2.1. INTRODUCTION

While considering Lipschitzian mappings, a natural question arises whether it is possible to weaken contraction assumption a little bit in Banach contraction principle and still ensures the existence of fixed point. In general the answer to this question is no. In order to substantiate this viewpoint, the following interesting example can be adopted which is available in Khamsi and Kirk [87].

Example 2.1.1. Let $C[0, 1]$ be the complete metric space of real valued continuous functions defined on $[0, 1]$ under supremum metric and consider the closed subspace $X$ of $C[0, 1]$ consisting of those functions $f \in C[0, 1]$ satisfying $f(1) = 1$. Since $X$ is a closed subspace of $C[0, 1]$, hence $X$ is also complete. Now, define $T : X \rightarrow X$ by $Tf(t) = tf(t)$, $\forall t \in [0, 1]$. Then one can easily verify that $d(Tf, Tg) < d(f, g)$ whenever $f \neq ghntT\ \\&\ no\ fixed\ point\ asTf = f =^\wedge tf = f \Rightarrow f(t) = 0, \ y t G(0, 1)$. On the other hand, $/(1) = 1$ which contradicts the continuity of $/$ and so $T$ cannot have a fixed point in $X$. Here one may note that $T$ is a contractive mapping on $X$. Let us recall that a self mapping $T$ on a metric space $(X, d)$ is said to be contractive (cf. [28]) if $d(Tx, Ty) < d(x, y), \ \forall x, y \in X$.

In view of above example, a contractive condition does not ensure the existence of fixed points unless the underlying metric space is compact (cf. [28]) or the contractive conditions are replaced by relatively stronger conditions such as Banach type contraction condition (cf. [72,75,109]) or Meir-Keeler type condition (cf. [66,74,103,104,107]). Recently, Aamri and Moutawakil [1] (also Pant and Pant [109]) obtained some relatively more general common fixed point theorems for strict contractive conditions in complete metric spaces. In fact, Aamri and Moutawakil [1] proved the following result in metric spaces.

THEOREM 2.1.1. Let A and S be two weakly compatible self mappings of a metric space \((X, d)\) such that

(61) the pair \((A, S)\) satisfies property \((E.A)\),

(62) for all \(x^y \in X\)

\[ d(Ax, Ay) < \max \{d(x, Sy), d(Ax, Sx) + d(Ay, Sy) + d(Ax, Sy) \} \]

(63) \(A(X)CS(X)\).

If \(A(X)\) or \(^\infty(X)\) is a complete subspace of \(X\), then \(A\) and \(S\) have a unique common fixed point.

The main objective of this chapter is to obtain some results on coincidence and common fixed points without continuity requirements satisfying a slightly more general contractive condition which also admits a nonmetric distance function \(d\) with the property that sequence \(\{a_n\}\) converges to \(x\) if and only if \(d(x_n, x) \rightarrow 0\). We choose symmetric spaces as well as semi-metric spaces as our underlying spaces. In process, some recent results due to Aamri and Moutawakil [1, Theorem 1], Pant and Pant [109, Theorems 2.1 and 2.3] and some others extended to symmetric (semi-metric) spaces. We also observe that results contained in [1,109] remain true (upto coincidence points) even in symmetric (semi-metric) spaces for a slightly general contractive condition besides the possibility of sharpening other conditions as well. In the end of the chapter, we derive some related results besides furnishing illustrative examples which establish the utility of the results proved in this chapter.

Before presenting our results, let us recall the relevant definitions and motivations.

DEFINITION 2.1.1. A symmetric on a set \(X\) is a function \(d : X \times X \rightarrow [0, \infty)\) such that for all \(x,y \in X\)

\(d(x,y) = 0\) if and only if \(x = y\),

\(d(x,y) = d(y,x)\).
If \( d \) is symmetric on the set \( X \), then for \( x \in A^n \) and \( c > 0 \), we write \( B(x, e) = \{ y \in X : d(x, y) < e \} \). The topology \( r(d) \) on \( X \) is given by \( \{ \emptyset \} \in T(r(d)) \) if and only if for each \( X \in U \), \( B(x, e) \subseteq U \) for some \( e > 0 \). A set \( S \subseteq X \) is a neighborhood of \( x \in G \) if and only if there exists \( U \in r(d) \) such that \( x \in U \subseteq S \). A symmetric \( d \) is a semi-metric if for each \( X \in X \) and each \( e > 0 \), \( S(x, e) \) is a neighborhood of \( x \) in the topology \( T(d) \).

**Definition 2.1.2.** A semi-metric space \( X \) is a topological space whose topology \( r(d) \) on \( X \) is induced by semi-metric \( d \). In what follows symmetric space as well as semi-metric space will be denoted by \( (X,d) \).

The distinction between a symmetric and a semi-metric is evident as one can easily construct a symmetric \( d \) such that \( B(x, e) \) need not be a neighborhood of \( X \) in \( r(d) \). For a symmetric \( d \) on \( X \) the following two axioms were given by Wilson [154].

\((be)\) Given \( \{x_n\} \), \( x \) and \( y \) in \( X \), \( d(x_n,x) \to 0 \) and \( d(x_n,y) \to 0 \) imply that \( x \neq y \).

\((67)\) Given \( \{x_n\}, \{y_n\} \) and an \( x \) in \( X \), \( d(x_n,x) \to 0 \) and \( d(x_n,y_n) \to 0 \) imply that \( d(y_n,x) \to 0 \).

Here it may be noted that for a semi-metric \( d \) if \( r(d) \) is Hausdorff, then \((be)\) holds.

Now we state some weak commutativity conditions from the existing literature which are relevant in the present context and can be naturally adopted to the setting of symmetric (semi-metric) spaces.

**Definition 2.1.3.** [102] A pair \( \{A, S\} \) of self mappings defined on a symmetric (semi-metric) space \( (X,d) \) is said to be \( \lambda \)-weakly commuting if there exists some real number \( R > 0 \) such that

\[
d(ASx, SAx) < R \cdot d(Ax, Sx)
\]

for all \( x \in X \), whereas the pair \( \{A, S\} \) is said to be pointwise \( \lambda \)-weakly commuting if for given \( x \in G \) there exists \( R > 0 \) such that

\[
d(ASx, SAx) < R \cdot d(Ax, Sx).
\]
Here it may be pointed out that on the set of coincidence points \( i^- \)-weak commutativity is equivalent to commutativity and remains a necessary minimal condition for the existence of common fixed points of contractive type mappings.

**Definition 2.1.4.** [72] A pair \( \{A, S\} \) of self mappings defined on a symmetric (semi-metric) space \( (X, d) \) is said to be compatible if

\[
\lim_{n \to \infty} d(ASx_n, SAx_n) = 0
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \in X. \)

Here it may be noted that pointwise \( i^- \)-weakly commuting mappings need not be compatible.

**Definition 2.1.5.** [1] A pair \( \{A, S\} \) of self mappings defined on a symmetric (semi-metric) space \( (X, d) \) is said to enjoy property \( (E.A) \) if there exists a sequence \( \{x_n\} \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \) for some \( t \in X. \)

Clearly compatible pairs satisfy property \( (E.A) \).

**Definition 2.1.6.** A pair \( \{A, S\} \) of self mappings defined on a nonempty set \( X \) is said to be weakly compatible if \( Ax = Sx \) for some \( x \in X \) implies \( ASx = SAx \).

The notion of weak compatibility is also defined by some other authors under the different names (e.g. coincidentally commuting (cf. [27]) and partially commuting mappings (cf. [133])). Here it may be pointed out that Jungck [74] defined this notion in metric space. But the notion of weak compatible mappings never involves the metric of underlying space.

The organization of this chapter is as follows: In Section 2.1, we have already collected the relevant definitions and results which serve as a backup material to the contents of this chapter. In Section 2.2, we present some results on coincidence and
common fixed points besides establishing the utility of our results with the aid of an illustrative example. In the last section, we prove results on coincidence and common fixed points for two pairs of self mappings satisfying a functional inequality control by a suitable control function. The genuineness of our results over the relevant ones is supported by an illustrative example.

2.2. RESULTS FOR A PAIR OF MAPPINGS

Our main result runs as follows:

**THEOREM 2.2.1.** Let \( (X,d) \) be a symmetric (semi-metric) space that enjoys (be) (the Hausdorffness of \( T(d) \)). Let \( A \) and \( S \) be two self mappings of \( X \) such that

\( (bs) \) the pair \( \{A,S\} \) enjoys the property \((\mathcal{L}^{4})\),

\( (bg) \) \( \forall x^y \in X \)

\[
\left\{ \begin{array}{c}
\{ k, k \} \\
\{ d(Sx, Sy), -[d(Ax, Sx) + dAy, Sy), -[d(Ay, Sx) + d(Ax, Sy)] \} \\
\end{array} \right.
\]

\( 1 < k < 2 \). If \( S(X) \) is a \( d \)-closed (\( T(d) \)-closed) subset of \( X \), then \( A \) and \( S \) have a point of coincidence.

**PROOF.** Firstly, one needs to note that a sequence \( \{x^n\} \) in a semi-metric space \( (X,d) \) converges to a point \( x \) in \( r(d) \) if \( \lim_{n \to \infty} d(x^n, x) = 0 \). To substantiate this, suppose and let \( e > 0 \). Since \( B(x,e) \) is a neighbourhood of \( x \) there exists \( U \in T(d) \) such that \( X \in U \subseteq B(x,e) \). Since \( x^n \to x \) there is an \( m \in \mathbb{N} \) (the natural number) such that \( x^n \in U \subseteq B(x,e) \) for \( n > m \) so \( d(x^n, x) < e \) for \( n > m \), i.e., \( d(a,\ldots, x) \to 0 \). The converse part is obvious in view of the definition of \( T(d) \).

Now in view of (fg), there must exist a sequence \( \{x^n\} \) in \( X \) with \( t \in X \) such that

\[
\lim_{n \to \infty} Ax^n = \lim_{n \to \infty} Sx^n = t \in X.
\]

As \( S(X) \) is \( d \)-closed, every convergent sequence of points of \( S(X) \) has a limit in \( S(X) \), therefore \( \lim_{n \to \infty} Sx^n = t = Sa = \lim_{n \to \infty} Ax^n \) for some \( a \in X \) which in turn yields that \( t = Sa \notin S(X) \). Now we assert that \( Sa = Aa \). If it is not so then in view of (2.1), one gets

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\[
d{A_n, A} < \max \{d(5^n, S), -[r(5^n, A_n) + d(A, S)], -[d(S, A_n) + d(S, A)]\}
\]
which on letting \( n \to \infty \), reduces to
\[
\left\{ \begin{array}{c}
k \\
k \\
-k
\end{array} \right\}
\]

\[
-d(A, S), -d(S, A) > < d(S, A)
\]
yielding thereby \( S = A \), which shows that \( a \) is a point of coincidence for \( A \) and \( S \).

The same proof works for the alternate statement. This completes the proof.

Since a pair of noncompatible mappings also satisfies the property \((E.A)\). Therefore, we have the following result for a pair of noncompatible mappings.

**Corollary 2.2.1.** Let \((X,d)\) be a symmetric (semi-metric) space that enjoys \((be)\) (the Hausdorffness of \(r(d)\)). Let \(A\) and \(S\) be two self mappings of \(X\) such that

\((610)\) the pair \((A, S)\) is noncompatible,

\((611)\) for all \(x, y \in X\)
\[
d(A, A) < \max \{d(S, S), d(A, A), d(A, S), d(S, A)\},
\]

If \(S(X)\) is a \(d\)-closed (\(r(d)\)-closed) subset of \(X\), then \(A\) and \(S\) have a point of coincidence.

The following variant of Theorem 2.2.1 also remains true.

**Theorem 2.2.2.** Theorem 2.2.1 remains true if \(d\)-closedness (\(r(d)\)-closedness) of \(S(X)\) is replaced by \(d\)-closedness (\(r(rf)\)-closedness) of \(A(X)\) along with \(A(X) \subseteq S(X)\) retaining the rest of the hypotheses.

**Proof.** since \(A\) and \(S\) enjoy property \((E.A)\), we have \( \lim S_n = \lim A_n = A = t \notin X \) iov some \(a \in E\) as \(A(X)\) is a \(d\)-closed subset of \(X\). Now due to \(A(X) \subseteq S(X)\) one can find some \(b \in E\). \(X\) such that \(A = Sb\). Suppose on contrary that \(A \neq Ab\); then using (2.1) one obtains
\[
d(A_n, Ab) < \max \{d(5^n, Sb), -[d(A_n, 5^n) + d(5^n, S)], -[d(5^n, Sb), -d(A, Sb)]\}
\]
which on letting \( n \to \infty \), reduces to

\[
\left\{ \begin{array}{ll}
    k \\
    -d(Ab, Aa), -d(Ab, Aa) > < d(Aa, Ab)
\end{array} \right.
\]

yielding thereby \( Aa = Ab - Sb \) as desired.

Like Pant and Pant [109], Theorems 2.2.1 and 2.2.2 ensure common fixed point instead of point of coincidence if contractive condition (2.1) is replaced by a slightly weaker condition. In this regard we have the following.

**Theorem 2.2.3.** In the setting of Theorems 2.2.1 and 2.2.2, \( A \) and \( S \) have a unique common fixed point provided \( A \) and \( S \) are weakly compatible and contraction condition (2.1) is replaced by the following: for all \( x, y \in X \)

\[
d(Ax, Ay) < \max \left| d(Sx, Sy), -[d(Ax, Sx) + d(Ay, Sy)], -[d(Ay, Sx) + d(Ax, Sy)] \right|, 1,
\]

(2.2)

where \( 1 < A; < 2 \).

**Proof.** In view of Theorems 2.2.1 and 2.2.2, \( A \) and \( S \) have a point of coincidence, say \( a \) i.e., \( Aa = Sa \). Now due to weak compatibility of the pair \( \{A, S\} \), one can write \( AAa = ASa = SAa = SSa \). If \( AAa \neq Aa \), then (2.2) implies

\[
d(Aa, AAa) < \max \left| d(Sa, Sa), -[d(Aa, Sa) + d(AAa, SAa)], -[d(AAa, Sa) + d(Aa, SAa)] \right| < 1,
\]

a contradiction. Hence \( Aa = AAa = ASa = SAa \), which shows that \( Aa \) is a common fixed point of \( A \) and \( S \). Uniqueness of the common fixed point follows easily.

**Remark 2.2.1.** Theorem 2.2.3 generalizes relevant fixed point theorems due to Aamri and Moutawakil [1] and Pant and Pant [109].

**Remark 2.2.2.** Theorems 2.2.1 and 2.2.2 remain true, if one replaces contractive condition (2.1), i.e.

\[
d(Ax, Ay) < \max \left| d(Sx, Sy), -[d(Ax, Sx) + d(Ay, Sy)], -[d(Ay, Sx) + d(Ax, Sy)] \right|,
\]

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where $1 < A; < 2$ by

$$d(Ax, Ay) < \max\{d(Sx, Sy), k[d(Ax, Sx) + d(Ay, Sy)], k[d(Ay, Sx) + d(Ax, Sy)]\},$$

where $0 < k < 1$.

We now furnish an example to demonstrate the validity of the hypotheses and degree of generality of our results over earlier ones especially those contained in [1,109]. Our example presents a nonmetric setting satisfying the hypotheses of Theorem 2.2.3 which in turn establishes the genuineness of our results over all relevant metrical fixed point theorems.

**Example 2.2.1.** Consider $X = [0,1]$ equipped with the symmetric $d(x, y) = (x - y)^+$. Define $A, S : X \rightarrow X$ as follows:

$$S(x) = \begin{cases} 
1 - x, & \text{if } 0 < x < 1, \\
0, & \text{if } 1 < x < 1.
\end{cases}$$

$$A(x) = \begin{cases} 
1, & \text{if } 0 < x < 1, \\
0, & \text{if } 1 < x < 1.
\end{cases}$$

Clearly $S(X) = \{0\} \cup [1, 1]$ is $d$-closed in $X$. The pair $(A, S)$ enjoys the property $(E, A)$ as for the sequence $\{| - X\} \subset (0, 1)$, we have

$$\lim_{n \rightarrow \infty} A(1 - X) = \lim_{n \rightarrow \infty} S(l - X) = le [0, 1].$$

By a routine calculation one can show that the contractive condition (2.1) holds for every $x, y \in X$. Also notice that the $A(X) = S(X) = AS(X) = SA(X)$. Since the topology induced by $d$ is usual on $[0,1]$, it will be Hausdorff and therefore condition $(be)$ is naturally satisfied. Thus all the conditions of Theorem 2.2.3 are satisfied and $X$ is the unique common fixed point of $A$ and $S$. Here, one needs to note that $d$ is not a metric as $d(0, l) = 1 + i + i = d(0, i) + d(l, 1)$. Thus all the available metrical common fixed point theorems cannot be used in the context of this example. Notice that both the mappings $A$ and $S$ are discontinuous at the unique common fixed point $X$.

Here it may be observed that Example 2.2.1 also satisfies the requirements of Theorems 2.2.1 and 2.2.2 as $A(X) = \{X\} \subset \{0\} \cup [1, 1] = S(X)$ and $A(X)$ is $d$-closed. Finally, it may be mentioned that condition (65) is also crucial as this
ensures the uniqueness of limit to convergent sequences. It is not difficult to find a symmetric which induces non-Hausdorff topology such as Ti-topology which permits the convergence of a sequence to more than one limit points (e.g., \( X = ^\wedge, d(x,y) = |x-y| \) when \( x \neq y \) and \( d(x,x) = 0 \) with \( X_n = n, n \in \mathbb{N} \)).

2.3. RESULTS FOR TWO PAIRS OF MAPPINGS

Our next theorem involves a function \((f):^\wedge, ^\wedge, ^\wedge\) which satisfies the following conditions:

(612) \((p \) is nondecreasing on \( \mathbb{R}^m \),

(613) \( 0 < (f)/t < t \) for each \( t \in (0, \infty) \).

**THEOREM 2.3.1.** Let \( A, B, S \) and \( T \) be self mappings of a symmetric (semi-metric) space \((X,d)\) that enjoy \((be)\) (the Hausdorffness of \( T(d) \)). Suppose that

(614) \( A(X)cT(X), B(X)cS(X), \)

(615) the pair \((B,T)\) (or alternatively the pair \((A,S)\)) enjoys the property \((E.A)\),

(fue) the following inequality holds:

\[
d(Ax,By) < \langle \langle 1 \rangle \rangle (m(x,y)) (2.3)
\]

\[
m(x,y) = \max Uxx,Ty, \wedge[d(Ax, Sx) + d(By, Ty)], \wedge[d(Ax,Ty) + d(By, Sx)]^\wedge
\]

where \( 1 < A; < 2 \),

(617) \( S(X) \) (or alternatively \( T(X) \)) is a \( d \)-closed (\( T(rf) \)-closed) subset of \( X \).

Then the pairs \((A,S)\) and \((B,T)\) have points of coincidence.

**PROOF.** since the pair \((B,T)\) enjoys the property \((E.A)\), therefore there exists a sequence \( \{x_n\} \subset X \) such that \( \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = teX. \) Since \( B(X) \subset C S(X) \) for \( X_n \) there exists \( y_n \) such that \( Bx_n = 5y_n^\wedge \). Thus in all \( BXn \wedge T \to t \) and \( Syx \to t \). Now we assert that \( Ayn \wedge t \). If not, there must exists a subsequence \( \{Aym\} \) of \( \{Ayn\} \), a positive number \( M \) and a number \( 7 > 0 \) such that for each \( m > M \), we have \( d(Ayn, t) > 7, d(Ayn, Bxm) > 7 \) and

\[
d(Aym, Bxm) < \langle \langle (\max \wedge[d(Sym,T xm)], -[d(Aym^\wedge Sym) + d(Bx,,^\wedge,x^\wedge)]) \rangle \)
\]
\[ k \旅游{d(Aym,Txm) + d(Bxm, Sym)}j \]
\[ < d(Aym, Sym) = d(Aym, Bxm), \]
a contradiction. Hence \( Aym \rightarrow t. \)

Suppose that \( S(X) \) is \( d \)-closed subset of \( X \) and \( 5j/., \rightarrow t \), then one can find a point \( u \in X \) such that \( Su = t. \) Now we suppose that \( Au ^ Su. \) Then inequality (2.3) implies

\[ d(Au,Bxn) < (\max U(Su,Txn), -[d(Au,Su) + d(Bxn,Txn)], -[d(Au,Txn) + d(Bx,n,5u)]) \]

which on letting \( n \rightrightarrows \infty \), yields

\[ d(Au, Su) < (\max U(5u), -d(Au, Su)) \]
a contradiction. Hence \( Au = Su. \) Also \( A(X) \subseteq T(X) \), there exists a point \( w \in X \) such that \( Au = Tw. \) We assert that \( Tw = Biu. \) If not, then using inequality (2.3), one gets

\[ d(Au,Bw) < (\max U(Su,Tw), -[d(Au,Su)+d(Bw,Tw)], -[d(Au,Tw) + d(Bu,,5u)]) \]

\[ = 0 \rightarrow d(Bw, Au) \]

(as \( 1 < k < 2 \)) a contradiction. Hence \( Au = Su = Bw = Tw, \) which shows that the pairs \( (A, S) \) and \( (B, T) \) have a point of coincidence \( u \) and \( K; \) respectively.

The proof is similar if we consider the case when pair \( (A, S) \) enjoys property \( (E.A) \) and \( T(X) \) is \( rf \)-closed subset of \( X. \) Hence it is omitted. This completes the proof.

On the lines of the Theorem 2.2.3, one can have the following in the context of Theorem 2.3.1.

**THEOREM 2.3.2.** in the setting of Theorem 2.3.1, \( A, B, S \) and \( T \) have a unique common fixed point provided one adds the weak compatibility of the pairs \((A, S)\) and
(B, T) besides replacing contractive condition (2.3) with a slightly weaker condition:

\[ m(x, y) = \max \{ d(Sx, Ty), d(Ax, Sx) + d(By, Ty), d(Ax, Ty) + d(By, Sx) \} \]

\[ d(Ax, By) < (f)(m(x, y)) \]

PROOF, in view of Theorem 2.3.1, one concludes that \( Au = Su = Bw = Tw \).

Now the weak compatibility of \( \{ A, S \} \) implies that \( ASu = SAu \) and \( AAu = ASu = SAu = SSu \). Suppose that \( Au \neq AAu \); then using (2.4), one gets

\[ d(Au, AAu) = d(AAu, Bw) < M \max \{ SAu, Tw \}, (d(AAu, SAu) + d(Bw, Tw)), \]

a contradiction. Thus \( Au = AAu = SAu \), then \( Au \) is the common fixed point of the mappings \( A \) and \( S \).

Since the pair \( \{ B, T \} \) is also weakly compatible, hence \( BBw = BTw = TBw = TTw \). Suppose that \( Bw \neq BBw \); then in view of (2.4), one gets

\[ d(Bw, BBw) = d(Au, BBw) < (f)(\max \{ Su, TBw \}, d(Au, Su) + d(BBw, TBw)) \]

\[ hd(Au, TBw) + d(BBw, Su) \]

\[ < l(d(Bw, BBw)) < d(Bw, BBw) \]

a contradiction. Hence \( Bw = BBw = TBw \) which shows that \( Bw \) is a common fixed point of the pair \( \{ S, T \} \). Therefore, \( Au = Bw \) is a common fixed point of the pair \( \{ B, T \} \). Similarly, one can show that \( Su, Bw \) and \( Tw \) are common fixed points of the mappings \( A, S, B \) and \( T \). Uniqueness of the common fixed point follows easily. This completes the proof.

By choosing \( A, B, S \) and \( T \) suitably, one can deduce corollaries for a pair as well as for a triod of mappings. The detail of possible corollaries for a pair of mappings is not included because we have stated it as Theorem 2.2.1. We now outline the following natural results for pairs of three self mappings.
COROLLARY 2.3.1. Let $y_1, y_5$ and $r$ be self mappings of a symmetric (semi-metric) space $(X,d)$ that enjoy $(b^\wedge)$ (the Hausdorffness of $r(d)$). Suppose that

\[ A(X) \subset T(X) US(X), \]

(6i8) the pair $(A,T)$ (or alternatively the pair $(A,S)$) enjoys the property $(E,A)$,

(6i9) the following inequality holds:

\[ d(Ax, Ay) < l \{m(x,y)\} \quad (2.5) \]

\[ m(x, y) = \max U(Sx, Ty), \wedge [d(Ax, Sx) + d(Ay, Ty)], -[d(Ax, Ty) + d(Ay, Sx)] \]

where $1 < fc < 2$,

(6i2) $S(X)$ (or alternately $T(X)$) is a rf-closed (r(rf)-closed) subset of $X$.

Then the pairs $(A, S)$ and $(A, T)$ have points of coincidence.

COROLLARY 2.3.2. Let $A, B$ and $S$ be self mappings of a symmetric (semi-metric) space $(X,d)$ that enjoy (65) (the Hausdorffness of $T(rf)$). Suppose that

\[ A(X) \subset SiX), BiX) \subset SiX), \]

(623) the pair $(B,S)$ (or alternatively the pair $(A,S)$) enjoys the property $(\wedge^\wedge 4)$,

(624) the following inequality holds:

\[ d(Ax, By) < 4 \{m(x,y)\} \quad (2.6) \]

\[ m(x, y) = \max lU(Sx, Sy), -[dAx, Sx) + d(By, Sy)], \wedge [d(Ax, Sy) + d(By, Sx)] \]

where $1 < fc < 2$,

(625) $S(X)$ is a rf-closed (r(d)-closed) subset of $X$.

Then the pairs $(A, S)$ and $(B, S)$ have points of coincidence.

REMARK 2.3.1. A remark similar to Remark 2.2.2 in context of Theorems 2.3.1 and 2.3.2 can be furnished. But due to repetition, details are omitted.

We now give an example to illustrate the above theorem.
Example 2.3.1. Consider \( X = [0,1] \) equipped with the symmetric \( d(x,y) = |x - y| \). Define

\[
Ax = Bx = \begin{cases} 
\frac{1}{1+x}, & \text{if } 0 < a; < 1, \\
5x = Tx = x, & \text{if } 0 < a < 1,
\end{cases}
\]

and \( \langle x, y \rangle : 3?+ \to 3?+ \) as

\[
m = \begin{cases} 
\frac{7}{i}, & \text{if } 0 < i < 1 \\
1, & \text{if } i > 1
\end{cases}
\]

Then \( A(X) = B(X) = [0,1] \subset [0,1] = S(X) = T(X), \) \( (f) \) is nondecreasing and \( 0 < (f)(t) < t \) for all \( t \in (0,\infty) \). Since \( d \) induces the usual topology therefore condition (be) is satisfied. The pair \( \langle A, S \rangle \) satisfies the property \( (E.A) \) as there is a sequence \{\( x^n \}\} \subset [0,1] \) such that

\[
\lim_{n \to \infty} A(x^n) = \lim_{n \to \infty} S(x^n) = 0 \in X.
\]

Also \( \langle A, S \rangle \) is weakly compatible as \( SO = AO = ASO = SAO \). \( S(X) = [0,1] \) is a rf-closed subset of \( X \).

In order to verify contractive condition (2.4), if \( x = 0 \) and \( 0 < y < 1 \), then

\[
d(Ax, Ay) = \frac{y}{1+x} < \frac{\sqrt{\frac{1+y}{1+2}}} {1+\frac{\sqrt{|x-y|}} {1+\frac{\sqrt{|x-y|}^2} {1+|x-y|^2}}}
\]

In case \( x \geq y \) and \( 0 < x < y < 1 \), then

\[
d(Ax, Ay) = \frac{y}{1+x} < \frac{\sqrt{\frac{1+y}{1+2}}} {1+\frac{\sqrt{|x-y|}} {1+\frac{\sqrt{|x-y|}^2} {1+|x-y|^2}}}
\]

Thus all the conditions of Theorems 2.3.1 and 2.3.2 are satisfied and 0 is the coincidence as well as common fixed point of the pair \( \langle A, S \rangle \).

Finally, one may note that Theorem 2.3 due to Pant and Pant [109] cannot be used in the context of this example due to nonmetric setting besides other improvements realized due to certain tight conditions.