Chapter 6

Wavelet Packet Approximation

6.1 Introduction

The aim of this chapter is to generalize some of the results of wavelets appeared in Bertoluzza [8] and Chambolle et al [13], for wavelet packets. We have proved the projection error estimates in Besov and $L_p$ norm and given some numerical results. We start with the sequences of functions due to Wickerhauser [81]. For $n = 0, 1, 2, 3, \cdots$

\begin{equation}
\Psi_{2n}(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} a_k \psi_n(2t - k) \tag{6.1}
\end{equation}

\begin{equation}
\Psi_{2n+1}(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} b_k \psi_n(2t - k) \tag{6.2}
\end{equation}

where $a = \{a_k\}$ and $b = \{b_k\}$ are the filters as defined in Section 1.5 of Chapter 1. The space $\Omega_n := \{f \mid f = \sum_{k \in \mathbb{Z}} c_k \psi_n(t - k)\}$ are the linear span of the integer translates of $\psi$'s. Taking $\delta f(t) = \sqrt{2} f(2t)$ we get $\delta \Omega_n = \Omega_{2n} \oplus \Omega_{2n+1}$, or more generally

\[ \delta^k \Omega_n = \Omega_{2^n} \oplus \cdots \oplus \Omega_{2^{(n+1)-1}} \ ; \ k \geq 0. \]

This follows to refine the decomposition $L_2(\mathbb{R}) = \oplus_n \Omega_n$ by scales, and for every partition $P$ of the non-negative integers into the sets of the form $I_{kn} = \{2^k n, \cdots, 2^k (n + 1) - 1\}$, the collection of functions

\[ \psi_{k,j} = 2^{k/2} \psi_n(2^k t - j), I_{kn} \in P, j \in \mathbb{Z}, \]

is an orthonormal basis of $L_2(\mathbb{R})$. These collection of functions give rise to many bases including Walsh, wavelet and subband basis. A Wavelet packet
basis of $L_2(\mathbb{R})$ is an orthonormal basis selected from among the functions $\Psi_{k,j}^n$.

Let $f(t)$ be a function in $L_2(\mathbb{R})$ and let $\{c_p : p \in \mathbb{Z}\}$ be the coefficients of $f(t)$ in $\delta^k \Omega_0$. Here, $c_p = \int 2^{L/2}\Psi_{0}(2^Lt - p)f(t)dt$ and the $L_2$ function given by the projection will be denoted by

$$P_L f(t) = \sum_{p \in \mathbb{Z}} c_p \Psi_{0}(2^L t - p).$$

(6.3)

From $\{c_p\}$ we may calculate the coefficients of $f(t)$ in any space $\delta^k \Omega_n$, for $0 \leq k \leq L$, and $0 \leq n < 2^{L-k}$, by applying the filters to the sequences $\{c_p\}$. Thus we have

$$c_{nk}^p = \int_{-\infty}^{\infty} 2^{k/2}\Psi_{n}(2^k t - p)f(t)dt$$

(6.4)

$p \in \mathbb{Z}, 0 \leq k \leq L, 0 \leq n < 2^{L-k}$. The coefficients of $f(t)$ in the subspace $\delta^k \Omega_n$ form a sequences $\{c_{nk}^p : p \in \mathbb{Z}\}$.

We shall consider the family of Besov spaces $B^\alpha_q(L_p(\mathbb{R}))$, $0 < \alpha < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$. These spaces have $\alpha$ derivatives in $L_p(\mathbb{R})$. If we set $p = q = 2$ then $B^\alpha_2(L_2(\mathbb{R}))$ is the Sobolev space $W^\alpha(L_2(\mathbb{R}))$ and for $\alpha < 1$, $1 \leq p \leq \infty$ and $q = \infty$, $B^\alpha_\infty(L_p(\mathbb{R}))$ is the Lipschitz space $\text{Lip}(\alpha, L_p(\mathbb{R}))$.

The following is the definition of wavelet packet of class $r$.

**Definition 6.1** A wavelet packet of class $r$, $r \in \mathbb{N}$, is a function $\Psi \in L_2(\mathbb{R})$ satisfying the following properties.

(i) $\partial^s \Psi \in L_\infty(\mathbb{R}), s = 0, 1, \ldots, r$

(ii) $\Psi$ is decreasing at infinity together with its derivatives of order lower or equal to $r$

(iii) $\int_R t^k \Psi(t) dt = 0$, for $0 \leq k \leq r$

(iv) The collection of functions

$$\Psi_{k,j}^n = 2^{k/2}\Psi_n(2^k t - j), j, k \in \mathbb{Z}$$

is an orthonormal basis of $L_2(\mathbb{R})$.

**6.2 Theoretical Results**

We prove here some results concerning the projection error estimates in Besov spaces. Assume that $\alpha$ and $p$ satisfy $\frac{1}{p} < \frac{\alpha}{2} + 1$, so that $B^\alpha_q(L_p(\mathbb{R}))$ is embedded in $L_1(\mathbb{R})$. The following is the characterization of Besov functions.
6.2. THEORETICAL RESULTS

Proposition 6.2 Let $\Psi_{j,k}^n$ be the wavelet packet of class $r$ and let $0 < \alpha < r$. Let $f \in L_p(\mathbb{R})$ then $\|f\|_{B_q^p} (L_p(\mathbb{R}))$ is equivalent to a norm of the sequences of wavelet packet coefficients $\{c_{j,k}^n\}$,

$$\|f\|_{B_q^p} (L_p(\mathbb{R})) \equiv \left( \sum_{j \geq 0} \left( \sum_{n,k} 2^{\alpha j p} 2^{j(p-2)} |c_{j,k}^n|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.$$

The above proposition can be used in the following theorem to obtain an error estimate.

Theorem 6.3 Let $\Psi_{j,k}^n$ be the wavelet packet of class $r$ and let $0 < \beta < \alpha < r$. Then for $L \geq 0$,

$$\|f - P_L f\|_{B_q^p} (L_p(\mathbb{R})) \leq C \|f\|_{B_q^p} (L_p(\mathbb{R})) 2^{-L(\alpha - \beta)}.$$

Proof. We have,

$$f - P_L f = \sum_{j \geq L} \sum_{n,k} c_{j,k}^n \Psi_{j,k}^n.$$

From proposition 6.2 we get,

$$\|f - P_L f\|_{B_q^p} (L_p(\mathbb{R})) = \left( \sum_{j \geq L} \left( \sum_{n,k} 2^{\beta j p} 2^{j(p-2)} |c_{j,k}^n|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq 2^{-L(\alpha - \beta)} \left( \sum_{j \geq L} \left( \sum_{n,k} 2^{\alpha j p} 2^{j(p-2)} |c_{j,k}^n|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq C \|f\|_{B_q^p} (L_p(\mathbb{R})) 2^{-L(\alpha - \beta)}.$$

Now we prove a result using wavelet packets to characterize $L_p$ spaces. As explained earlier, if we take $p = q$ then $B_q^p (L_p(\mathbb{R})) = W^\alpha (L_p(\mathbb{R}))$, which measures smoothness of order $\alpha$ in ($L_p(\mathbb{R})$). In case of wavelet packets,

$$\|f\|_{W^\alpha (L_p(\mathbb{R}))} \equiv \left( \sum_{j} \sum_{n,k} 2^{2 \alpha j} |c_{j,k}^n|^p \right)^{\frac{1}{p}}.$$
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**Theorem 6.4** Let $\Psi_{j,k}^n$ be the wavelet packet of class $r$. Then for $L \geq 0,$

\[
\|f - P_Lf\|_{L^p(\mathbb{R})} \leq 2^{-\alpha L} \|f\|_{W^a(L^p(\mathbb{R}))}.
\]

**Proof.**

\[
\|f - P_Lf\|_{L^p(\mathbb{R})} = \left( \sum_{j \geq L} \sum_{n,k} |c_{n,k}^{n_j}|^p \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{j \geq L} \sum_{n,k} \frac{2^{2\alpha j}}{2\alpha L} |c_{n,k}^{n_j}|^p \right)^{\frac{1}{p}}
\]

\[
\leq 2^{-2\alpha L} \left( \sum_{j} \sum_{n,k} 2^{2\alpha j} |c_{n,k}^{n_j}|^p \right)^{\frac{1}{p}}
\]

\[
\leq 2^{-2\alpha L} \|f\|_{W^a(L^p(\mathbb{R}))}.
\]

**6.3 Numerical Experiments**

We take $f(x) = e^{-2x} \sin(40x) + e^{-x}$, see figure 6.1, a $C^\infty$ function defined on the interval $[0, 1]$. The result of the approximation of our test function is given in the following table. $\|f - P_Lf\|_2$ for different value of $L$ and $N$ is computed, where $N$ represents the number of non-zero coefficients of the wavelet packet coefficients.

<table>
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<th>$L$</th>
<th>$N$</th>
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Figure 6.1: The test function