$. Let $C$ be a $T$-invariant subset of $X$ and let $\bar{x} \in F(T)$. If $D \subseteq X$ is non-empty, compact, and starshaped, then $D \cap F(T) \neq \emptyset$. 

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Later, he observed that only the nonexpansiveness of $T$ on $D' = D \cup \{\bar{x}\}$ is enough for his earlier result. Hicks and Humphries [34] noted that Singh’s earlier result (Theorem A) remains true if $T(C) \subset C$ is replaced by $T(\partial C) \subset C$. Smoluk [98] noted that the finite dimensionality of $C$ in Subrahmanyam’s result can be replaced by the assumptions that $T$ is linear and $T(D)$ is compact for every bounded subset $D$ of $C$. Habiniak [31] removed the linearity of $T$ from Smoluk’s result whereas, Sahab et al. [85] improved and generalized the results of Hicks and Humphries [34] and the results of Singh [95, 96] by proving the following:

**Theorem B.** Let $X$ be a normed space, $I$ and $T$ self-maps of $X$ with $\bar{x} \in F(T, I), C \subset X$ with $T(\partial C) \subset C$, and $q \in F(I)$. If $D = B_{C}(\bar{x})$ is compact and $q$-starshaped, $I(D) = D, I$ is continuous and linear on $D, I$ and $T$ are commuting on $D$ and $T$ is $I$-nonexpansive on $D \cup \{\bar{x}\}$, then $I$ and $T$ have a common fixed point in $D$.

Since then several interesting results have been given in approximation theory using the fixed point theory by many researchers. Applications of the fixed point theorems to simultaneous best approximation were given by several authors (for instance see Sahney and Singh [86]). For further references and a survey of the subject, we refer to Brosowski [8] and Cheney [12].

In this section using a fixed point theorem of Jungck [41], we first derive a common fixed point theorem in compact metric spaces involving six mappings, which is then used to prove yet another extension of Theorem B employing the notion of best approximation. But before proving our results, first recall the following basic definitions:

In an attempt to generalize the notion of convex sets Doston [18] introduced
the notion of starshaped set which asserts that a subset $C$ of a normed linear space $X$ is said to be starshaped with respect to a point $q \in C$ if for all $x \in C$ and all $\lambda, 0 \leq \lambda \leq 1$, $\lambda x + (1 - \lambda)q \in C$. Clearly, a convex set is starshaped with respect to each of its points but the converse is not always true.

The set of all best approximants to $\bar{x}$ is denoted by

$$D(\bar{x}, C) = \{y \in C : \| \bar{x} - y \| = d(\bar{x}, C)\} = B_C(\bar{x})$$

A point $x \in C$, $C$ is a subset of a normed linear space $X$, is said to be a best approximation for $fx$, where $f : C \to X$, if for $x \in C$

$$\| x - fx \| = d(fx, C) = \inf \{\| fx - y \| : y \in C\}$$

The following theorem is due to Jungck [47].

**Theorem 4.3.1.** Let $A, S, I$ and $J$ be continuous self-mappings of a compact metric space $(X, d)$ with $A(X) \subset J(X)$ and $S(X) \subset I(X)$. If $(A, I)$ and $(S, J)$ are compatible and satisfy

$$d(Ax, Sy) < M(x, y),$$

where

$$M(x, y) = \max\{d(Ix, Jy), d(Ix, Ax), d(Jy, Sy), 1/2[d(Ix, Sy) + d(Jy, Ax)]\},$$

for all $x, y \in X$, with $M(x, y) > 0$, then $A, S, I$ and $J$ have a unique common fixed point.

First of all, as an application of Theorem 4.3.1, we derive a common fixed point theorem for six self-mappings, as follows:

**Theorem 4.3.2.** Let $A, B, S, T, I$ and $J$ be self-mappings of a compact metric space $(X, d)$ such that $A(X) \subset TJ(X)$, $S(X) \subset BI(X)$ with $A, S, TJ$ and $BI$ being continuous. If $(A, BI)$ and $(S, TJ)$ are compatible and satisfy

$$d(Ax, Sy) < M(x, y),$$
where
\[ M(x,y) = \max \{d(BIx,TJy),d(BIx,Ax),d(TJy,Sy)\} \frac{1}{4} \left[ d(BIx,Sy) + d(TJy,Ax) \right] \]
for all \( x, y \in X \) with \( M(x,y) > 0 \), then \( A, S, BI \) and \( TJ \) have a unique common fixed point \( z \) in \( X \). Moreover, if the pairs \( (B,I), (IB,I), (T,J), (JT,J), (A,B), (A,I), (S,T) \) and \( (S,J) \) commute at the fixed point \( z \), then \( z \) remains the unique common fixed point of \( A, B, S, T, I \) and \( J \) separately.

**Proof.** We begin by noting that the continuity of \( BI \) (resp. \( TJ \)) does not demand the continuity of \( B \) or \( I \) or both (resp. \( T \) or \( J \) or both). But for maps \( A, S, BI \) and \( TJ \) all the conditions of Theorem 4.3.1 are satisfied ensuring the existence of unique common fixed point \( z \) of \( A, S, BI \) and \( TJ \). Here it is worth noting that \( z \) is the common fixed point of both the pairs \( (A, BI) \) and \( (S, TJ) \) respectively.

Now it remains to show that \( z \) is also a common fixed point of \( A, B, S, T, I \) and \( J \). For this let \( z \) is the unique common fixed point of both the pairs \( (A, BI) \) and \( (S, TJ) \), then
\[ Bz = B(BIz) = B(IBz) = BI(Bz), \quad Bz = B(Az) = A(Bz), \]
\[ Iz = I(BIz) = IB(Iz) = BI(Iz), \quad Iz = I(Az) = A(Iz), \]
\[ Tz = T(TJz) = T(JTz) = TJ(Tz), \quad Tz = T(Sz) = S(Tz), \]
\[ Jz = J(TJz) = JT(Jz) = TJ(Jz), \quad Jz = J(Sz) = S(Jz), \]
which show that \( Bz \) and \( Iz \) (resp. \( Tz \) and \( Jz \)) are other fixed points of the pair \( (A, BI) \) (resp. \( (S, TJ) \)). Now in view of the uniqueness of common fixed point of the pairs \( (A, BI) \) and \( (S, TJ) \), we get
\[ z = Bz = Iz = Tz = Jz = BIz = TJz = Az = Sz, \]
which shows that \( z \) also remains the common fixed point of \( A, B, S, T, I \) and \( J \) separately. This evidently completes the proof.
**Remark 4.3.1.** By restricting $A, B, S, T, I$ and $J$ suitably and modifying the remaining hypotheses accordingly, one can derive a multitude of known and unknown fixed point theorems. So far we are not familiar of any fixed point theorem involving five or six mappings in compact metric spaces.

Following the lines of Popa [76] and Theorem 4.3.2 one can predict the following:

**Theorem 4.3.3.** The conclusions of Theorem 4.3.2 remain true if we replace the condition (4.3.2.1) by the following (retaining the rest of the hypotheses and notations):

$$d(Ax, Sy) < M(x, y),$$

where

$$M(x, y) = \max \{d(BIx, TJy), 1/2[d(BIx, Ax) + d(TJy, Sy)], 1/2[d(BIx, Sy) + d(TJy, Ax)]\}.$$ 

As an application of Theorem 4.3.2, we now prove our main result (employing the notion of best approximation) which generalizes earlier results due to Brosowski [8], Hicks-Humphries [34], Singh [95], Sahab et al. [85] and others.

**Theorem 4.3.4.** Let $A, B, S, T, I$ and $J$ be self-mappings of a normed space $X$ and $C$ be a subset of $X$ such that $A, S : \partial C \to C$ with $\bar{x} \in F(A, B, S, T, I, J)$. Let $A, B, S, T, I$ and $J$ satisfy the condition

$$\| Ax - Sy \| < M(x, y),$$

with $A$ and $S$ being continuous where $BC(\bar{x}) = D$ and

$$M(x, y) = \max \left\{ \| BIx - TJy \|, 1/2[\| BIx - Ax \| + \| TJy - Sy \|], 1/2[\| BIx - Sy \| + \| TJy - Ax \|]\right\},$$

for all $x, y \in D' = D \cup \{\bar{x}\}$.

Further, suppose that the pairs $(A, BI)$ and $(S, TJ)$ are compatible with $BI$ and $TJ$ being linear and continuous on $D$. If $D$ be a nonempty, compact and starshaped
with respect to a point \( q \in F(BI, TJ) \) and \( BI(D) = D = TJ(D) \), then

\[ D \cap F(A, B, S, T, I, J) \neq \emptyset, \]

provided the pairs \( (B, I), (IB, I), (T, J), (JT, J), (A, B), (S, T), (A, I) \) and \( (S, J) \) commute at the common fixed point of \( BI, TJ, A \) and \( S \).

**Proof.** Let \( y \in D \), then \( BIy \in D \) as \( BI(D) = D \). Also if \( y \in \partial C \) then \( Ay \in C \) as \( A(\partial C) \subset C \).

Using condition (4.3.4.1), we obtain

\[ \| Ay - \bar{x} \| = \| Ay - S\bar{x} \| < M(y, \bar{x}), \]

giving thereby \( Ay \in D \). Thus \( A \) is a self-mapping of \( D \). Similarly \( S \) is also a self-mapping of \( D \).

Let \( \{ t_n \} \) be a sequence of real number such that \( 0 < t_n < 1 \) and converging to \( '1' \). Define sequences \( \{ A_n \} \) and \( \{ S_n \} \) of mappings by

\[ A_n x = t_n Ax + (1 - t_n)q \]
\[ S_n x = t_n Sx + (1 - t_n)q, \]

for all \( x \in D \) and for each \( n \).

Since \( D \) is starshaped with respect to \( q \) hence \( \{ A_n \} \) maps \( D \) into itself and also so does \( \{ S_n \} \). Since \( BI \) is linear and \( q = BIq \) (due to \( q \in F(BI, TJ) \)), one can have

\[ \left(\begin{array}{c} A_n(BI) \\ \end{array}\right) x_m = t_n \left(\begin{array}{c} A(BI) \\ \end{array}\right) x_m + (1 - t_n)q, \]

and

\[ \left(\begin{array}{c} (BI)A_n \\ \end{array}\right) x_m = t_n \left(\begin{array}{c} (BI)A \\ \end{array}\right) x_m + (1 - t_n)BIq. \]

Since \( (A, BI) \) are compatible, therefore making \( m \rightarrow \infty \) (keeping \( n \) fixed), we get

\[ 0 \leq \lim_{m \rightarrow \infty} \| (BI)A_n x_m - A_n(BI)x_m \| \leq \lim_{m \rightarrow \infty} \left[ \| (BI)Ax_m - A(BI)x_m \| + (1 - t_n) \| q - BIq \| \right] = 0, \]

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whenever \( \lim_{m \to \infty} Ax_m = \lim_{m \to \infty} BI x_m = t \in D \), for all \( m \). Hence \( (BI, A_n) \) are compatible on \( D \). Similarly it can be shown that \( (TJ, S_n) \) are compatible on \( D \).

Further from (4.3.4.1), we have

\[
\| A_n x - S_n y \| = t_n \| Ax - Sy \| < t_n M(x, y) < M(x, y),
\]

for all \( x, y \in D \). Since \( BI \) and \( TJ \) are continuous and \( D \) is compact, therefore by Theorem 4.3.2 one gets

\[
F(A_n) \cap F(BI) \cap F(S_n) \cap F(TJ) = \{ x_n \},
\]

for each \( n \). Also, since \( D \) is compact, \( \{ x_n \} \) has a convergent subsequence \( \{ x_{n_i} \} \) converging to \( z \) in \( D \). Since

\[
x_{n_i} = A_{n_i} x_{n_i} = t_{n_i} A x_{n_i} + (1 - t_{n_i}) q,
\]

and \( A \) is continuous, we have, as \( i \to \infty \) that \( Az = z \), giving thereby \( z \in D \cap F(A) \). Similarly, it can be shown that \( z \in D \cap F(S) \). Since \( BI \) and \( TJ \) are continuous, we have

\[
BIz = BI \lim_{i \to \infty} x_{n_i} = \lim_{i \to \infty} BI x_{n_i} = \lim_{i \to \infty} x_{n_i} = z,
\]

\[
TJz = TJ \lim_{i \to \infty} x_{n_i} = \lim_{i \to \infty} TJ x_{n_i} = \lim_{i \to \infty} x_{n_i} = z,
\]

yielding thereby \( BIz = TJz = Az = S z = z \).

Let the pairs \( (A, BI) \) and \( (S, TJ) \) have different fixed point \( u \) and \( v \) respectively, then

\[
\| u - v \| = \| Au - Sv \| < \max \left\{ \| BI u - TJ v \|, 1/2[ \| BI u - Au \| + \| TJ v - Sv \|], \right. \\
\left. 1/2 [ \| BI u - Sv \| + \| TJ v - Au \|] \right\},
\]

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which is a contradiction, implying thereby $u = v$, thus both the pairs have the same common unique fixed point $u = v = z$.

Now on the lines of the proof of Theorem 4.3.2 it can be easily shown that $z$ remains the unique common fixed point of $A, B, S, T, I$ and $J$ separately.

Hence, we conclude that

$$D \cap F(A, B, S, T, I, J) \neq \emptyset.$$  

This completes the proof.

Remark 4.3.2. (i) Theorem 4.3.4 extends the result of Sahab et al. [85] as we use generalized contractions along with compatibility (cf. [46]) instead of commutativity (cf. [88]). Also, Theorem 4.3.4 involves six mappings instead of two mappings. In process, related results due to Hicks-Humphries [34], Singh [95], Brosowski [8] and others are modified and improved either partially or completely.

(ii) If we use a fixed point theorem in complete spaces corresponding to Theorem 4.3.3 then the continuity requirement of any one of the maps $A, S, BI$ or $TJ$ can serve the purpose which is possible due to the fact that compact metric spaces are always complete. But due to a shorter proof we opt to utilize Theorem 4.3.3.

Remark 4.3.3. We conclude our discussion by observing that any common fixed point theorem for four mappings can be used to prove common fixed point theorems for six, eight or any finite numbers of mappings. For details, we refer to Imdad [36].