4.1. INTRODUCTION

Motivated by the fact that a fixed point of any mapping on metric spaces can always be viewed as a common fixed point of that mapping and identity mapping on the same space. Jungck [70] proved the following interesting generalization of celebrated Banach contraction principle. While proving his result, Jungck [70] replaced identity mapping with a continuous mapping.

**Theorem 4.1.1.** Let $T$ be a continuous mapping of a complete metric space $(X,d)$ into itself. Then $T$ has a fixed point in $X$ if there exists $k \in (0,1)$ and a mapping $S : X \rightarrow X$ which commutes with $T$ and satisfies $S(X) \subset T(X)$ and $d(Sx,Sy) \leq kd(Tx,Ty)$, for all $x,y \in X$.

As reflected in Theorem 4.1.1, a metrical common fixed point theorem generally involves conditions on commutativity, continuity, completeness and suitable containment of ranges of the involved mappings besides an appropriate contraction condition and researchers in this domain are aimed at weakening one or more of these conditions.

As discussed in earlier chapters, after the evolution of weak commutativity of Sessa [136] and compatibility of Jungck [72], researchers started utilizing weak conditions of commutativity as a tool to improve common fixed point theorems. Consequently, the recent literature of metric fixed point theory has witnessed the evolution of several weak conditions of commutativity whose lucid survey and illustration (upto 2001) is available in Murthy [100]. In what follows, we choose to utilize the most natural of these weak conditions namely weak compatibility due to Jungck [74].

The contents presented in Sections 4.1-4.3 and 4.4 of this chapter have been published online in Acta Math. Sinica, English Series 24(2008).
With a view to improve the continuity requirement in fixed point theorems, Kannan [84] proved a result for self mappings (without continuity) and shown that there do exist mappings which are discontinuous at their fixed points. However, common fixed point theorems without any continuity requirement were established by Singh and Mishra [144] (also Pant [105]). Here, we opt a method which is essentially inspired by Singh and Mishra [144] wherein the completeness of the space is alternately replaced by the completeness of range of one or more mappings as required.

The tradition of improving contraction conditions in fixed and common fixed point theorems is still in fashion and continues to be most effective tool to improve such results. For an extensive collection of contraction conditions one can be referred to Rhoades [126,128] and references cited therein. Most recently, with a view to accommodate many contraction conditions, Popa [122] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contraction conditions. In this chapter, we also utilize implicit functions to prove our results because of their versatility of deducing several contraction conditions at the same time. This fact will be substantiated by furnishing several examples in Sections 4.2 and 4.5.

Most recently, Aamri and Moutawakil [1] and Liu et al. [94] introduced the notions of property \( (E.A) \) and common property \( (E.A) \), respectively. As mentioned earlier, a pair of compatible as well as noncompatible self mappings of a metric space \((X,d)\) satisfies the property \( (E.A) \). In general, pairs enjoying property \( (E.A) \) and common property \( (E.A) \) need not follow the pattern of containment of range of one mapping into the range of other as utilized in common fixed point considerations but still it relaxes such requirements.

**Example 4.1.1.** Consider \( X = [-1,1] \) with the usual metric. Define the self mappings \( T \) and \( S' \) on \( X \) as follows:

\[
T(x) = \begin{cases} 
\frac{3}{5}, & \text{if } x = -1 \\
\frac{1}{2}, & \text{if } -1 < x < 1 \\
1, & \text{if } x = 1 
\end{cases}
\]

\[
S(x) = \begin{cases} 
\frac{1}{2}, & \text{if } -1 < x < 1 \\
-1, & \text{if } x = 1 
\end{cases}
\]
Consider the sequence $X_n = \wedge$. Clearly,

$$\lim_{n \to \infty} T_{x_n} = \lim_{n \to \infty} 5_{x_n} = 0.$$  

Then $T$ and $S$ satisfy property $(E.A)$. Also, $T(X) = \wedge \cup \{\wedge, \} \cup \{^\wedge, \}$ and $S(X) = \{-, |\}$. Here one needs to note that neither $T(X)$ is contained in $S(X)$ nor $S(X)$ is contained in $T(X)$.

In this chapter, we observe that up to a pair(s) of mappings newly introduced property $(E.A)$ (common property $(E.A)$) buys containment conditions on ranges without any continuity requirement besides limiting commutativity requirement to the points of coincidence. Moreover, the completeness requirement of the space is weakened to alternative natural conditions and the involved contraction condition is replaced by implicit functions. In Section 4.5, we also define a new implicit function to enhance the domain of applicability which includes several well known contraction conditions such as: Ciric quasi-contraction, generalized contraction, $\phi$-type contraction, rational inequality and others besides admitting new unknown contraction conditions which is used to prove a general common fixed point theorem for two pairs of weakly compatible self mappings satisfying the common property $(E.A)$. In process, many known results are enriched and improved. Some related results are also derived besides furnishing illustrative examples.

The details of implicit function due to Popa [122] will be presented in Section 4.2. In Section 4.3, we prove our results using the implicit function (due to Popa [122]) where Section 4.4 is devoted to some illustrative examples to the results proved in Section 4.3. This chapter concludes with Section 4.7 wherein some examples illustrating the results presented in Section 4.6.

**4.2. IMPLICIT FUNCTION I**

Recently, Popa [122] introduced the idea of implicit functions to prove new common fixed point theorems. To describe the implicit functions of Popa [122], let $\Phi$ be the family of real lower semi-continuous functions $F(f_1, t_2, \cdots, \wedge, e) : \mathbb{R}^\wedge \times 3\mathbb{R}$ satisfying the following conditions:

$F_\Phi$: $F$ is nonincreasing in the variables $t^\wedge$ and $fe$. 

i * 2 : there exists \( hG (0,1) \) such that for every \( u,v >0 \) with
\[
F_2 u \quad F(u, v, u + v, 0) < 0 \text{ or }
\]
\[
\text{we have } u < hv, \text{ and }
\]
\[
F_3 \quad F(u, u, 0, 0, u) > 0, \text{ for all } u > 0.
\]

The following examples of such functions appeared in Popa \[122\] with details and verifications.

**EXAMPLE 4.2.1.** Define \( F(i, i_2, \ldots, i_6) : \mathbb{R}^6 \rightarrow \mathbb{R} \) as
\[
F(i, i_2, \ldots, i_6) = ti - k\max t_2, t_3, U, \{-t_5 + t_6\}, \text{ where } A : G(0, 1).
\]

**EXAMPLE 4.2.2.** Define \( F(i, i_2, \ldots, k) : \mathbb{R}^k \rightarrow \mathbb{R} \) as
\[
F(i, i_2, \ldots, t, t_e) = ti - C_1 \max \{t_1, t_2, t_3\} - C_2 \max \{t_4, t_5, t_6\} - C_3 t_e,
\]
where \( C_1 > 0, C_2, C_3 > 0, C_1 + 2C_2 < 1 \) and \( C_1 + C_3 < 1 \).

**EXAMPLE 4.2.3.** Define \( F(t, t_2, \ldots, i_6) : \mathbb{R}^{+6} \rightarrow \mathbb{R} \) as
\[
\text{where } a > 0, b, c, d > 0, a + b + c < 1 \text{ and } a + rf < 1.
\]

**EXAMPLE 4.2.4.** Define \( F(i, i_2, \ldots, i_6) : \mathbb{R}^6 \rightarrow \mathbb{R} \) as
\[
F(i, i_2, \ldots, t, t_e) = ti - at_2 - \cdots - c d t_e,
\]
where \( a > 0, b, c, d > 0, a + c + d < 1 \) and \( a + 6 < 1 \).

**EXAMPLE 4.2.5.** Define \( F(i, i_2, \ldots, i_6) : \mathbb{R}^6 \rightarrow \mathbb{R} \) as
\[
F(i, i_2, \ldots, i_6) = ti - A - B - C - D t_e,
\]
where \( A : e(0, 1) \).

**EXAMPLE 4.2.6.** Define \( F(i, t_2, \ldots, i_6) : \mathbb{R}^{+6} \rightarrow \mathbb{R} \) as
\[
F(i, t_2, \ldots, i_6) = ti - a i 2 - \cdots - 1 - 72 t e \quad \text{where } a > 0, 6 > 0 \text{ and } a + 6 < 1.
\]
Implicit functions are quite fruitful in deducing many known contraction conditions besides admitting new ones. To substantiate this viewpoint we add some examples described as follows:

**EXAMPLE 4.2.7.** \(De \& ae F(tu t2, ..., h)\):

\[
F(tu t2, ..., h) = ti - k \max \{t2, t3, t4, ..., h\}, \quad \text{where} \quad A; G(0,1).
\]

**Fi** : Obviously.

**F2a** : Let \(u > 0\) and \(F(u, v, u, u + v, 0) = u - k \max \{Vy, u, \wedge\{u + v\}\} < 0\). If \(w > u\), then \(u < ku < u\), a contradiction. Thus \(u < v\) and \(u < kv\), where \(k \in (0,1)\).

**F2b** : Similar argument as in F2a-

**Fa** : \(F(u, u, 0, 0, u, u) = u - ku = (1 - k)u > 0\), for all \(u > 0\).

**EXAMPLE 4.2.8.** Define \(F(tu <2, ..., te)\) as

\[
F(tu t2, ..., t6) = ti - k \max \{t2, t3, t4, t5, t6\}, \quad \text{where} \quad fc G(0,1).
\]

**EXAMPLE 4.2.9.** Define \(F(f1, t2, ..., te)\) as

\[
F(ti, t2, ..., te) = *i - \{o - it2 + \wedge2t3 + aiU + (\wedge ih + o - bk)\}, \quad \text{where} \quad \wedge a j < 1.
\]

**EXAMPLE 4.2.10.** Define \(F(ti, t2, ..., t6)\) as

\[
F(ti, t2, ..., te) = ti - \max\{t2, t3, U, t5, t6\}, \quad \text{where} \quad A; G(0,1).
\]

Since verifications of requirements (Fi, F2 and F3) for Examples 4.2.8-4.2.10 are straightforward, hence details are omitted.

### 4.3. RESULTS VIA PROPERTY (E.A)

Here, we prove a general common fixed point theorem for a pair of self mappings using the implicit function due to Popa [122] without requiring the condition on containment of ranges of the involved mappings. We also utilize our main theorem to highlight how several fixed point theorems can be unified by using an implicit function.
THEOREM 4.3.1. Let 5 and \( T \) be self mappings of a metric space \( (X,d) \) such that

(di) the pair \( (S^\wedge T) \) satisfies the property \( (E.A) \),

(6.2) for all \( a, 2, 6 X \) and \( F G * \),

\[
F(d(Tx,Ty),d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Sy,Tx)) < 0, \quad (4.1)
\]

(da) \( S(X) \) is a complete subspace of \( X \).

Then the pair \( (5, T) \) has a point of coincidence. Moreover, the pair \( (5, T) \) has a common fixed point provided it is weakly compatible.

PROOF, in view of (di), there exists a sequence \( \{xn\} \) in \( X \) such that

\[
\lim_{n \to \infty} Txn = \lim_{n \to \infty} 5x_n = t \in X.
\]

As \( S(X) \) is a complete subspace of \( X \), every convergent sequence of points of \( S(X) \) has a limit in \( S(X) \). Therefore

\[
\lim_{n \to \infty} Sxn = t = Sa = \lim_{n \to \infty} Txn, \quad \text{for some } a \in X
\]

which in turns yields that \( t = Sa \in S(X) \). Now assert that \( Sa = Ta \). If it is not, then \( d(Ta, Sa) > 0 \). Using (4.1)

\[
F(d(Ta, Txn), d(Sa, 5z,,), d(Sa, Ta), d(Sx, Txn), d(Sa, Txn), d(Sx,, Ta)) < 0
\]

which on making \( n \to \infty \), reduces to

\[
F(d(Ta, t), d(Sa, t), d(Sa, Ta), d(t, t), d(Sa, t), d(t, Ta)) < 0
\]

or

\[
F(d(Ta, Sa), 0, d(Sa, Ta), 0,0, d(Sa, Ta)) < 0,
\]

yielding thereby (due to \( F2b \)), \( d(Ta, Sa) < 0 \). Hence \( Ta = Sa \) which shows that \( a \) is a coincidence point of \( S \) and \( T \).

Since \( S \) and \( T \) are weakly compatible, then

\[ St = STa = TSa = Tt. \]
Now assert that $Tt = t$. If not, then $d(Tt, t) > 0$. Again using (4.1)

$$F[d(Tt, Ta), d(S, Sa)](d(Tt, Ta), d(S, Sa)) < 0$$

or

$$F(d(Tt, t), d(Tt, t), 0, 0, d(Tt, t), d(t, Tt)) < 0,$$

which contradicts $F3$. Hence $Tt = t$ which shows that $f$ is a common fixed point of $S$ and $T$. The uniqueness of the common fixed point is an easy consequence of inequality (4.1). This completes the proof.

**Remark 4.3.1.** Theorem 4.3.1 is a generalized and improved form of Theorem 4.1.1 due to Jungck [70] without any continuity requirement besides relaxing the containment of the range of one map into the range of the other (i.e $T(X) \subset S(X)$). Also the commutativity requirement is reduced to points of coincidence along with replacement of the completeness of the space with alternate natural condition. In fact Theorem 4.3.1 refines all existing results established for a pair of mappings.

**Corollary 4.3.1.** The conclusions of Theorem 4.3.1 remain true if for all $x, y \in X$ inequality (4.1) is replaced by any one of the following:

1. $(^4) d(Tx, Ty) < k \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{1}{2}(d(Sx, Ty) + d(Sy, Tx))\}$

where $k \in (0, 1)$.

2. $(^5) d(Tx, Ty) < \max\{d(Sx, Ty), d(Sy, Tx)\} + c^\alpha d(Sx, Ty) + d(Sx, Tx)$

where $c_1 > 0$, $C2,03 > 0$, $C1 + 2C2 < 1$ and $C1 + C3 < 1$.

3. $(^6) d(Tx, Ty) < d(Tx, Ty)(ad(Sx, Sy) + bd(Sx, Tx) + cd(Sy, Ty)) + d(Sx, Ty)d(Sy, Tx)$

where $a > 0$, $b, c, d > 0$, $a + b + c < 1$ and $a + d < 1$.

4. $(^7) d(Tx, Ty) < ad^\alpha(Tx, Ty)d(Sx, Sy) + hd(Tx, Ty)d(Sx, Tx)d(Sy, Ty)$

where $a > 0$, $6, c, d, e > 0$, $a + 6 + c < 1$ and $a + d < 1$.

5. $(^H) H(T, T, A) < \frac{1}{2}(g(T, T) + f(T, T))$
where \( a > 0, \ b > 0 \) and \( a + b < 1 \).

\[(\text{dio}) \ d(r, T) < \text{fmax} \ U_{s}, S_{y}, d(S_{x}, T_{x}), d(S_{y}, T_{y}), d(S_{y}, T_{x}), d(S_{x}, T_{y}), \ d(S_{x}, T_{y}), d(S_{y}, T_{x}) \]

where \( A; G (0,1) \).

\[(\text{dn}) \ <T_{x}, T_{y}/) < k_{\text{max}}[d(S_{x}, S_{y}), d(S_{x}, T_{x}) + d(S_{y}, T_{y})], d(S_{x}, T_{x}) \]

where \( A; G (0,1) \).

\[(\text{dia}) \ d(T_{x}, T_{y}) < a_{t=1} d(S_{x}, S_{y}) + a_{s} d(S_{x}, T_{x}) + a_{t} d(S_{y}, T_{y}) + a_{s} d(S_{x}, T_{y}) + a_{s} d(S_{y}, T_{x}) \]

where \( A; G (0,1) \).

\[(\text{di3}) \ d(T_{x}, T_{y}) < \text{max} \{d(S_{x}, S_{y}), d(S_{x}, T_{x}), d(S_{y}, T_{y}), d(S_{x}, T_{y}), d(S_{y}, T_{x})\}, \]

where \( A; G (0,1) \).

**Proof.** The proof of the Corollary 4.3.1 follows from Theorem 4.3.1 and Examples 4.2.1-4.2.10.

**Remark 4.3.2.** Corollaries corresponding to (\(^4\)) to (dis) are new results as they are free from any condition on containment of range of the involved mappings. Also, no existing result proved for four mappings can deduce them as corollaries due to obvious reason. Also notice that some of our corollaries are seeming new to the literature (e.g. corollary corresponding to \(d_{5, d_{s}, d_{g} \text{ and } r_{f}}\).

As an application of Theorem 4.3.1, we prove a common fixed point theorem for two finite families of mappings which runs as follows:

**Theorem 4.3.2.** Let \( \{S_{1}, S_{2}, \ldots, S_{p}\} \) and \( \{T_{1}, T_{2}, \ldots, T_{m}\} \) be two finite families of self mappings of a metric space \( (X, d) \) with \( S = S_{1} S_{2} \ldots S_{p} \) and \( T = T_{1} T_{2} \cdots T_{m} \) satisfying property \( \{E.A\} \) and condition (4.1). If \( S(X) \) is a complete subspace of \( X \), then

\(^{14}\) \( (S, T) \) has a point of coincidence.

53
Moreover, if \( T_iT^j = T_jT^i \), \( SiSk = SiSk \) and \( TiSk = SkTi \) for all \( i,j \in \{1,2,\ldots,m\} \) and \( \mathcal{A}_i / 2 = \{1,2,\ldots,p\} \), then (for all \( H \in \mathcal{A} \) and \( f \in \mathcal{G} \)) \( Sk \) and \( Ti \) have a common fixed point.

**Proof.** The conclusion \( \{du\} \) is immediate as 5 and \( T \) satisfy all the conditions of Theorem 4.3.1. Now appealing to componentwise commutativity of various pairs, one can immediately prove that \( TS = ST \) and hence, obviously the pair \((5, T)\) is weakly compatible. Note that all the conditions of Theorem 4.3.1 (for mappings \( S \) and \( T \)) are satisfied ensuring the existence of unique common fixed point, say \( t \). Now one needs to show that \( t \) remains the fixed point of all the component mappings. For this consider

\[
T(T(t)) = ((T_1,T_2,\ldots,T_m)T)_t = (T,T_2,\ldots,T_m)T^t = (T_1,T_2,\ldots,T_{m-1})(T_{m-1})(T_{m-2})\ldots(T_1)T = T_1T_2\ldotsT_mT = T(t).
\]

Similarly, one can show that,

\[
T(S(t)) = Sk(T(t)) = Sk(t), \quad SiSk(t) = SiSk(t) = Sk(t)
\]

and

\[
S(T(t)) = Ti(S(t)) = Ti(t)
\]

which show that \((i = k) Ti(t) \) and \( Sk(t) \) are other fixed points of the pair \((5, T)\). Now appealing to the uniqueness of common fixed points of the pair separately, we get

\[
t = lit = \bigwedge Okt
\]

which shows that \( t \) is a common fixed point of \( Ti \) and \( Sk \) for all \( i \) and \( k \).

By setting \( Ti = Ta = \ldots = T^i = F \) and \( 5i = 5^2 = \ldots = 5p = B \) in Theorem 4.3.2, we deduce the following:

**Corollary 4.3.2.** Let \( F \) and \( B \) be self mappings of a metric space \( (X, d) \) satisfying property \((E.A)\) along with the inequality (4.1) for all distinct \( x,y \in X \). If \( B''(X) \) is a complete subspace of \( X \), then \( F \) and \( B \) have a unique common fixed point provided \( FB = BF \).
By setting $\% = F$ (for all $i$) and $Sk - Ix$ (identity listing) in Theorem 4/1.2, one deduces the following fixed point theorem which can be viewed as a significant of Bryant's theorem (cf. [16]).

**Corollary 4.3.3.** Let $F$ be a self mapping of a metric space $(X, d)$ such that there exists some $m \in \mathbb{N}$ satisfying

$$F(d(F^x, F^y), d(x, y), d(x, F^x), d(y, F^y), d(x, F^y), d(y, F^x)) < 0,$$

for all $x, y \in X$ and $F \in \mathcal{F}$. If $F^2(X)$ is a complete subspace of $X$, then $F$ has a unique fixed point.

### 4.4. Illustrative Examples

In this section, we present two examples illustrating the results proved in Section 4.3. The following examples illustrate the validity of the hypotheses of Theorem 4.3.1 besides establishing its utility over earlier results due to Jungck [70] and others proved for a pair of mappings.

**Example 4.4.1.** Consider $X = [-1,1]$ with the usual metric. Define self mappings $S$ and $T$ on $A^\prime$ as in Example 4.1.1.

Notice that all the mappings are discontinuous even at their unique common fixed point '0' which is their common coincidence point as well. Also the pair $(S, T)$ is commutative at coincidence point '0'. Define a continuous function $F : 3 \rightarrow 3$ such that $F(t_1, t_2, ..., t_e) = t_i - kmQx(t_2, h, U, \{t^x + te, \})$, where $k \in (0,1)$, then one can verify that $F$ satisfies $F_1, F^2$ and $F_3$. By a routine calculation one can also show that inequality (4.1) is satisfied for $fc = ^\prime$.

Example 4.4.1 exhibits that Theorem 4.1.1 due to Jungck [70] cannot be used in this context as all the involved mappings are discontinuous whereas Theorem 4.1.1 requires the continuity of at least one of the involved mappings besides $T(X) \subset S(X)$ which is not met in the present example.

Finally, we show that requirement of completeness of $S(X)$ is essential in Theorem 4.3.1 and cannot be relaxed even if the space $X$ is complete.
**Example 4.4.2.** Let $X = \{0, 1, 1/2, 1/2^2, 1/2^3, \ldots\}$ be a metric space with the usual metric $d(x,y) = |x - y|$ for all $x, y \in X$. Define mappings $T, S : X \rightarrow X$ by $r(0) = 1/2^2, T(1/2^n) = 1/2^{n+2}, S(0) = 1/2$ and $S(1/2^n) = 1/2^{n+^2}$ for $n = 0, 1, 2, \ldots$ respectively. Clearly, pair $(S, T)$ enjoys property $(E.A)$ (e.g. $X_n \rightarrow ^\wedge$). Define a continuous function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ as follows

$$F(h_1, \ldots, h_n) = t_l - a t_l - r^{\wedge^\wedge^\wedge}$$

(with $a = ^\wedge$ and $6 = ^\wedge^\wedge$)

then $F$ satisfies $F_1, F_2$ and $F_3$ (see [122]).

By a routine calculation one can verify that all the conditions of Theorem 4.3.1 are satisfied except the completeness of the subspaces $S(X)$ and $T(X)$. Note that $S$ and $T$ have no point of coincidence. Here it is fascinating to note that in the set up of Theorem 4.3.1 even the completeness of the space cannot ensure the existence of coincidence point as the space $X$ is complete in the present example. Notice that $S$ and $T$ are not continuous at 0.

### 4.5. Implicit Function II

In this section, we define a new class of implicit functions and furnish a variety of examples which include most of the well known contractions of the existing literature besides admitting several new ones. Here it is fascinating to note that some of the presented examples are of nonexpansive type (e.g. Examples 4.5.16 and 4.5.19) and Lipschitzian type (e.g. Examples 4.5.12, 4.5.14 and 4.5.15). Here, it may be pointed out that most of the following examples do not meet the requirements of implicit function due to Popa [122]. In order to describe the present implicit function, let $S$ be the family of lower semi-continuous functions $F : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the following conditions.

1. **(F1)**: $F(t, 0, t, 0, \ldots, t) > 0$, for all $t > 0$,
2. **(F2)**: $F(t, 0, 0, t, 0) > 0$, for all $t > 0$,
3. **(F3)**: $F(t, t, 0, 0, t, t) > 0$, for all $t > 0$.

**Example 4.5.1.** Define $F(t_1, t_2, \ldots, k) \cdot ^\wedge^\wedge^\wedge$ as

$$F(t_1, t_2, \ldots, h) = t_l - A; \max\{2, 3, 4, 5, 6\}, \quad \text{where } k \in [0,1).$$
\( F(t, 0, t, 0, 0, t) = t(f - fc) > 0 \), for all \( t > 0 \),

\( F(t, 0, 0, t, t, 0) = t(l - fc) > 0 \), for all \( t > 0 \),

\( F(t, t, 0, 0, t, t) = f(l - fc) > 0 \), for all \( t > 0 \).

**Example 4.5.2.** Define \( F(\xi, f_2, \ldots, \xi) : \mathbb{R}^5 \to \mathbb{R} \) as

\[
F(\xi, f_2, \ldots, \xi) = \xi - A; \max\{f_2, \xi, M_5, M_6\}, \quad \text{where } k \in [0,1).
\]

\( F(t, 0, t, 0, 0, t) = t(l - fc) > 0 \), for all \( t > 0 \),

\( F(t, 0, 0, f, t, 0) = 0 \), for all \( t > 0 \),

\( F(t, t, 0, 0, t, t) = (l - fc) > 0 \), for all \( t > 0 \).

**Example 4.5.3.** Define \( F(t_1, t_2, \ldots, t_e) : \mathbb{R}^e \to \mathbb{R} \) as

\[
F(t_1, t_2, \ldots, t_e) = t_1 - k\{\max\{t_1, t_3, t_5, t_3t_5, t_4\}\}^\alpha,
\]

where \( k \in [0,1) \) and \( P > 0 \).

**Example 4.5.4.** Define \( F(\xi f_1, f_2, \ldots, f_e) : \mathbb{R}^e \to \mathbb{R} \) as

\[
F(\xi f_1, f_2, \ldots, f_e) = \xi - a\{\max\{t_1, t_1, t_2, \xi, t_3, t_4, t_5, t_6\}\} - 7\max\{t_3, t_4, t_5, t_6\} - 7\max\{t_1, t_2, \xi, t_3, t_4, t_5, t_6\} - 7\max\{t_1, t_2, \xi, t_3, t_4, t_5, t_6\},
\]

where \( a, /3, 7 > 0 \) and \( a + 7 < 1 \).

**Example 4.5.5.** Define \( F(\xi f_1, f_2, \ldots, f_e) : \mathbb{R}^e \to \mathbb{R} \) as

\[
F(\xi f_1, f_2, \ldots, f_e) = \xi - a\{\max\{t_1, t_1, t_2, \xi, t_3, t_4, t_5, t_6\}\} - 7\max\{t_3, t_4, t_5, t_6\} - 7\max\{t_1, t_2, \xi, t_3, t_4, t_5, t_6\} - 7\max\{t_1, t_2, \xi, t_3, t_4, t_5, t_6\},
\]

where \( a, /3, 7 > 0 \) and \( a + 7 < 1 \).

**Example 4.5.6.** Define \( F(t_1, t_2, \ldots, t_e) : \mathbb{R}^e \to \mathbb{R} \) as

\[
F(t_1, t_2, \ldots, t_e) = (1 + a^2)\xi - a\max\{t_3, t_4, t_5, t_6\} - 7\max\{t_3, t_4, t_5, t_6\},
\]

where \( a > 0 \) and \( 3/3 G [0,1) \).

**Example 4.5.7.** Define \( F(i_1, i_2, \ldots, t_e) : \mathbb{R}^e \to \mathbb{R} \) as

\[
F(i_1, i_2, \ldots, t_e) = h - ai_2 - 7\max\{f_3, f_4\} - 7\max\{t_3, t_4, t_5, t_6, t_7\},
\]

where \( a, /3, 7 > 0 \) and \( a + 9/3 + 27 < 1 \).
EXAMPLE 4.5.8. Define $F(t_1,t_2,\cdots,t_e)$ as

$$F(t_1,t_2,\cdots,t_e) = t_1 - \langle \langle \max\{t_2,t_4,t_5,t_6\} \rangle \rangle,$$

where $\langle : \mathbb{R}^n \rightarrow \mathbb{R}$ is an upper semi-continuous function such that $\langle(0) = 0$ and $4(t) < t$ for all $t > 0$.

EXAMPLE 4.5.9. Define $F(t_1,t_2,\cdots,t_e)$ as

$$F(t_1,t_2,\cdots,t_e) = t_1 - \langle \langle \delta_2, \langle \delta_7 U, h, h \rangle \rangle,$$

where $\delta : \mathbb{R}^n \rightarrow \mathbb{R}$ is an upper semi-continuous and nondecreasing function in each coordinate variable such that $\langle(t,u,at,bt,ct) < t$ for each $t > 0$ and $a, 5, c > 0$ with $a + b + c < 3$.

EXAMPLE 4.5.10. Define $F(t_1,t_2,\cdots,t_e)$ as

$$F(t_1,t_2,\cdots,t_e) = t_1 - \langle \langle \phi(t_2,\delta) \rangle \rangle,$$

where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is an upper semi-continuous and nondecreasing function in each coordinate variable such that $\langle(t,u,at,bt,ct) < t$ for each $t > 0$ and $a, 6, c > 0$ with $a + b + c < 3$.

EXAMPLE 4.5.11. Define $F(t_1,t_2,\cdots,t_e)$ as

$$F(t_1,t_2,\cdots,t_e) = \langle \langle \phi(t_3+t_4) \rangle \rangle,$$

where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is an upper semi-continuous and nondecreasing function in each coordinate variable such that $\langle(t,u,at,bt,ct) < t$ for each $t > 0$ and $a, 6, c > 0$ with $a + b + c < 3$.

EXAMPLE 4.5.12. Define $F(t_1,t_2,\cdots,t_e)$ as

$$F(t_1,t_2,\cdots,t_e) = \langle \langle \phi(t_3+t_4) \rangle \rangle,$$

where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is an upper semi-continuous and nondecreasing function in each coordinate variable such that $\langle(t,u,at,bt,ct) < t$ for each $t > 0$ and $a, 6, c > 0$ with $a + b + c < 3$.

EXAMPLE 4.5.13. Define $F(t_1,t_2,\cdots,t_e)$ as

$$F(t_1,t_2,\cdots,t_e) = \langle \langle \phi(t_3+t_4) \rangle \rangle,$$

where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is an upper semi-continuous and nondecreasing function in each coordinate variable such that $\langle(t,u,at,bt,ct) < t$ for each $t > 0$ and $a, 6, c > 0$ with $a + b + c < 3$.
where $Q, \gamma, 7 > 0$ and $\gamma + 7 < 1$.

**Example 4.5.14.** Define $F(t_i,t_2,-\ldots ,t_e) : 31^\gamma ^3\gamma$ as

$$[h, \quad ah + te = o,$$

where $fc > 0$.

**Example 4.5.15.** Define $F(t_i,t_2,\cdots ,t_e) : 3t^4 -> 3t^\gamma$ as

$$[ t_i, \quad i + 1t^4 = 0oTh + t6^40,$$

where $A; > 0$.

**Example 4.5.16.** Define $F(t_i,t_2,\cdots ,t_e) : 3f^t ^3t$ as

**Example 4.5.17.** Define $F(t_i,t_2,\cdots ,t_e) : 3t^9 ^\gamma ^3$ as

$$F(tu2. \cdots ,te) = tl- atl - ^\gamma _._^-^\gamma 2^\gamma 2.$$

where $a, \gamma > 0$ and $a + P < 1$.

**Example 4.5.18.** Define $F(t_i,t_2,\cdots ,t_e) : 3t^7 -> 3t$ as

**Example 4.5.19.** Define $F(t_i,t_2,\cdots ,t_e) : 3t^7 -> 3t$ as

$$F(t_i, t2,\cdots , te) = tl- alti - PUUU - itlu - vtstl$$

where $a, 5, 7, 7; > 0$ and $a + 7 + r / < 1$.

Since verification of requirements (F1,F2 and F3) for Examples 4.5.3-4.5.20 are easy, hence details are not included.
4.6. RESULTS VIA COMMON PROPERTY (E.A)

We begin with the following observation.

**Lemma 4.6.1.** Let \( A, B, S \) and \( T \) be self mappings of a metric space \( (X, d) \) such that

\[
\text{(dis)} \quad \text{the pair } \{A, S\} \text{ (or } \{B, T\} \text{) satisfies the property (E.A)},
\]

\[
\text{(die)} \quad A(X) \subseteq T(X) \quad \text{(or } B(X) \subseteq S(X)\text{)},
\]

\[
\text{(dn)} \quad \text{for all } x, y \in X \text{ and } F \subseteq G \subseteq \mathbb{F} \quad F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)) < 0. \quad (4.2)
\]

Then the pairs \( \{A, S\} \) and \( \{B, T\} \) satisfy the common property (E.A).

**Proof.** if the pair \( \{A, S\} \) enjoys property (E.A), then there exists a sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t, \quad \text{for some } t \in X.
\]

Since \( A(X) \subseteq T(X) \), hence for each \( \{x_n\} \) there exists \( \{y_n\} \) in \( X \) such that \( Ax_n = Tyn \). Therefore, \( \lim_{n \to \infty} Ty_n = \lim_{n \to \infty} Ax_n = t \). Thus, in all we have \( Ax_n \to t, Sx_n \to t \) and \( Tyn \to t \). Now, we assert that \( 5y_n \to f \). If not, then using (4.2), we have

\[
F(d(Ax_n, By_n), d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), d(Sx_n, By_n), d(Ty_n, Ax_n)) < 0.
\]

which on making \( n \to \infty \), reduces to

\[
F(d(t, By_n), 0, 0, d(By_n, t), d(t, Ty_n), 0) < 0
\]

a contradiction to \( \{F2\} \). Hence \( By_n \to t \) which shows that the pairs \( \{A, S\} \) and \( \{B, T\} \) satisfy the common property (E.A).

**Remark 4.6.1.** The converse of Lemma 4.6.1 is not true in general. For a counter example, one can see Example 4.7.1.

Now, we state and prove our main result for two pairs of weakly compatible mappings satisfying an implicit function.

**Theorem 4.6.1.** Let \( A, B, S \) and \( T \) be self mappings of a metric space \( (X, d) \) which satisfy inequality (4.2). Suppose that
(dis) the pairs \( \{A, S\} \) and \( \{B, T\} \) enjoy the common property \( \{E.A\} \),

\( (dig) \) \( S(X) \) and \( T(X) \) are closed subsets of \( X \).

Then the pair \( \{A, S\} \) as well as \( \{B, T\} \) have a coincidence point. Moreover, \( A, B, S \) and \( T \) have a unique common fixed point provided both the pairs \( \{A, S\} \) and \( \{B, T\} \) are weakly compatible.

\textbf{Proof.} since the pairs \( \{A, S\} \) and \( \{S, T\} \) enjoy common property \( \{E.A\} \), then there exist two sequences \( \{x^n\} \) and \( \{y^n\} \) in \( X \) such that

\[
\lim_{n \to \infty} Ax^n = \lim_{n \to \infty} Sx^n = \lim_{n \to \infty} By^n = \lim_{n \to \infty} Ty^n = t, \quad \text{for some } t \in X.
\]

If \( S(X) \) is a closed subset of \( X \), then \( \lim_{n \to \infty} Sx^n = t \in S(X) \). Therefore, there exists a point \( u \in X \) such that \( Su = t \). Now we assert that \( Au = Su \). If not, then using (4.2), we have

\[
F(d(Au, By^n), d(Su, Ty^n), d(Au, Su), d(By^n, Ty^n), d(Su, By^n), d(Ty^n, Au)) < 0
\]

which on making \( n \to \infty \), reduces to

\[
F(d(Au, t), d(Su, t), d(Au, Su), d(t, t), d(Su, t), d(t, Au)) < 0
\]

or

\[
F(d(Au, Su), 0, d(Au, Su), 0, d(Su, Au)) < 0
\]

a contradiction to (Fi). Hence \( Au = Su \). Therefore, \( u \) is a coincidence point of the pair \( \{A, S\} \).

If \( T(X) \) is a closed subset of \( X \), then \( \lim_{n \to \infty} Ty^n = t \in T(X) \). Therefore, there exists a point \( w \in X \) such that \( Tw = t \). Now we assert that \( Bw = Tw \). If not, then again using (4.2), we have

\[
F(d(Axn, Bw), d(Sxn, Tw), d(Axn, Sxn), d(Bw, Tw), d(Sxn, Bw), d(Tw, Axn)) < 0
\]

which on making \( n \to \infty \), reduces to

\[
F(d(t, Bw), d(t, Tw), d(t, t), d(Bw, Tw), d(t, Bw), d(Tw, t)) < 0
\]

or

\[
F(d(Tw, Bw), 0, 0, d(Bw, Tw), d(Tw, Bw), 0) < 0
\]
a contradiction to (F2). Hence $Bw = Tw$, which shows that $lo$ is a coincidence point of the pair $(B, T)$.

Since the pair $(A, S)$ is weakly compatible and $Au = Su$, hence $At = ASu = SAu = St$. Now we assert that $t$ is a common fixed point of the pair $(A, S)$. Suppose that $At \neq t$, then using $(4.2)$, we have

$$F(d(At, Bw), d(St, Tw), d(At, St), d(Bw, Tw), d(St, Bw), d(Tw, At)) < 0$$

or

$$F(d(At, t), d(At, t), 0, 0, d(At, t), d(t, At)) < 0$$
a contradiction to (F2).

Also the pair $(S, T)$ is weakly compatible and $Bw = Tw$, then $Bt = BTw = TBw = Tt$. Suppose that $Bt \neq t$, then using $(4.2)$, we get

$$F(d(Au, Bt), d(Su, Tt), d(Au, Su), d(m, Tt), d(Su, Bt), d(Tt, Au)) < 0$$

or

$$F(d(Bt, t), d(Bt, t), 0, 0, d(Bt, t), d(t, Bt)) < 0$$
a contradiction to (F3). Therefore, $Bt = t$ which shows that $t$ is a common fixed point of the pair $(B, T)$. Hence $t$ is a common fixed point of both the pairs $(A, S)$ and $(B, T)$. Uniqueness of common fixed point is an easy consequence of inequality $(4.2)$ (in view of condition (F3)). This completes the proof.

**Theorem 4.6.2.** The conclusions of Theorem 4.6.1 remain true if the condition (dig) of Theorem 4.6.1 is replaced by the following.

(raf2o) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

As a corollary of Theorem 4.6.2, we can have the following result which is also a variant of Theorem 4.6.1.

**Corollary 4.6.1.** The conclusions of Theorems 4.6.1 and 4.6.2 remain true if the conditions (dig) and (raf2o) are replaced by the following.

(raf2i) $A(X)$ and $B(X)$ are closed subsets of $X$ provided $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

**Theorem 4.6.3.** Let $A, S, S$ and $T$ be self mappings of a metric space $(X, d)$ satisfying inequality $(4.2)$. Suppose that
(d22) the pair \(\{A,S\}\) (or \(\{B,T\}\)) has property \((E.A)\),

(\(^23\)) \(A(X) \subset T(X)\) (or \(B(X) \subset S(X)\)),

(d24) \(S(X)\) (or \(T(X)\)) is a closed subset of \(X\).

Then the pair \(\{A, S\}\) as well as \(\{B, T\}\) have a coincidence point each. If the pairs \(\{A, S\}\) and \(\{B, T\}\) are weakly compatible, then \(A, B, S\) and \(T\) have a unique common fixed point.

**Proof**, in view of Lemma 4.6.1, the pairs \((A^S)\) and \((B,T)\) satisfy the common property \((\mathcal{E}^{\wedge}4)\), i.e. there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} 5x_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = t \in X.
\]

If \(S(X)\) is a closed subset of \(X\), then on the lines of Theorem 4.6.1, the pair \(\{A, S\}\) has coincidence point, say \(u\), i.e. \(Au = Su\). Since \(\{4(X) \subset T(X)\}\) and \(\exists M \in A(X)\), there exists \(w \in X\) such that \(Au = Tiu\). Now we assert that \(Bw = Tw\). If not, then using (4.2), we have

\[
F(d(Ax_n, Bw), d(Sx_n, Tw), d(Ax_n, Sx_n), d(Bw, Tw), d(Sx_n, Bw), d(Tw, Ax_n)) < 0
\]

which on making \(n \to \infty\), reduces to

\[
F(d(t, Bw), d(t, Tw), d(t, t), d(Bw, Tw), d(t, Bw), d(Tw, t)) < 0
\]

or

\[
F(d(Tw, Bw), 0,0, d(Bw,Tw), d(Tw, Bw), 0) < 0
\]

a contradiction to \((F2)\). Hence \(Bw = Tw\), which shows that \(w\) is a coincidence point of the pair \(\{B,T\}\). Rest of the proof can be completed on the lines of the proof of Theorem 4.6.1.

By choosing \(A, B, S\) and \(T\) suitably, one can deduce corollaries for a pair as well as for a triod of mappings. The detail of two possible corollaries for a triod of mappings are not included. As a sample, we outline the following natural result for a pair of self mappings.

**Corollary 4.6.2.** Let \(A\) and \(S\) be self mappings of a metric space \((X,d)\). Suppose that
the pair \((A, B)\) has property \((E.A)\),

for all \(x, y \in X\) and \(F \in G\)

\[
F(d(Ax, Ay), diSx, Sy), d(Ax, Sx), d(Ay, Sx), (5y, Ay), d(5y, Ax)) < 0 \quad (4.3)
\]

\((S)\) is a closed subset of \(X\).

Then \(A\) and \(S\) have a coincidence point. Moreover, if the pair \((A, B)\) is weakly compatible, then \(A\) and \(S\) have a unique common fixed point.

**Corollary 4.6.3.** The conclusions of Theorem 4.6.1 remain true if inequality (4.2) is replaced by one of the following contraction conditions. For \(eHx, y \in X\),

\[
d(Ax, By) < k \max\{d(5x, Ty), d(Ax, Sx), d(Ay, Ty), d(Sx, By), d(Ty, Ax)\},
\]

where \(k \in [0, 1)\).

\[
d(Ax, By) < k \max\{d(Sx, Ty), d(Ax, Sx), d(Ax, Sx), d(Sx, By), d(Ty, Ax)\},
\]

where \(k \in [0, 1)\).

\[
d(Ax, By) < a \max\{d(5x, Ty), d(Ax, Sx), d(Ax, Sx), d(Ax, Sx), d(Ty, Ax)\}
\]

where \(a \in [0, 1)\) and \(\gamma > 0\).

\[
d(Ax, By) < \gamma \max\{d(5x, Ty), d(Ax, Sx), d(Ax, Sx), d(Ax, Sx), d(Ty, Ax)\}
\]

where \(a, \gamma, 7 > 0\) and \(a + 7 < 1\).
\[ P \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}, \]

where \( a > 0 \) and \( \beta \in [0,1) \).

\[ d(Ax, By) < ad(Sx, Ty) + 7\max\{d(ylx, Sx), d(By, Ty)\} + 7\max\{d(Ax, Sx) + d(By, Ty), d(Sx, By) + d(Ty, Ax)\}, \]

where \( a, \gamma, 7 > 0 \) and \( a + \gamma + 27 < 1 \).

\[ d(Ax, By) < (\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}), \]

where \((^\gamma : \mathbb{R}^n \to \mathbb{R}\) is an upper semi-continuous function such that \((^\gamma)(0) = 0\) and \((^\gamma)(t) < t\) for all \( t > 0 \).

\[ d(Ax, By) < (\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}), \]

where \((: \mathbb{R}^n \to \mathbb{R}\) is an upper semi-continuous and nondecreasing function in each coordinate variable such that \((\cdot)(t, t, at, bt, ct) < t\) for each \( t > 0 \) and \( a, b, c > 0 \) with \( a + b + c < 3 \).

\[ d(Ax, By) < (\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}), \]

where \((^\gamma : \mathbb{R}^n \to \mathbb{R}\) is an upper semi-continuous and nondecreasing function in each coordinate variable such that \((^\gamma)(t, t, at, bt, ct) < t\) for each \( t > 0 \) and \( a, b, c > 0 \) with \( a + b + c < 3 \).

In following contraction conditions, we denote \( D = d(Ax, Sx) + d(By, Ty) \) and \( D_i = d(Sx, By) + d(Ty, Ax) \).

\[ (dJ) d(Ax, By) < (\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}), \]

where \( (dJ : \mathbb{R}^n \to \mathbb{R}\) is an upper semi-continuous and nondecreasing function in each coordinate variable such that \((dJ)(t, t, at, bt, ct) < t\) for each \( t > 0 \) and \( a, b, c > 0 \) with \( a + b + c < 3 \).

\[ (dJ) d(Ax, By) < (\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}), \]

where \( (dJ : \mathbb{R}^n \to \mathbb{R}\) is an upper semi-continuous and nondecreasing function in each coordinate variable such that \((dJ)(t, t, at, bt, ct) < t\) for each \( t > 0 \) and \( a, b, c > 0 \) with \( a + b + c < 3 \).
where \( p > l \) and \( fc > 0 \).

\[
\begin{align*}
&\text{if } Di^O \\
\end{align*}
\]

where \( a, l3 > 0 \) and \( l? + 7 < 1 \).

\[
\begin{align*}
&\text{if } Di = 0, \\
\end{align*}
\]

where \( A; > 0 \).

\[
\begin{align*}
&\text{if } D = 0 \text{ or } Di = 0, \\
\end{align*}
\]

where \( a, l? \in [0,1) \).

\[
\begin{align*}
&\text{if } D = 0 \text{ or } Di = 0, \\
\end{align*}
\]

where \( a, l? > Q \) and \( a + y5 < 1 \).

\[
\begin{align*}
&\text{if } D = 0 \text{ or } Di = 0, \\
\end{align*}
\]

where \( a, l3 > Q \) and \( a + y5 < 1 \).
Proof. Proof follows from Theorem 4.6.1 and Examples 4.5.1-4.5.20.

Remark 4.6.2. Corollaries corresponding to contraction conditions (4.28) to (4.47) are new results as these never require any conditions on containment of ranges. Some contraction conditions (e.g. 4.28, 4.31, 4.33—4.42) in above corollary are well known and generalized relevant results from [2,21,23-25,34,36,46,50,53,57-59,61,69,72,83,89, 94,99,105,138,145] while some others are new ones (e.g. 4.29, 4.30, 4.32, 4.43 - 4.47).

As an application of Theorem 4.6.1, we have the following result for four finite families of self mappings.

Theorem 4.6.4. Let \( \{A_1, A_2, \ldots, A_m\}, \{B_1, B_2, \ldots, B_p\}, \{S_1, S_2, \ldots, S_n\} \) and \( \{T_1, T_2, \ldots, T_q\} \) be four finite families of self mappings of a metric space \( (X, d) \) with \( A = A_1 A_2 \cdots A_m, B = B_1 B_2 \cdots B_p, S = S_1 S_2 \cdots S_n, \) and \( T = T_1 T_2 \cdots T_q, \) satisfying condition (4.2),

(4.43) the pairs (4.45) and \( (B, T) \) share common property \( \{E.A\}, \)

(4.49) \( S(X) \) and \( T(X) \) are closed subsets of \( X. \)

Then the pair \( \{A, S\} \) as well as \( \{B, T\} \) has a coincidence point.

Moreover, if \( A_i A_j = A_j A_i, B_k B_i = B_i B_k, S_r S_s = S_s S_r, T_i T_u = T_u T_i \) \( AB_k = B_k A_i \) and \( S_r T_t = T_t S_r \) for all \( i, j \in I = \{1, 2, \ldots, m\}, k, l \in H = \{1, 2, \ldots, p\}, r, s \in S = \{1, 2, \ldots, n\} \) and \( i, u \in A = \{1, 2, \ldots, q\}, \) then (for all \( i \in I, k \in H, r \in r \) and \( t \in T^\wedge \)) \( A_i B_k, S_r, \) and \( T_t \) have a common fixed point.

Proof. Proof follows on the lines of a result due to Imdad et al. [63, Theorem 2.2].

By setting \( A_1 = A_2 = \ldots = A_m = G, B_1 = B_2 = \ldots = B_p = H, S_1 = S_2 = \ldots = S_n = I, \) and \( T_1 = T_2 = \ldots = T_q = J \) in Theorem 4.6.4, we deduce the following.

Corollary 4.6.4. Let \( G, H, I \) and \( J \) be self mappings of a metric space \( (X, d) \) such that the pairs \( (G^*, I^*) \) and \( (H^*, J^*) \) have common property \( (E^\wedge, A^\wedge) \) and also satisfy the condition

\[
Fidi G^\wedge x, H^\wedge y, dirx, J^\wedge y), d(G^\wedge x, P_x), d(H^\wedge y, J^\wedge y),
\]

67
for all $x, y \in X$ and $F, G, H, I$ where $m, n, p$ and $q$ are fixed positive integers. If $P(X)$ and $J^\wedge(X)$ are closed subsets of $X$, then $G, H, I$ and $J$ have a unique common fixed point provided $GI = IG$ and $HJ = JH$.

**Remark 4.6.3.** By restricting four families as $\{AuA2\}, \{Bj, B2\}, \{Si\}$ and $\{rj$ in Theorem 4.6.4, we deduce a substantial but partial generalization of the main results of Imdad and Khan [57, 58] as such a result will deduce stronger commutativity condition besides relaxing continuity requirements and weakening completeness requirement of the space to the closedness of subspaces.

**Remark 4.6.4.** Corollary 4.6.4 is a slight but partial generalization of Theorem 4.6.1 as the commutativity requirements (i.e. $GI = IG$ and $HJ = JH$) in this corollary are stronger as compared to weak compatibility in Theorem 4.6.1.

**Remark 4.6.5.** Results similar to Corollary 4.6.3 can be derived from Theorems 4.6.2-4.6.3 and Corollaries 4.6.2 and 4.6.4. For the sake of brevity, we have not included the details.

### 4.7. Illustrative Examples

Now we furnish examples demonstrating the validity of the hypotheses and degree of generality of our results presented in Section 4.6 over the majority of previously known results proved till date with some possible exceptions.

**Example 4.7.1.** Consider $X = [-1, 1]$ equipped with the usual metric. Define self mappings $A, B, S$ and $T$ on $X$ as

- $A(-1) = A1 = 3/5$, $Ax = x/4$, $-1 < x < 1$,
- $B(-1) = Bl = 3/5$, $Bx = -x/4$, $-1 < x < 1$,
- $S(-1) = 1/2$, $Sx = x/2$, $-1 < x < 1$, and $S1 = -1/2$, and
- $T(-1) = -1/2$, $Tx = -x/2$, $-1 < x < 1$, and $T1 = 1/2$.

Consider sequences $\{x^n\} = ^\wedge$ and $\{y^n\} = ^\wedge$ in $X$. Clearly,

\[
\lim_{n \to \infty} Axn = \lim_{n \to \infty} Sx^n = \lim_{n \to \infty} By^n = \lim_{n \to \infty} Ty^n = 0
\]
which shows that pairs \( \{A,S\} \) and \( \{B,T\} \) satisfy the common property \((E.A)\).

Define a continuous implicit function \( F : \mathbb{R}^n \to \mathbb{R}^n \) such that 
\[
F(t_1, t_2, \ldots, t_n) = t_1 - k \max\{t_2, h, U, h, tf\}
\]
where \( k \in [0,1) \) and \( F \in \mathcal{F} \). By a routine calculation, one can verify the inequality (4.2) with \( \epsilon = 1 \). Also, 
\[
A(X) = B(X) = \{\emptyset\} \cup \{\neq f, \neq \} \quad S(X) = T(X) = \{=^\neq, \}.
\]
Therefore, all the conditions of Theorem 4.6.1 are satisfied and 0 is a unique common fixed point of the pairs \((-4,5)\) and \((B,T)\) which is their coincidence point as well.

Here it is worth noting that none of the theorems (with some possible exceptions) can be used in the context of this example as Theorem 4.6.1 never requires any condition on the containment of ranges of the involved mappings while the completeness condition is replaced by the closedness of subspaces. Moreover, the continuity requirements of involved mappings are completely relaxed whereas all earlier theorems (prior to 1997) require the continuity of at least one involved mapping.

Now, we furnish an example which presents a situation applicable to Theorems 4.6.1, 4.6.2 and 4.6.3.

**Example 4.7.2.** Consider \( X = [2,20] \) equipped with the usual metric. Define self mappings \( A, B, S \) and \( T \) on \( X \) as
\[
A(x) = \begin{cases} 2, & x \in \{2\} \cup (5,20], \\ 4, & 2 < x < 5, \\ 0, & x > 5. 
\end{cases}
\]
\[
B(x) = \begin{cases} 2, & x \in \{2\} \cup (5,20], \\ 3, & 2 < x < 5, \\ 0, & x > 5. 
\end{cases}
\]
\[
S(x) = \begin{cases} 8, & 2 < x < 5, \\ (x+1)/3, & x > 5. 
\end{cases}
\]
\[
T(x) = \begin{cases} 12 - x, & 2 < x < 5, \\ x - 3, & x > 5. 
\end{cases}
\]

Clearly, both the pairs \( \{A,S\} \) and \( \{B,T\} \) satisfy the common property \((E.A)\) as there exist two sequences \( \{x_n, \neq 5 + \} \), \( \{y_n, \neq 5 + \} \) \( C \in \) such that
\[
\lim_{n \to \infty} A x_i = \lim_{n \to \infty} 5 x_n = \lim_{n \to \infty} B y_n = \lim_{n \to \infty} T y_n = 2.
\]

Also 
\[
A(X) = \{2,4\} \quad C \quad [2,17] = T(X) \quad \text{and} \quad B(X) = \{2,3\} \quad C \quad [2,7] \cup \{8\} = S(X).
\]
Define 
\[
F(t_1, t_2, \ldots, t_n) \quad U \quad ^\infty \quad \text{with} \quad ^\infty 3 + ^\infty 4 \quad 7^\infty 0
\]
as
\[
F(t,u,t_2, \ldots, t_n) = \text{if} \quad t - j \quad ^\infty \quad \frac{\text{tl} + \text{tl}}{U} = \text{ih} + \text{te}
\]
where \( a, /3, 7 > 0 \) with at least one is nonzero and \( /3 - i - 7 < 1 \)
By a routine calculation one can verify that contraction condition (4.2) is satisfied for \( a = 7 = i \) and \( P = i^* \). If \( x, y \in \{2\} \cup (5,20] \), then \( d(Ax, By) = 0 \) and verification is trivial. If \( x \in (2,5] \) and \( y > 5 \), then
\[
\begin{align*}
\text{ad}(Sx, Ty) + P &= \frac{\langle f(Sx, Ax) \rangle + \langle P(Ty, By) \rangle}{d(Sx, Ax) + d(Ty, By)} + \frac{jd(Sx, By) + d(Ty, Ax)}{6 + |y-7|} \\
&= \frac{4^4 + \sqrt{|y-5|}}{4 + |y-5|} + \frac{4 + |y-7|}{6 + |y-7|}
\end{align*}
\]
\[
\text{if } y \in (5,7,] \quad \text{and} \quad \text{if } y \in (7,11] \]

Similarly, one can verify the other cases. One may note that the pairs \( \{A, S\} \) and \( \{B, T\} \) commute at \( 2 \) which is their common coincidence point. All the needed pair-wise commutativity at coincidence point \( 2 \) are immediate. Thus all the conditions of Theorems 4.6.1, 4.6.2 and 4.6.3 are satisfied and \( 2 \) is the unique common fixed point of \( A, B, S \) and \( T \). Here one may notice that all the mappings in this example are even discontinuous at their unique common fixed point \( 2 \).

Example 4.7.2 may lead an impression that Theorems 4.6.1, 4.6.2 and 4.6.3 are not different results. In what follows, we show that these results can be situationally useful, i.e. there do exist situations when one theorem is applicable whereas others are not. In order to substantiate this view point, we furnish the following examples.

**Example 4.7.3.** In the setting of Example 4.7.2 retain the same \( A, B, T \) and implicit function \( F \) and modify \( S \) as follows.

\[
S2 = 2, \quad S20 = 2, \quad Sx = S, \quad 2 < x < 5, \quad Sx = (x + 1)/3, \quad 5 < x < 20.
\]

Clearly, \( S(X) = \{2,7\} \cup \{8\} \) which is not a closed subset of \( X \). Here, Theorems 4.6.2 and 4.6.3 are applicable but not Theorem 4.6.1.

**Example 4.7A.** In the setting of Example 4.7.2 retain the same \( A, B \) and implicit function \( F \) and modify \( S \) and \( T \) as follows.
52 = 2, 520 = 2, \ Sx = 8, 2 < x < 5, \ Sx = (x + 1)/3, 5 < x < 20, \
r2 = 2, \ T20 = 2, \ Tx = 12 + x, 2 < x < 5, \ Tx = x - 3, 5 < x < 20.

Clearly, \( S(X) = [2, 7) \cup \{8\} \) and \( T(X) = [2, 17) \) which are not closed subsets of \( X \). Here Theorem 4.6.2 is applicable but not Theorems 4.6.1 and 4.6.3.