CHAPTER 1
PRELIMINARIES

1.1 INTRODUCTION

Theory of special functions plays an important role in mathematical physics. These functions commonly arise in such areas of applications as heat conduction, communication systems, electro-optics, non-linear wave propagation, electromagnetic theory, quantum mechanics, approximation theory, probability theory and electrocircuit theory, among others (see Luke [53], Morse and Feshback [59]).

Special functions are sometimes discussed in certain engineering and physics courses, as we can find in Andrews [4], Bell [11] and mathematics courses like partial differential equations [14].

The advent of large, fast and sophisticated computing machines did not diminish the importance of special functions within the context of applied sciences. They provide a unique tool for developing simplified yet realistic models of physical problems, thus allowing for analytic solutions and hence a deeper insight into the problem under study. A vast mathematical literature has been devoted to the theory of these functions as constructed in the works of Euler, Gauss, Legendre, Hermite, Riemann, Hardy, Littlewood, Ramanujan and other classical authors, for example, Erdélyi, A. et al [24],[25],[26],[27] and [28], McBride [56] Szegô [100], Watson [104], Bailey [9], Bell [11], Exton [30],[31], Braun [14], Grosswald [37] and Slater [79].

As soon as the group concepts and methods had penetrated into physics, it became natural and desirable to present in the same way special functions employed in the description of physical models. This theory appeared in mathematics in the work of Miller [57],[58], Sasaki [75], Vilenkin [103], Wawrzynczyk [106] Aleksander [3], Askey [8], Talman [101] and Wigner [108].
Brief historical synopsis can be found in Aksenov [2], Chihara [22], Klein [46], Lavrent’ev and Shabat [51], Watson and Whittaker [105].

Each special function can be defined in a variety of ways and different researchers may choose different definitions (Rodrigues formula, generating functions etc). Most of the special functions have a common root in their relation to the hypergeometric function.

The aim of the present chapter is to introduce the several classes of special functions which occur rather more frequently in the study of Gaussian hypergeometric series and needed for presentation of the subsequent chapters. First, we recall some definitions and important properties of such elementary functions as Gamma and Beta functions and related functions and then proceed to the hypergeometric functions in one, two and more variables. We present definitions of the classical orthogonal polynomials and some hypergeometric representations of polynomials. A concept of generating function and their classification is also given in this chapter.

The Gamma Function

The familiar factorial function \( n! \) is well defined when \( n \) is a positive integer. In an attempt to give a meaning to \( \tau! \) when \( \tau \) is any positive number, Euler in 1729, undertook the problem of interpolating \( n! \) between the positive integral values of \( n \). He was thus led to what is now well known as the gamma function which is encountered fairly frequently in the study of special functions. It has several equivalent definitions.

We define the gamma function due to Euler by

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \, dt, \quad \text{Re}(z) > 0
\]
Upon integration by parts, definition (1.1.1) yields the recurrence relation for $\Gamma(z)$:

$$\Gamma(z + 1) = z\Gamma(z) \quad (1.1.2)$$

From the relations (1.1.1) and (1.1.2) it follows that:

$$\Gamma(z) = \begin{cases} 
\int_0^\infty e^{-t} t^{z-1} dt, & \text{Re}(z) > 0 \\
\frac{\Gamma(z + 1)}{z}, & \text{Re}(z) < 0, \ z \neq 0, -1, -2, -3, \ldots
\end{cases} \quad (1.1.3)$$

By repeated applications of the recurrence relation (1.1.2), it follows that

$$\Gamma(n + 1) = n! \quad (1.1.4)$$

The Beta Function

The Beta function $B(\alpha, \beta)$ is a function of two complex variables $\alpha$ and $\beta$, defined by the Eulerian integral of the first kind

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1 - t)^{\beta-1} dt, \ \text{Re}(\alpha) > 0, \ \text{Re}(\beta) > 0 \quad (1.1.5)$$

The Beta function is closely related to the Gamma function; in fact, we have

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}, \ \alpha, \beta \neq 0, -1, -2, \ldots \quad (1.1.6)$$

We may write by analogy with (1.1.3)
\[ B(\alpha, \beta) = \begin{cases} \int_0^\infty t^{\alpha-1} (1-t)^{\beta-1} dt, & \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0 \\ \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}, & \text{Re}(\alpha) < 0, \text{Re}(\beta) < 0, \alpha, \beta \neq 0, -1, -2, -3, \ldots \end{cases} \]

Pochhammer’s Symbol and the Factorial Function

Throughout this work we shall find it convenient to employ the Pochhammer symbol \((\lambda)_n\) defined by

\[(\lambda)_n = \begin{cases} 1, & \text{if } n = 0 \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & \text{if } n = 1, 2, 3, \ldots \end{cases} \quad (1.1.8)\]

Since \((1)_n = n!\), \((\lambda)_n\) may be looked upon as a generalization of the elementary factorial; hence the symbol \((\lambda)_n\) is also referred to as the factorial function.

In terms of Gamma functions, we have

\[(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}, \quad \lambda \neq 0, -1, -2, \ldots \quad (1.1.9)\]

which can be easily verified. Furthermore, the binomial coefficient may now be expressed as

\[
\binom{\lambda}{n} = \frac{\lambda(\lambda - 1) \cdots (\lambda - n + 1)}{n!} = \frac{(-1)^n(\lambda)_n}{n!} \quad (1.1.10)
\]

or, equivalently as

\[
\binom{\lambda}{n} = \frac{\Gamma(\lambda + 1)}{n! \Gamma(\lambda - n + 1)} \quad (1.1.11)
\]
It follows from (1.1.10) and (1.1.11) that

$$\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - n + 1)} = (-1)^n (-\lambda)_n$$  \hspace{1cm} (1.1.12)

which for \( \lambda = \alpha - 1 \), yields

$$\frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1 - \alpha)_n}, \quad \alpha \neq 0, \pm 1, \pm 2, \ldots$$  \hspace{1cm} (1.1.13)

Equations (1.1.9) and (1.1.13) suggest the definition

$$(-\lambda)_n = \frac{(-1)^n}{(1 - \lambda)_n}, \quad n = 1, 2, 3, \ldots; \quad \lambda \neq 0, \pm 1, \pm 2, \ldots$$  \hspace{1cm} (1.1.14)

Equation (1.1.9) also yields

$$(\lambda)_{m+n} = (\lambda)_m (\lambda + m)_n$$  \hspace{1cm} (1.1.15)

which, in conjunction with (1.1.14), gives

$$(\lambda)_{n-k} = \frac{(-1)^k (\lambda)_n}{(1 - \lambda - n)_k}, \quad 0 \leq k \leq n$$  \hspace{1cm} (1.1.16)

For \( \lambda = 1 \), we have

$$(n - k)! = \frac{(-1)^k n!}{(-n)_k}, \quad 0 \leq k \leq n,$$  \hspace{1cm} (1.1.17)

which may alternatively be written in the form

$$(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n - k)!}, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$  \hspace{1cm} (1.1.18)
Gauss’s Multiplication Theorem

For every positive integer \( m \), we have

\[
(\lambda)_m = m^m \prod_{j=1}^{m} \left( \frac{\lambda + j - 1}{m} \right)_n, \quad n = 0, 1, 2, \ldots
\]  

(1.19)

Starting from (1.19) with \( \lambda = mz \), it can be proved that

\[
\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{-mz-\frac{1}{2}} \prod_{j=1}^{m} \Gamma\left( z + \frac{j-1}{m} \right),
\]

(1.120)

\( z \neq 0, -\frac{1}{m}, -\frac{2}{m}, \ldots; \ m = 1, 2, 3, \ldots \)

which is known in the literature as Gauss’s multiplication theorem for the Gamma function.

1.2 GAUSSIAN HYPERGEOMETRIC FUNCTION AND ITS GENERALIZATIONS

The Hypergeometric Function

The term ‘hypergeometric’ was first used by Wallis in Oxford as early as 1655 in his work “Arithmetica Infinitorum” when referring to any series which could be regarded as a generalization of ordinary geometric series

\[
\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots
\]

(1.21)

Because of the many relations connecting the special functions to each other and to the elementary functions, it is natural to enquire whether more general functions can be developed so that the special functions and elementary functions are merely specializations of these general functions.
General functions of this nature have in fact been developed and are collectively referred to as functions of the hypergeometric type.

There are several varieties of these functions, but the most common are the standard hypergeometric function.

Some important results concerning the hypergeometric function had been developed earlier by Euler and others, but it was Gauss who made the first systematic study of the series that define this function. Gauss’s work was of great historical importance because it initiated for reaching developments in many branches of analysis not only in infinite series, but also in the general theories of linear differential equations and functions of a complex variable. The hypergeometric function has retained its significance in modern mathematics because of its powerful unifying influence, since many of the principal special functions of higher analysis are also related to it.

The main systematic development of what is now regarded as the hypergeometric function of one variable

\[
\begin{align*}
2F_1 \left[ \begin{array}{c} a, b \\ c 
\end{array} \right] & = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad c \neq 0, -1, -2, \ldots
\end{align*}
\]

was undertaken by Gauss in 1812.

In (1.2.2), \((a)_n\) denotes the Pochhammer symbol defined by (1.1.8), \(z\) is a real or complex variable, \(a, b\) and \(c\) are parameters which can take arbitrary real or complex values and \(c \neq 0, -1, -2, \ldots\). If \(c\) is zero or a negative integer, the series (1.2.2) does not exist and hence the function \(2F_1 \left[ \begin{array}{c} a, b \\ c 
\end{array} \right] \) is not defined unless one of the parameters \(a\) or \(b\) is also a negative integer such that \(-c < -a\). If either of the parameters \(a\) or \(b\) is a negative integer \(-m\) then in this case (1.2.2) reduces to the hypergeometric polynomial defined by
\[ _2F_1 \left[ \begin{array}{c} -m, b \\ c \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(-m)_n (b)_n z^n}{(c)_n n!}, \quad -\infty < z < \infty. \tag{1.2.3} \]

By d’Alembert’s ratio test, it is easily seen that the hypergeometric series in (1.2.2) converges absolutely within the unit circle, that is, when \(|z| < 1\), provided that the denominator parameter \(c\) is neither zero nor a negative integer. Notice, however, that if either or both of the numerator parameters \(a\) and \(b\) in (1.2.2) is zero or a negative integer, the hypergeometric series terminates, and the question of convergence does not enter the discussion.

Further tests show that the hypergeometric series in (1.2.2), when \(|z| = 1\), (i.e on the unit circle), is

(i) absolutely convergent, if \(\text{Re}(c - a - b) > 0;\)

(ii) conditionally convergent, if \(-1 < \text{Re}(c - a - b) \leq 0, z \neq 1\)

(iii) divergent, if \(\text{Re}(c - a - b) \leq -1\)

\[ _2F_1 \left[ \begin{array}{c} a, b \\ c \end{array} ; z \right] \] is a solution, regular at \(z = 0\), of the homogeneous second order linear differential equation

\[ z(1-z)\frac{d^2u}{dz^2} + [c - (a + b + 1)z] \frac{du}{dz} - abu = 0 \tag{1.2.4} \]

where \(a\), \(b\) and \(c\) are independent of \(z\), (1.2.4) is called the hypergeometric equation and has atmost three singularities, 0, \(\infty\) and 1 which are all regular (see for example, [78]).
Confluent Hypergeometric Function

Since the Gauss function \( _2F_1 \left[ \begin{array}{c} a, b \\ z \\ c \end{array} \right] \) is a solution of the differential equation (1.2.4), replacing \( z \) by \( \frac{z}{b} \) in (1.2.4) we have

\[
\frac{z}{b} \left( 1 - \frac{z}{b} \right) \frac{d^2 u}{dz^2} + \left[ c - \left( 1 + \frac{a}{b} \right) z \right] \frac{du}{dz} - au = 0
\]  

(1.2.5)

Obviously \( _2F_1 \left[ \begin{array}{c} a, b \\ \frac{z}{b} \\ c \end{array} \right] \) is a solution of (1.2.5) as \( b \to \infty \)

\[
\lim_{b \to \infty} _2F_1 \left[ \begin{array}{c} a, b \\ \frac{z}{b} \\ c \end{array} \right] = _1F_1 \left[ \begin{array}{c} a \\ c \end{array} \right]
\]

is a solution of differential equation

\[
\frac{d^2 u}{dz^2} + (c - z) \frac{du}{dz} - au = 0
\]  

(1.2.6)

where

\[
_1F_1 \left[ \begin{array}{c} a \\ c \\ z \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}
\]  

(1.2.7)

and is called the confluent hypergeometric function or Kummer's function given by E. E. Kummer in 1836 [49]. It is also denoted by Humbert’s symbol \( \phi(a; c; z) \).

The differential equation (1.2.6) has a regular singularity at \( z = 0 \) and an irregular singularity at \( z = \infty \) (see [78]).

**Generalized Hypergeometric Function**

The hypergeometric function defined in (1.2.2) has two numerator parameters \( a \) and \( b \) and one denominator parameter \( c \). It is a natural generalization to move...
from the definition (1.2.2) to a similar function with any number of numerator and denominator parameters.

We define a generalized hypergeometric function by

\[
pFq \left[ \begin{array}{c}
  a_1, \ldots, a_p \\
  b_1, \ldots, b_q
\end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n n!} z^n
\]

(1.2.8)

where \( p \) and \( q \) are positive integers or zero. The numerator parameters \( a_1, a_2, \ldots, a_p \) and the denominator parameters \( b_1, b_2, \ldots, b_q \) take on complex values, provided that \( b_j \neq 0, -1, -2, \ldots; j = 1, 2, \ldots, q \).

An application of the elementary ratio test to the power series on the right in (1.2.8) shows at once that:

(i) if \( p \leq q \); the series converges for all finite \( z \);

(ii) if \( p = q + 1 \); the series converges for \( |z| < 1 \) and diverges for \( |z| > 1 \);

(iii) if \( p > q + 1 \); the series diverges for \( z \neq 0 \). If the series terminates, there is no question of convergence, and the conclusions (ii) and (iii) do not apply;

(iv) if \( p = q + 1 \); the series in (1.2.8) is absolutely convergent on the circle \( |z| = 1 \) if \( \text{Re} \left( \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i \right) > 0 \).

Also, for \( p = q + 1 \), the series is conditionally convergent for \( |z| = 1, z \neq 1 \), if

\[-1 < \text{Re} \left( \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i \right) \leq 0 \]

and divergent for \( |z| = 1 \) if \( \text{Re} \left( \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i \right) \leq -1 \).

1.3 HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

Appell’s Functions

In addition to increasing the number of parameters, hypergeometric functions may be generalized along the lines of increasing the number of variables. Appell
[5; p.296(1)] defined the four hypergeometric functions of two variables which follows:

$$F_1[a, b, b'; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)(b')_n x^m y^n}{(c)_{m+n} m! n!}, \quad \max\{|x|, |y|\} < 1;$$  \hspace{1cm} (1.3.1)

$$F_2[a, b, b'; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)(b')_n x^m y^n}{(c)_{m+c'}_n m! n!}, \quad \max\{|x|, |y|\} < 1;$$  \hspace{1cm} (1.3.2)

$$F_3[a', b', b'; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a')_m(a')(b)(b')_n x^m y^n}{(c)_{m+n} m! n!}, \quad \max\{|x|, |y|\} < 1;$$  \hspace{1cm} (1.3.3)

$$F_4[a, b; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)(b')_n x^m y^n}{(c)_{m+c'}_n m! n!}, \quad \sqrt{|x|} + \sqrt{|y|} < 1.$$  \hspace{1cm} (1.3.4)

The functions $F_1, F_2, F_3$ and $F_4$ given above are all generalizations of the Gauss hypergeometric function $2F_1$. Here as usual, the denominator parameters $c$ and $c'$ are neither zero nor a negative integers. The standard work on the theory of Appell series is the monograph by Appell and Kampé de Feriét [6]. See Erdélyi et al [24; p.222-245] for a review of the subsequent work on the subject (see also Slater [80; Ch. 8], Exton [30; p.23-28]) and Srivastava and Karlsson [93]

Horn Functions

In the year 1931, J. Horn defined ten hypergeometric functions of two variables and denoted them by $G_1, G_2, G_3, H_1, \cdots, H_7$; he thus completed the set of
all possible second order (complete) hypergeometric functions of two variables in the terminology given in Appell and Kampé de Feriet [6; p.143] (see also [24; pp. 224-228], [95; pp. 56-57]). Some of them are

\[ G_1[\alpha, \beta, \beta'; x, y] = \sum_{m,n=0}^{\infty} (\alpha)_{m+n}(\beta)_{n-m}(\beta')_{m-n} \frac{x^m y^n}{m! n!} \]  

\[ |x| < r, \quad |y| < s, \quad r + s = 1; \]

\[ G_2[\alpha, \alpha', \beta, \beta'; x, y] = \sum_{m,n=0}^{\infty} (\alpha)_{m}(\alpha')_{n}(\beta)_{n-m}(\beta')_{m-n} \frac{x^m y^n}{m! n!} \]  

\[ |x| < 1, \quad |y| < 1; \]

\[ G_3[\alpha, \alpha'; x, y] = \sum_{m,n=0}^{\infty} (\alpha)_{2n-m}(\alpha')_{2m-n} \frac{x^m y^n}{m! n!} \]  

\[ |x| < r, \quad |y| < s, \quad 27r^2s^2 + 18rs \pm (r-s) - 1 = 0; \]

\[ H_1[\alpha, \beta, \gamma; \delta; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_{n} x^m y^n}{(\delta)_m m! n!} \]  

\[ |x| < r, \quad |y| < s, \quad 4rs = (s-1)^2; \]

\[ H_2[\alpha, \beta, \gamma; \delta; \epsilon; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m}(\gamma)_{n}(\delta)_n x^m y^n}{(\epsilon)_m m! n!} \]  

\[ |x| < r, \quad |y| < s, \quad (r+1)s = 1; \]
\[ H_3[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{\alpha_{2m+n} \beta_n}{\gamma_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.10) \]

\[ |x| < r, \quad |y| < s, \quad r + \left( s - \frac{1}{2}\right)^2 = \frac{1}{4}; \]

\[ H_4[\alpha, \beta; \gamma, \delta; x, y] = \sum_{m,n=0}^{\infty} \frac{\alpha_{2m+n} \beta_n}{\gamma_m \delta_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.11) \]

\[ |x| < r, \quad |y| < s, \quad 4r = (s - 1)^2; \]

\[ H_5[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{\alpha_{2m+n} \beta_n}{\gamma_m} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.12) \]

\[ |x| < r, \quad |y| < s, \quad 16r^2 - 36rs \pm (8r - s + 27rs^2) + 1 = 0; \]

\[ H_6[\alpha, \beta, \gamma; x, y] = \sum_{m,n=0}^{\infty} (\alpha)_{2m-n} (\beta)_{n-m} (\gamma)_{m} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.13) \]

\[ |x| < r, \quad |y| < s, \quad rs^2 + s - 1 = 0; \]

and

\[ H_7[\alpha, \beta, \gamma, \delta; x, y] = \sum_{m,n=0}^{\infty} \frac{\alpha_{2m-n} (\beta)(\gamma)_{m} (\delta)}{m!} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3.14) \]

\[ |x| < r, \quad |y| < s, \quad 4r = (s^{-1} - 1)^2. \]

**Humbert Functions**

In 1920, Humbert [39] has studied seven confluent forms of the four Appell functions and denoted them by \( \phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \Xi_1, \Xi_2 \) and are defined as follows (see [95; p.58 and 59])
\[ \phi_1[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!} \]  
\[ |x| < 1, \quad |y| < \infty; \]  

\[ \phi_2[\beta, \beta'; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!} \]  
\[ |x| < \infty, \quad |y| < \infty; \]  

\[ \phi_3[\beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!} \]  
\[ |x| < \infty, \quad |y| < \infty; \]  

\[ \psi_1[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)(\gamma')_n} \frac{x^m y^n}{m! n!} \]  
\[ |x| < 1, \quad |y| < \infty; \]  

\[ \psi_2[\alpha; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)(\gamma')_n} \frac{x^m y^n}{m! n!} \]  
\[ |x| < \infty, \quad |y| < \infty; \]  

\[ \Xi_1[\alpha, \alpha', \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m}(\alpha')_n(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!} \]  
\[ |x| < 1, \quad |y| < \infty; \]  

and

\[ \Xi_2[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!} \]  
\[ |x| < 1, \quad |y| < \infty. \]
Kampé de Fériet’s Function

In an attempt to generalize the four Appell functions $F_1$ to $F_4$, Kampé de Fériet [41] defined a general hypergeometric series in two variables (see [6; p.150(29)]) Kampé de Fériet function is denoted by

$$F_{E,G,H}^{A,B,D} \left[ \begin{array}{c} (a) : (b) ; (d) \\ (e) : (g) ; (h) \end{array} ; x, y \right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (d)_n}{(e)_m (g)_n} \frac{x^m y^n}{m! n!}$$

(1.3.22)

where for convergence

(i) $A + B < E + G, A + D < E + H$ for $\max\{|x| < \infty, |y| < \infty$

(ii) $A + B = E + G + 1, A + D = E + H + 1,$ and

$$\frac{|A-E|}{1-E} < 1, \quad \text{if} \quad A > E,$$

$$\max\{|x|, |y|\} < 1, \quad \text{if} \quad A \leq E.$$

1.4 HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

Lauricella’s Functions of n-Variables

In 1893, Lauricella [50] further generalized the four Appell functions $F_1, F_2, F_3, F_4$ to functions of n-variables and defined his functions as follows [95; p. 60]

$$F_A^{(n)}[a, b_1, b_2, \ldots, b_n; c_1, \ldots, c_n; x_1, x_2, \ldots, x_n]$$

$$= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}$$

| $x_1| + \ldots + |x_n| < 1;$$

(1.4.1)
\[
F_B^{(n)}[a_1, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a_1)^{m_1} \cdots (a_n)^{m_n} (b_1)^{m_1} \cdots (b_n)^{m_n}}{(c)^{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \quad (1.4.2)
\]

\[
\max\{|x_1|, \ldots, |x_n|\} < 1;
\]

\[
F_C^{(n)}[a, b; c_1, \ldots, c_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)^{m_1} \cdots (b)^{m_n}}{(c)^{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \quad (1.4.3)
\]

\[
\sqrt{|x_1|} + \cdots + \sqrt{|x_n|} < 1;
\]

\[
F_D^{(n)}[a, b; c; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)^{m_1} \cdots (b)^{m_n}}{(c)^{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \quad (1.4.4)
\]

\[
\max\{|x_1|, \ldots, |x_n|\} < 1;
\]

In particular, we have

\[
F_A^{(2)} = F_2; \quad F_B^{(2)} = F_3; \quad F_C^{(2)} = F_4; \quad F_D^{(2)} = F_1
\]

and

\[
F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} = 2F_1
\]
Confluent forms of Lauricella functions

Two important confluent hypergeometric functions of $n$ variables are the functions $\phi_2^{(n)}$ and $\psi_2^{(n)}$ defined by (see [95; p.62(10) and (11)])

$$\phi_2^{(n)}[b_1, \cdots, b_n; c; x_1, \cdots, x_n] = \sum_{m_1, \cdots, m_n=0}^{\infty} \frac{(b_1)_{m_1} \cdots (b_n)_{m_n} x_1^{m_1} \cdots x_n^{m_n}}{(c)_{m_1 + \cdots + m_n} m_1! \cdots m_n!}$$ (1.4.5)

and

$$\psi_2^{(n)}[a; c_1, \cdots, c_n; x_1, \cdots, x_n] = \sum_{m_1, \cdots, m_n=0}^{\infty} \frac{(a)_{m_1 + \cdots + m_n} x_1^{m_1} \cdots x_n^{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n} m_1! \cdots m_n!}$$ (1.4.6)

Clearly, we have

$$\phi_2^{(2)} = \phi_2, \quad \psi_2^{(2)} = \psi_2$$

where $\phi_2$ and $\psi_2$ are Humbert’s confluent hypergeometric functions of two variables.

Generalized Lauricella Function

A further generalization of Kampe de Fériet function $F_{E;G;H}^{A:B:C}$ and the Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ is due to Srivastava and Daoust (see [90] and [91]) who indeed defined an extension of Wright’s $\psi$ function (see [95; p.50(21)]) in two variables.

The generalized Lauricella function is defined as follows:

$$F_{C:D;\cdots;D}^{A:B;\cdots;B^{(n)}}[(a) : \theta', \cdots, \theta^{(n)}] : [(b') : \phi'] ; \cdots ; [(b^{(n)}) : \phi^{(n)}] ; x_1, \cdots, x_n$$

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The coefficients

\[ \theta_j^{(k)}, \quad j = 1, 2, \ldots, A; \quad \phi_j^{(k)}, \quad j = 1, 2, 3 \ldots, B^{(k)}; \quad \psi_j^{(k)}, \quad j = 1, 2, \ldots, c; \]

\[ \delta_j^{(k)}, \quad j = 1, 2, \ldots, D^{(k)}; \quad \phi_j^{(k)}, \quad \forall \quad k \in \{1, \ldots, n\} \]

are real and positive and \((b^{(k)})\) abbreviates the array of \(B^{(k)}\) parameters \(b_j^{(k)}\), \(j = 1, 2, \ldots, B^{(k)}\), \(\forall k \in \{1, 2, \ldots, n\}\) with similar interpretations for \((d^{(k)})\), \(k = 1, 2, \ldots, n\) etc. A detailed discussion of the conditions of convergence of the multiple series occurring in (1.4.7) is given in a paper of Srivastava and Daoust [91]

If the positive constants \(\theta\)'s, \(\psi\)'s, \(\phi\)'s and \(\delta\)'s are all chosen as unity, then (1.4.7) reduces to the generalized Kampé de Fériet function given by Karlsson [43] in its more general form

\[
F_{A, D', C, D'}^{B', B^{(n)}, D^{(n)}} \left[ \begin{array}{c}
(a) : (b') ; \cdots ; (b^{(n)}) ; \\
(c) : (d') ; \cdots ; (d^{(n)}) ;
\end{array} \right] x_1, \ldots, x_n \\
= \sum_{m_1, m_2, m_n = 0}^{\infty} \frac{((a))_{m_1 \theta_j^{(k)} + m_2 \phi_j^{(k)} + m_n \psi_j^{(k)}} \cdot ((b^{(n)})_{m_1 \delta_j^{(k)} + m_2 \phi_j^{(k)} + m_n \psi_j^{(k)}} \cdot \frac{\tau_1^{m_1}}{m_1!} \cdots \frac{\tau_n^{m_n}}{m_n!}. \\
(1.4.7)
\]

clearly, we have

\[
F_{0,1}^{1,1} = F_A^{(n)}, \quad F_{1,0}^{0,2} = F_B^{(n)} \\
F_{0,1}^{2,0} = F_C^{(n)}, \quad F_{1,0}^{1,1} = F_D^{(n)} \\
(1.4.9)
\]
Lauricella function of three variables

Lauricella [50; p.114] introduced 14-complete hypergeometric functions of three variables and of second order. He denoted his triple hypergeometric functions by the symbols \( F_1, F_2, F_3, \ldots, F_{14} \) (see [95; p.66-68]) of which \( F_1, F_2, F_5 \) and \( F_9 \) correspond, respectively to the three variables Lauricella functions \( F_A^{(3)}, F_B^{(3)}, F_C^{(3)} \) and \( F_D^{(3)} \) defined by (1.4.1), (1.4.2), (1.4.3) and (1.4.4) with \( n = 3 \). The remaining ten functions \( F_3, F_4, F_6, F_7, F_8, F_{10}, \ldots, F_{14} \) of Lauricella’s set apparently fell into oblivion (except that there is an isolated appearance of the triple hypergeometric function \( F_8 \) in a paper by Mayr [55; p. 265] who came across this function while evaluating certain infinite integrals). Saran [73] initiated a systematic study of these ten triple hypergeometric functions of Lauricella’s set.

For the purpose of our present work, we shall require only three functions. We give below the definitions of these functions using Saran’s notations \( F_E, F_G \) and \( F_P \) and also indicating Lauricella’s notations:

\[
F_4 : F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)m(\beta_2)n+p}{(\gamma_1)m(\gamma_2)n(\gamma_3)p} \frac{x^m y^n z^p}{m! n! p!} \quad (1.4.10)
\]

\( |x| < r, \quad |y| < s, \quad |z| < t, \quad r + (\sqrt{s} + \sqrt{t})^2 = 1; \)

\[
F_8 : F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)m(\beta_2)n+p(\beta_3)p}{(\gamma_1)m(\gamma_2)n+p} \frac{x^m y^n z^p}{m! n! p!} \quad (1.4.11)
\]

\( |x| < r, \quad |y| < s, \quad |z| < t, \quad r + s = 1 = r + t; \)
\[ F_{14} : F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) \]

\[
= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m (\beta_1)_m + p (\beta_2)_n^n}{} \frac{(\gamma_1)_m (\gamma_2)_n + p}{} x^m y^n z^p \quad m! \ n! \ p!
\]

\(|x| < r, \ |y| < s, \ |z| < t, \ (1 - s)(s - t) = rs; \)

Srivastava’s Triple Hypergeometric Function

In 1967, a unification of Lauricella fourteen hypergeometric functions \(F_1, \ldots, F_{14}\) of three variables [50] and Srivastava’s three additional functions \(H_A, H_B, H_C\) [81] was introduced by Srivastava (see for example [84; p.428] and [95; p.69]) in the form of a general triple hypergeometric series \(F^{(3)}[x, y, z]\) defined as

\[
F^{(3)}[x, y, z] = F^{(3)} \left[ \begin{array}{c}
(a) :: (b) ; (b') ; (b'') ; (c) ; (c') ; (c'') ; \\
(e) :: (g) ; (g') ; (g'') ; (h) ; (h') ; (h'') ; \\
\end{array} \right] x, y, z
\]

\[
= \sum_{m,n,p=0}^{\infty} \frac{(a)_m (b)_m + p (b')_n + p (b'')_n + p}{} \frac{(c)_m (c')_n + p (c'')_n + p}{} \frac{x^m y^n z^p}{m! \ n! \ p!}
\]

For the convergence of the series (1.4.13) see [95; p. 70(41)]

Triple Hypergeometric Series of Horn’s Type

While transforming Pochhammer’s double loop contour integrals associated with the functions \(F_8\) and \(F_{14}\) (that is \(F_G\) and \(F_F\), respectively) belonging to the Lauricella’s set of hypergeometric functions of three variables, the following
two interesting triple hypergeometric series of Horn’s type were encountered by Pandey [60; pp.115-116]:

\[
G_A[\alpha, \beta, \beta'; \gamma; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta)_{m+p}(\beta')_n}{(\gamma)_{n+p-m}} \frac{x^m y^n z^p}{m! n! p!} 
\]  

which provides a generalization of Appell’s functions \(F_1\) and Horn’s functions \(G_1\) and \(G_2\);

\[
G_B[\alpha, \beta_1, \beta_2, \beta_3; \gamma; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_{m}(\beta_2)_{n}(\beta_3)_p}{(\gamma)_{n+p-m}} \frac{x^m y^n z^p}{m! n! p!} 
\]  

which generalizes the Appell’s functions \(F_1\) and Horn’s function \(G_2\).

Srivastava [85; p.104] observe that, in terms of the function \(G_B\) defined by (1.4.15), a solution of the system of partial differential equations associated with the Lauricella function

\[
F_D^{(3)}[\alpha, \beta_1, \beta_2, \beta_3; \gamma; x, y, z] \text{ can be expressed in the form}
\]

\[
y^{-\beta_2}G_B(\alpha - \beta_2, \beta_2, \beta_1, \beta_3; \gamma - \beta_2; y^{-1}, x, z)
\]

which is valid near the singularity \(x = y^{-1} = z = 0\).

A similar investigation of the system of partial differential equations associated with the triple hypergeometric function \(H_C\) defined by [95; p.69(38)] led him to the new function [85; p.105]

\[
G_C[\alpha, \beta, \beta'; \gamma; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+n}(\beta')_{n-p}}{(\gamma)_{m+n-p}} \frac{x^m y^n z^p}{m! n! p!} 
\]  

which evidently furnishes a generalization of Appell’s function \(F_1\) and Horn’s
functions $G_2$ and $H_1$.

Pathan’s Function

In 1979, a general quadruple hypergeometric series $F_p^{(4)}$ was considered by Pathan [62; p.172(1.2), (see also [63; p. 51(1)]) in the form

\[
F_p^{(4)} \left[ \begin{array}{c}
(a_A) \colon (b_B); (d_D); (e_E); (g_G) \colon (h_H); (k_K); (m_M); (n_N);\\
(a_{A'},) \colon (b_{B'}); (d_{D'}); (e_{E'}); (g_{G'}) \colon (h_{H'}); (k_{K'}); (m_{M'}); (n_{N'});
\end{array} \right] = \sum_{q,r,s,t=0}^{\infty} \frac{[(a_A)]_{q+r+s+t}[(b_B)]_{q+r+s+t}[(d_D)]_{r+s+t}[(e_E)]_{s+t+q}}{[(a_{A'})]_{q+r+s+t}[(b_{B'})]_{q+r+s+t}[(d_{D'})]_{r+s+t}[(e_{E'})]_{s+t+q}} \\
\frac{[(g_G)]_{j+t+r}[(h_H)]_{j+t+r}[(m_M)]_{j}[(n_N)]_{j}}{[(g_{G'})]_{j+t+r}[(h_{H'})]_{j+t+r}[(m_{M'})]_{j}[(n_{N'})]_{j}} \cdot x^q y^r z^t u^j
\]

(1.4.17)

It being understood that $|x|, |y|, |z|$ and $|u|$ are sufficiently small to ensure the convergence of the concerned quadruple series.

By suitable adjustment of parameters and variables in $F_p^{(4)}$, we can easily find that $F_p^{(4)}$ is the unification of triple hypergeometric series $F^{(3)}$ of Srivastava, Exton’s $E_0^{(4)}$, $E_1^{(4)}$, $E_2^{(4)}$, $E_3^{(4)}$, $E_0^{(4)}$, $E_1^{(4)}$, $E_2^{(4)}$, $E_3^{(4)}$, $E_4^{(4)}$, $K_2$, $K_{11}$, $K_{15}$, Lauricella’s $F_A^{(4)}$, $F_B^{(4)}$, $F_C^{(4)}$, $F_D^{(4)}$, Erdélyi’s $\phi_2^{(4)}$, Chandel’s $E_0^{(4)}$, $E_1^{(4)}$, $E_2^{(4)}$, $E_3^{(4)}$, and Humbert’s $\phi_2^{(4)}$.

For the definition of the above functions we refer the book of Exton [30] and Srivastava and Manocha [95].

1.5 ORTHOGONAL POLYNOMIALS

Orthogonal polynomials are of great importance in mathematical physics, approximation theory, the theory of mechanical quadratures, etc. This class contains many special functions commonly encountered in the applications, e.g.
Legendre, Hermite, Gegenbauer, Jacobi and Rice polynomials. Orthogonal polynomials are treated in many excellent books such as Rainville [70], Lebedev [52] and Prudnikov et al (see [67],[68] and [69]). Some of the orthogonal polynomials used in our work are given below.

Laguerre Polynomials

Laguerre polynomials \( L_n(x) \) of order \( n \) are defined by means of generating relation

\[
\sum_{n=0}^{\infty} L_n(x) t^n = \exp \left( -\frac{xt}{1-t} \right) \quad (1.5.1)
\]

\( L_n(x) \) can also be written in series form as

\[
L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r n! x^r}{(r!)^2 (n-r)!} \quad (1.5.2)
\]

Associated Laguerre Polynomials

We define, for \( n \) a non-negative integer,

\[
L_n^{(\alpha)}(x) = \sum_{r=0}^{n} \frac{(-1)^r (n+\alpha)! x^r}{(n-r)! (\alpha+r)! r!} \quad (1.5.3)
\]

where \( L_n^{(\alpha)}(x) \) are associated Laguerre polynomials. This is also called generalized Laguerre or Sonine polynomials [70].

When \( \alpha = 0 \), equation (1.5.3) becomes simple Laguerre polynomials given by equation (1.5.2).
Bessel Polynomials

The Bessel polynomials are among the most important cylinder polynomials, with very diverse applications to physics, engineering and mathematical analysis. In 1949, Krall and Frink [48] initiated serious study of what they called Bessel polynomials.

Exton [33; p.4(3.1)] has introduced a Bessel polynomial in several variables. This is defined as follows

\[
y_{m_1,\ldots,m_n}(x_1,\ldots,x_n,a) = \sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} (a - 1 + m_1 + \cdots + m_n)_{k_1+\cdots+k_n} \times (-1)^{k_1} \cdots (-1)^{k_n} \frac{(x_1)^{k_1}}{k_1!} \cdots \frac{(x_n)^{k_n}}{k_n!}
\]  

(1.5.4)

If all but one of the variables are suppressed, we recover the Bessel polynomial

\[
y_m(a;x) = 2F_0 \left[ \begin{array}{c} -m, a - 1 + m \\ -x \\ \end{array} \right] 
\]  

(1.5.5)

which on replacing \( x \) by \( \frac{x}{b} \), gives us Bessel polynomial

\[
y_m(a,b;x) = 2F_0 \left[ \begin{array}{c} -m, a - 1 + m \\ -\frac{x}{b} \\ \end{array} \right] 
\]  

(1.5.6)

The Bessel polynomials (1.5.6) were introduced by Krall and Frink [48] in connection with solution of the wave equation in spherical co-ordinates.

The multivariable Bessel polynomial \( y_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) is defined as follows
In (1.5.7), we set \( \alpha_j = 1 \ (j = 1, 2, \ldots, n) \) and \( \beta = a - 2 \), we shall readily obtain Exton’s Bessel polynomial (1.5.4)

**Jacobi polynomials**

The classical Jacobi polynomial \( P_n^{(\alpha, \beta)}(x) \) of order \( (\alpha, \beta) \) and degree \( n \) in \( x \), defined (in terms of the Gauss hypergeometric function \( {}_2F_1 \)) (see [70; p.254 (1)]) by

\[
P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha)_n}{n!} \ {}_2F_1 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n \\ 1 + \alpha \end{array} ; \frac{1 - x}{2} \right] \quad (1.5.8)
\]

When we take \( \alpha = \beta = 0 \), in equation (1.5.8), we get the Legendre polynomials \( P_n(x) \) given by (see [70; p.166(2)])

\[
P_n(x) = \ {}_2F_1 \left[ \begin{array}{c} -n, n + 1 \\ 1 \end{array} ; \frac{1 - x}{2} \right] \quad (1.5.9)
\]

The Laguerre polynomials and the generalized Bessel polynomials \( y_n(a, b; x) \) are, in fact, limiting cases of the Jacobi polynomials (see [95; p.131(1)] and [1; p.411(2)])

\[
L_n^{(\alpha)}(x) = \lim_{|\beta| \to \infty} \left\{ P_n^{(\alpha, \beta)} \left( 1 - \frac{2x}{\beta} \right) \right\} \quad (1.5.10)
\]

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1.6 GENERATING FUNCTIONS

The term "generating function" was introduced by Laplace in 1812. Since then the theory of generating functions has been developed into various directions and found wide applications in various branches of science and technology. A generating function may be used to define a set of functions, to determine a differential recurrence relation or a pure recurrence relation, to evaluate certain integrals, et cetera.

Linear Generating Functions

Consider a two-variable function \( F(x, t) \) which possesses a formal (not necessarily convergent for \( t \neq 0 \)) power series expansion in \( t \) such that

\[
F(x, t) = \sum_{n=0}^{\infty} f_n(x) t^n,
\]

where each member of the coefficient set \( \{f_n(x)\}_{n=0}^{\infty} \) is independent of \( t \). Then the expansion (1.6.1) of \( F(x, t) \) is said to have generated the set \( \{f_n(x)\} \) and \( F(x, t) \) is called a linear generating function (or, simply, a generating function) for the set \( \{f_n(x)\} \).

The foregoing definition may be extended slightly to include a generating function of the type:

\[
G(x, t) = \sum_{n=0}^{\infty} c_n g_n(x) t^n,
\]

where the sequence \( \{c_n\}_{n=0}^{\infty} \) may contain the parameters of the set \( g_n(x) \), but is independent of \( x \) and \( t \).

\[
y_n(a, b; x) = \lim_{\beta \to 0} \frac{\Gamma(n + 1)\Gamma(\beta)}{\Gamma(\beta + n)} P_n^{(\beta-1, a-\beta-1)} \left(1 - \frac{2x\beta}{b}\right)
\]
If \( c_n \) and \( g_n(x) \) in (1.6.2) are prescribed, and if we can formally determine the sum function \( G(x, t) \) in terms of known special functions, we shall say that the generating function \( G(x, t) \) has been found.

**Bilinear Generating Functions**

If a three-variable function \( G(x, y, t) \) possesses a formal power series expansion in \( t \) such that

\[
G(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n,
\]

where the sequence \( \{\gamma_n\} \) is independent of \( x, y \) and \( t \), then \( G(x, y, t) \) is called a bilinear generating function for the set \( \{f_n(x)\} \).

More generally, if \( G(x, y, t) \) can be expanded in powers of \( t \) in the form

\[
G(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) f_{\beta(n)}(y) t^n,
\]

where \( \alpha(n) \) and \( \beta(n) \) are functions of \( n \) which are not necessarily equal, we shall still call \( G(x, y, t) \) a bilinear generating function for the set \( \{f_n(x)\} \).

**Bilateral Generating Functions**

Suppose that a three variable function \( H(x, y, t) \) has a formal power series expansion in \( t \) such that

\[
H(x, y, t) = \sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n
\]

where the sequence \( \{h_n\} \) is independent of \( x, y \) and \( t \) and the sets of functions \( \{f_n(x)\}_{n=0}^{\infty} \) and \( \{g_n(x)\}_{n=0}^{\infty} \) are different. Then \( H(x, y, t) \) is called a bilateral generating function for the set \( \{f_n(x)\} \) or \( \{g_n(x)\} \).
The above definition of a bilateral generating function, used earlier by Rainville [70; p.170] and McBride [56; p.19] may be extended to include bilateral generating function of the type:

\[ H(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) g_{\beta(n)}(y) t^n, \]  

where the sequence \( \{\gamma_n\} \) is independent of \( x, y \) and \( t \), the sets of functions \( \{f_n(x)\}_{n=0}^{\infty} \) and \( \{g_n(x)\}_{n=0}^{\infty} \) are different, and \( \alpha(n) \) and \( \beta(n) \) are functions of \( n \) which are not necessarily equal.

**Multivariable Generating Functions**

In each of the above definitions, the sets generated are functions of only one variable. Suppose now that \( G(x_1, x_2, \ldots, x_r; t) \) is a function of \( r + 1 \) variables, which has a formal expansion in powers of \( t \) such that

\[ G(x_1, \ldots, x_r; t) = \sum_{n=0}^{\infty} c_n g_n(x_1, x_2, \ldots, x_r) t^n, \]  

where the sequence \( \{c_n\} \) is independent of the variables \( x_1, x_2, \ldots, x_r \) and \( t \). Then we shall say that \( G(x_1, \ldots, x_r; t) \) is a generating function for the set \( \{g_n(x_1, \ldots, x_r)\}_{n=0}^{\infty} \) corresponding to the non zero coefficients \( c_n \).

It is not difficult to extend the definitions of bilinear and bilateral generating functions to include such multivariable generating functions as

\[ F(x_1, \ldots, x_r; y_1, \ldots, y_r; t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x_1, \ldots, x_r) f_{\beta(n)}(y_1, \ldots, y_r) t^n \]  

and

\[ H(x_1, \ldots, x_r; y_1, \ldots, y_s; t) = \sum_{n=0}^{\infty} h_n f_{\alpha(n)}(x_1, \ldots, x_r) g_{\beta(n)}(y_1, \ldots, y_s) t^n, \]  

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respectively.

Multilinear and Multilateral Generating Functions

A multivariable generating function \( G(x_1, \cdots, x_r; t) \) given by (1.6.7), is said to be multilinear generating function if

\[
g_n(x_1, \cdots, x_r) = f_{\alpha_1(n)}(x_1) \cdots f_{\alpha_r(n)}(x_r),
\]

where \( \alpha_1(n), \cdots, \alpha_r(n) \) are functions of \( n \) which are not necessarily equal. More generally, if the functions occurring on the right hand side of (1.6.10) are all different, the multivariable generating function (1.6.7) will be called a multilateral generating function.

Generating Functions involving Laurent series

We now extend our definition of a generating function to include functions which possess Laurent series expansions. Thus, if the set \( \{f_n(x)\} \) is defined for \( n = 0, \pm 1, \pm 2, \cdots \) the definition (1.6.2) may be extended in terms of the Laurent series expansion:

\[
F^*(x, t) = \sum_{n=-\infty}^{\infty} r_n f_n(x)t^n
\]

where the sequence \( \{r_n\}_{n=-\infty}^{\infty} \) is independent of \( x \) and \( t \). Similar extensions of the generating functions (1.6.7) is

\[
G^*(x_1, \cdots, x_r; t) = \sum_{n=-\infty}^{\infty} c_n g_n(x_1, \cdots, x_r)t^n
\]