CHAPTER - FOUR

COMPRESSOR IN AN ABELIAN GROUP

1. Introduction: If $S$ be a subset of a group $G$, then the set of all elements $x$ in $G$, such that $xS = Sx$ is called the normalizer of $S$ in $G$. Since the normalizer of any subset of an abelian group is the group itself, the notion when associated with abelian groups reduces to be trivial one. We, therefore, utilise the concept of self compression in defining an analogous notion of 'compressor' of a subset in a group, which plays an important part in abelian groups specially. The set of all those elements of a group $G$, with respect to which a given set $S$ is self compressed, we call to be the 'compressor of $S$ in $G$' and denote it by $Q_g(S)$. This chapter studies the power of this concept in abelian groups. We first show that the compressor of any subset in a group is a subgroup of the group and find out some fundamental properties such as: 'The compressor of a set and of any of its $C$-transforms in the same', 'The compressor of a subset and its compliment is the same', 'Intersection (or union) of the compressors of two subsets is contained in the compressor of the intersection (or union) of those subsets', 'The product of compressors of two subsets is contained in the compressor of product of subsets', 'Compliment of the compressor of a subset is the compliment of the compressor of the compliment of the subset', 'The compressor of any subgroup of index 2, in a group, is the whole group but not conversely', 'Any two finite subgroups containing no elements of order two such that the compressor of one contains the other, coincide with each other' etc. We note that the isomorphic image of the compressor of a subset in a group under an isomorphism of the group is the
compressor of the image subset in the image group, and also prove that in a direct product the direct product of the compressors of subsets of direct factors in respective direct factors is the compressor of the product of these subsets in the direct product group. We have established that the smallest c.s.c-subgroup of the compressor of a subgroup is contained in the compressor of the smallest c.s.c-subgroup of the subgroup. We further show that in a cyclic group, the compressor of the anticenter of a subgroup coincides with the anticenter of the compressor of the subgroup, but not in general. We have also investigated that any subset is the union of all distinct cosets of the smallest c.s.c-subgroup of its compressor with respect to all elements of the subset, and also determine the conditions under which the compressor of a set in a group given as a union of cosets of a subgroup \( H \) of the smallest c.s.c-subgroup of the group coincides with the maximal subgroup of the group, whose smallest c.s.c-subgroup be the given subgroup \( H \). We introduce the concept of c-power of an element of a group in a subgroup of the group and utilise it in obtaining a criterion of equality of a subgroup with its compressor. Finally, since compressor of a subgroup contains the subgroup so the compressor of compressor contains the compressor and in this way we get an ascending chain of subgroups in a group; and obtain the circumstances when it breaks off.
2. Definition And Basic Properties of Compressor:

We define below the notion of compressor of a subset in a group, and give some criteria for an element of a group to belong to the compressor of a finite or infinite subset or of a subgroup. We find out that the compressor of any subset in a group is a subgroup. We also observe that the compressor of a subset and of its compliment in a group is the same, and note that every non identity element of the compressor of a subset of two elements including identity and an element of order other than two is of order two only.

Def. 4.1 - For a subset $S$ of a group $G$, the set of all elements $x \in G$ such that $xSx = S$ is called 'compressor' of $S$ in $G$, in symbols $C_G(S)$.

The following observations can be easily checked.

(i) For any subgroup $H$ of $G$, $H \subseteq C_G(H)$

(ii) The compressor of any element of $G$ is $O_2$, the subgroup of all elements of order 2 in $G$.

(iii) A subset $S$ of $G$ is completely self compressed if and only if $C_G(S) = G$.

(iv) The Compressor of $G$ in $G$ is the group $G$ itself.

(v) For any subset $S$ of $G$, $C_G(S) \supseteq O_2$, the subgroup of all elements of order 2 in $G$. 

(vi) If \( H \) be any subgroup of \( G \), which does not contain the subgroup of all elements of order 2 in \( G \), then, for any subset \( S \) of \( G \), \( C_G(S) \neq H \).

Theorem 4.1 - Let \( S \) be a finite subset of a group \( G \), then \( x \in C_G(S) \) if and only if to every \( s \in S \) there exists \( s^i \in S \) such that \( s = s^i \).

(Proof is immediate from theorem 2.1.)

Remark: By our remark in chapter 2 after theorem 2.1, it is immediate that the above theorem is not true in general for infinite subsets; however, if instead of \( S \), we take an arbitrary subgroup \( H \) of \( G \), then the theorem holds in view of theorem 1.8. We have, in general, the following theorem:

Theorem 4.2 - If \( S \) be an arbitrary subset of a group \( G \), then \( x \in C_G(S) \) if and only if \( x^i \in C_G(S) \) for all \( i \in I \).

(Proof follows from theorem 2.2).

Theorem 4.3 - Let \( S \) be any subset of a group \( G \), then \( C_G(S) \) is a subgroup and is the largest subset with respect to which \( S \) is self compressed.

Proof. For any \( x, y \in C_G(S) \),

\[
    xSx = S \text{ and } ySy = S
\]

\[
    \implies xy^{-1}Sxy^{-1} = y^{-1}xSx \cdot y^{-1} = y^{-1}S \cdot y^{-1} = y^{-1}ySy \cdot y^{-1} = S
\]
Hence $C_G(S)$ is a subgroup of $G$.

Finally, if $K$ be any subset of $G$ with respect to which $S$ is self compressed, then evidently

$$K \subseteq C_G(S)$$

This completes the proof.

Theorem 4.4 - For any subset $S$ of a group $G$,

$$C_G(S) = C_G(G - S)$$

where $G - S$ is the compliment of $S$ in $G$.

Proof. For any $x \in C_G(S)$,

$$S_x = S \implies (G - S)_x = G - S \quad (\text{Theorem 2.5})$$

$$\implies C_G(S) \subseteq C_G(G - S)$$

Similarly, we can see that

$$C_G(G - S) \subseteq C_G(G - (G - S))$$

or $$C_G(G - S) \subseteq C_G(S)$$

Hence

$$C_G(S) = C_G(G - S)$$

This completes the theorem.

Theorem 4.5 - Let $S = \{g, e\}$ be a subset in a group $G$, then if $x(\neq e) \in C_G(S)$, either $O(x) = 2$ or $O(g) = 2$. 
Proof. Let \( x(\neq e) \in C_G(S) \), then
\[
xSx = S
\]
\[\Rightarrow xgx = g \text{ or } xgx = e\]

If \( xgx = g \), then evidently, \( O(x) = 2 \).

In other case
\[
xgx = e
\]
\[\Rightarrow xex = g\]

Thus
\[
x^2 = g \text{ and also } x^{-2} = g
\]
\[\Rightarrow g^2 = e\]

i.e. \( O(g) = 2 \)

Hence the theorem is complete.

3. Relation Between A Set And Its Compressor:

In this section, we show that if a subgroup \( H \) of a group \( G \) be compressor of a subset, then the subset is the union of all distinct cosets of \( H^* \) with respect to elements of the subset; on the other hand, if a subset be a union of cosets of a subgroup \( H_1^* \) of \( G^* \), then the compressor of the subset is the maximal subgroup of \( G \) whose smallest c.s.c-subgroup is \( H_1^* \), provided that to every element of compressor, there exists at least one coset of \( H_1^* \) in the representation of the subset.
which is self compressed with respect to that element.

Theorem 4.6-(Correspondence theorem) - If for an arbitrary subset $S$ of a group $G$, $C_G(S) = H$, then $S = \bigcup s_i H^*$ where $s_i$'s are all distinct coset representatives of $H^*$ in $S$. Conversely, if for a subgroup $K^*$ of $G^*$, we have $S = \bigcup s_i K^*$ for $s_i$'s in $S$ then $H'$, the largest subgroup for which $H'^* = K^*$ coincides with $C_G(S)$, provided for every $x$ in $C_G(S)$, $(s_i K^*)_x = s_i K^*$ for some $s_i K^*$.

Proof. Let $C_G(S) = H$

$$\Rightarrow hSh = S \quad \forall \ h \in H$$

$$\Rightarrow sh^2 \in S \quad \forall \ s \in S, \ h \in H$$

$$\Rightarrow sH^* \subseteq S \quad \forall \ s \in S$$

$$\Rightarrow S = \bigcup_{s \in S} sH^* = \bigcup_{s \in S} s_i H^*$$

where $s_i$'s are all distinct representatives in $S$.

Conversely, let $S = \bigcup s_i K^*$. We have for any $h' \in H'$

$$(s_i K^*)_h' = h'(s_i K^*) h'$$

$$= s_i (h'^2 K^*)$$

$$= s_i K^* \quad \text{since } h'^2 \in H'^* = K^*$$

$$\Rightarrow (S)_{h'} = \bigcup_{s_i \in S} (s_i K^*)_h'$$

$$= \bigcup_{s_i \in S} s_i K^* = S \quad \text{(Theorem 2.5)}$$

$$\Rightarrow H' \subseteq C_G(S)$$
Also, if \( x \in C_G(S) \) be arbitrary, then by hypothesis

\[
x(s_1 K^*)x = s_1 K^* \quad \text{for some } s_1 K^*
\]

\[
\implies x^2 \in K^*
\]

Thus since \( H^* \) is the largest subgroup for which \( H^* = K^* \), we have

\[
x \in H^*
\]

\[
\implies C_G(S) \subseteq H^*
\]

Hence

\[
C_G(S) = H^*
\]

This proves the theorem.

Cor. 4.1 - For a subset \( S \) and a subgroup \( H \) of a group \( G \),

\[
(C_G(S))^* = H^* \implies H \subseteq C_G(S).
\]

(Proof follows immediately from the fact that \( S = \cup s_i H^* \) for \( s_i \)'s in \( S \)).

Cor. 4.2 - For any two subsets \( S_1, S_2 \) of a group \( G \), \( C_G(S_1) = C_G(S_2) = H \) (say) implies \( S_1 \cap S_2 = \cup s_i H^* \) where \( s_i \)'s are only those elements of \( S_1 \) or \( S_2 \) for which there exist \( s_j \)'s \( \in S_2 \) or \( S_1 \) respectively such that \( s_j \in s_i H^* \) and \( S_1 \cup S_2 = \cup s_i H^* \) where \( s_i \)'s \) are distinct representatives of cosets of \( H^* \) in \( S_1 \cup S_2 \).

The following example illustrates and explains the imposition of the additional condition in the converse statement of the theorem.
Example. Let \( G = \{a\} \) be the cyclic group of order 8. Evidently \( e \) is a subgroup of \( G^* \) and the largest subgroup \( H \) of \( G \) for which \( H^* = e \), is \( H = \{a^4, e\} \). Take \( S = \{a, a^5\} \) a subset in \( G \). We have

\[
S = \{aH^* \cup a^5H^*\}
\]

But, \( C_G(S) = \{a^2, a^4, a^6, e\} \neq H \)

Hence our condition is necessary in general.

4. Properties Of Compressors Of Subgroups:

In this section, we derive several properties of compressors of subgroups. We first determine a condition, under which the compressor of any subgroup of a group coincides with the subgroup itself, and further show that the compressor of a subgroup contains the compressor of any subgroup of the subgroup. We observe that the compressor of a subgroup whose index in its group is 2, coincides with the whole group but not conversely. It is interesting to note that if for any two finite subgroups containing no element of order 2 in a group, the compressor of one contains the other subgroup, the subgroups must coincide. Finally, we establish that the compressor of any subgroup of a group is the set theoretical union of all elements of the subgroup of elements of order 2 in the factor group of the group by the subgroup.

Def. 4.2 - Let \( H \) be a subgroup of a group \( G \). We call c-power of an element \( x \in G \) in \( H \) to be the least +ve integer \( n \), for which \( x^{2n} \in H \). The c-power of \( x \) is taken to be \(-\infty \) in \( H \) if there exists no such +ve integer \( n \).
It is immediate that c-power of any \( x \in C_G(H) \) is 1 in \( H \) and conversely. If c-power of an element \( x \) in \( G \) be \( m \) in \( H \), then \( x^{m-1} \in C_G(H) \).

**Theorem 4.7** - For any subgroup \( H \) of a group \( G \), \( C_G(H) = H \) if and only if c-power of all \( g \in G - H \) be \(-\infty \) in \( H \).

**Proof.** If \( C_G(H) = H \), then there exists no \( g \in G \), \( g \notin H \) for which \( g^2 \in H \), since otherwise

\[
g H g = g^2 H = H
\]

\( \implies g \in C_G(H) \)

A contradiction that \( C_G(H) \neq H \).

Thus

\( g \in G - H \implies g^2 \notin H \).

Hence, by repeating the argument on \( g^2 \in G - H \) etc. we get, in general,

\( g \in G = H \implies g^{2^i} \notin H \) for any integer \( i > 0 \)

\( \implies \) c-power of all \( g \in G - H \) is \(-\infty \) in \( H \).

Conversely, if c-power of all \( g \in G - H \) be \(-\infty \) in \( H \), then there exists no element \( g \in G - H \), such that \( g \in C_G(H) \) since

\( g \in C_G(H) \implies g^2 \in H \)

Hence,
\[ C_G(H) \subseteq H \]

\[ \implies C_G(H) = H \text{ since } H \subseteq C_G(H) \]

This proves the theorem.

Cor. 4.3 - For any subgroup \( H \) of a group \( G \), \( H \subseteq C_G(H) \) if and only if \( c \)-power of some \( g \in G - H \) be other than \( -\infty \).

Theorem 4.8 - If for two subgroups \( H_1, H_2 \) of a group \( G \), \( H_1 \supseteq H_2 \) then \( C_G(H_1) \supseteq C_G(H_2) \) but not conversely.

Proof. Let \( H_1 \supseteq H_2 \)

\[ \implies H_1 = \bigcup h_1^{(1)} H_2 \text{ where } h_1^{(1)}'s \in H_1 \]

For any \( x \in C_G(H_2) \),

\[ xH_1 x = x \bigcup h_1^{(1)} H_2 x \]

\[ = \bigcup x(h_1^{(1)} H_2) x \text{ (Theorem 1.2, (ii))} \]

\[ = \bigcup h_1^{(1)} H_2 \]

\[ = H_1 \]

\[ \implies C_G(H_1) \supseteq C_G(H_2) \]

To prove the falsity of converse, let

\[ G = [a], \text{ the cyclic group of order 12.} \]

\[ H_1 = [a^2], \quad H_2 = [a^3] \]
Then, we have
\[ C_G(H_1) = G, \quad C_G(H_2) = H_2 \]
Thus here, \( C_G(H_1) \supseteq C_G(H_2) \) but \( H_1 \neq H_2 \)

This completes the proof.

Theorem 4.9 - If for a subgroup \( H \) of a group \( G \), \( [G : H] = 2 \), then \( C_G(H) = G \) but not conversely.

Proof. Let \( [G : H] = 2 \)

\[ \Rightarrow H \text{ is a c.s.c-subgroup, by theorem 2.10} \]

\[ \Rightarrow C_G(H) = G \]

The converse is false.

Consider, \( G = \{a, b, ab, e\} ; \quad a^2 = b^2 = e, ab = ba \)
and \( H = e \)
We find \( C_G(H) = G \) but \( [G : H] = 4 \)

This proves the result.

Theorem 4.10 - If \( H_1, H_2 \) be any two periodic subgroups of a group \( G \), containing no element of order 2, then \( H_1 \subseteq C_G(H_2) \) together with \( H_2 \subseteq C_G(H_1) \) implies \( H_1 = H_2 \).

Proof. Let \( h_1 \in H \), then

\[ h_1 e h_1 = h_1^2 \in H_2 \text{ since } h_1 \in C_G(H_2) \]

\[ \Rightarrow H_1^2 \subseteq H_2 \]
But, evidently, $H^* = H_1$

$$\Rightarrow \quad H_1 \subseteq H_2$$

Similarly, we can see that

$$H_2 \subseteq H_1$$

Consequently

$$H_1 = H_2$$

Hence the proof is complete.

Theorem 4.11 - For any subgroup $H$ of a group $G$, $C_G(H)$ equals the set theoretical union of all cosets in $\bar{O}_2$, the subgroup of all elements of order 2 in $G/H$.

Proof. Let a coset $gH \in \bar{O}_2$, then

$$gHg = g^2H$$

$$= (gH)^2$$

$$= H$$ since $gH \in O_2$

$$\Rightarrow g \in C_G(H)$$

$$\Rightarrow gH \subseteq C_G(H)$$ since $H \subseteq C_G(H)$

$$\Rightarrow \bigcup_{gH \in \bar{O}_2} gH \subseteq C_G(H)$$

On the contrary, let $x \in C_G(H)$, then

$$xHx = H$$
Thus the proof is complete.

Cor. 4.4 - For any two subgroups \( H_1, H_2 \) of a group \( G \), \( C_G(H_1) = C_G(H_2) \) if and only if the set theoretical unions of cosets in subgroups of all elements of order 2 in \( G/H_1 \) and \( G/H_2 \) is the same.

5. Calculus Of Compressors:

In the following, we determine the nature of intersection, product and complimentation of compressors of subsets in a group.

Theorem 4.12 - If \( S_1, S_2 \) be any two subsets in a group \( G \), then \( C_G(S_1) \cap C_G(S_2) \) will be contained in both \( C_G(S_1 \cap S_2) \) and \( C_G(S_1 \cup S_2) \).

Proof. Let \( x \in C_G(S_1) \cap C_G(S_2) \).

\[ \Rightarrow x \in C_G(S_1), x \in C_G(S_2) \]
\[ \Rightarrow xS_1x = S_1, \quad xS_2x = S_2 \]
\[ \Rightarrow x(S_1 \cap S_2)x = S_1 \cap S_2, \quad x(S_1 \cup S_2)x = S_1 \cup S_2 \]

(Theorem 2.5)

\[ \Rightarrow C_G(S_1) \cap C_G(S_2) \subseteq C_G(S_1 \cap S_2) \text{ and also } C_G(S_1 \cup S_2) \]

This completes the proof.

Note: The above theorem also holds for an arbitrary family of subsets.

Cor. 4.5 - For any two subgroups \( H_1, H_2 \) of a group \( G \),

\[ C_G(H_1) \cap C_G(H_2) = C_G(H_1 \cap H_2) \]

(Proof is immediate from the above theorem and theorem 4.8)

Cor. 4.6 - Given two subsets \( S_1, S_2 \) of a group \( G \),

(i) \( \text{If } C_G(S_1 \cap S_2) = O_2, \text{ the subgroup of all elements of order 2 in } G, \text{ then} \)

\[ C_G(S_1) \cap C_G(S_2) = C_G(S_1 \cap S_2) \]

(ii) \( \text{If } C_G(S_1 \cup S_2) = O_2, \text{ then } C_G(S_1) \cap C_G(S_2) = C_G(S_1 \cup S_2). \)

Theorem 4.13 - Let \( S_1, S_2 \) be any two subsets of a group \( G \), then

\[ C_G(S_1) \cdot C_G(S_2) = C_G(S_1 \cdot S_2) \]

Proof. For an arbitrary element \( x \in C_G(S_1) \),

\[ x(S_1 S_2)x = (xS_1x)S_2 \]
\[= s_1 s_2\]

\[\Rightarrow C_G(s_1) \subseteq C_G(s_1 s_2)\]

Similarly

\[C_G(s_2) \subseteq C_G(s_1 s_2)\]

Hence

\[C_G(s_1) \cdot C_G(s_2) \subseteq C_G(s_1 s_2)\]

This proves the theorem.

Note: If in the above theorem \(C_G(s_1 s_2) = 0_2\), the subgroup of all elements of order 2 in \(G\), then \(C_G(s_1) \cdot C_G(s_2) = C_G(s_1 s_2)\).

Theorem 4.14 - For any subset \(S\) of a group \(G\), \(G = C_G(S) = G - C_G(G - S)\).

(Proof follows from theorem 4.4)

6. Compressor of A Set And Of Its C-Transform:

Here, we note that the compressors of a subset and of any of its c-transforms are the same in a group; however, if the compressors of any two subsets in a group be the same, one need not be c-transform of other.

Theorem 4.15 - For any subset \(S\) of a group \(G\) and any \(x \in G\),

\[C_G(S) = C_G(S_x)\]. Conversely, if for two subsets \(S_1, S_2\) of \(G\),

\[C_G(S_1) = C_G(S_2)\], then \(S_1\) is not a C-Transform of \(S_2\) in general.
Proof. One can easily verify that
\[ C_G(S_x) = C_G(S) \]

To show that the converse is not true,

Consider \( G = [a] \), a cyclic group of order 8

If \( S_1 = \{a^2, a^5\} \)
\[ C_G(S_1) = C_G(S_2) = \{a^4, e\} \]

But evidently,
\[ S_1 \not\subseteq S_1 \]

This completes the theorem.

Cor. 4.7 - Let \( \{S_\alpha\}_{\alpha \in A} \) be any family of subsets in a group \( G \),
then for any \( x \in G \)

(i) \( C_G(\bigcap S_\alpha) = C_G(\bigcap (S_\alpha)_x) \)

\[ \forall \alpha \in A \]
(ii) \( C_G(\bigcup S_\alpha) = C_G(\bigcup (S_\alpha)_x) \)

\[ \forall \alpha \in A \]

(Proof follows immediately from theorems 4.15 and 1.2)

Cor. 4.8 - For any finite family \( \{S_\alpha\}_{\alpha \in A} \) of subsets in a group \( G \)
and elements \( \{x_\alpha\}_{\alpha \in A} \) in \( G \), we have

\[ C_G(\bigcap S_\alpha) = C_G(\bigcap (S_\alpha)_x) \]

\[ \forall \alpha \in A \]

(Proof follows from theorems 4.15 and 1.3)
7. Fundamental Mappings And Compressors:

We observe that the isomorphic image of the compressor of a subset in a group under an isomorphism of the group coincides with the compressor of the image-subset in the image group; however, this does not hold in general in case of a homomorphism, but it does hold if the subset be a subgroup containing the kernel of the homomorphism.

Theorem 4.16 - Let \( \phi \) be an isomorphism of a group \( G \) onto a group \( G' \), then for any subset \( S \) of \( G \),

\[
(C_G(S)\phi) = C_{G'}(S\phi)
\]

Proof. Evidently, \( S \) is self compressed with respect to \( C_G(S) \), hence by theorem 2.16, \( S\phi \) is self compressed with respect to \( (C_G(S)\phi) \). Thus

\[
(C_G(S)\phi) \subseteq C_{G'}(S\phi) \quad \text{(Theorem 4.3)}
\]

Again, since \( S\phi \) is self compressed with respect to \( C_{G'}(S\phi) \), we have by theorem 2.16 that \( S \) is self compressed with respect to \( (C_{G'}(S\phi))^{-1} \). Thus by theorem 4.3,

\[
(C_{G'}(S\phi))^{-1} \subseteq C_G(S)
\]

\[
\Rightarrow C_{G'}(S\phi) \subseteq (C_G(S)\phi)
\]

Hence

\[
(C_G(S)\phi) = C_{G'}(S\phi)
\]

This proves the theorem.
Remark: If \( \phi \) be a homomorphism of \( G \), then for any set \( S \) only the restricted result \( (C_G(S)) \phi \subseteq C_{G'}(S \phi) \) holds true, but if \( S \) be a subgroup \( H \) containing kernel of \( \phi \), the character of theorem (4.16) is maintained as proved in the following theorem.

Theorem 4.17 - Let \( \phi \) be a homomorphism of a group \( G \) onto a group \( G' \), then for any subgroup \( H \) of \( G \) containing \( K \), the kernel of \( \phi \), we have

\[
C_{G'}(H \phi) = (C_G(H)) \phi
\]

Proof. Evidently, by theorem 2.14 and 4.3

\[
(C_G(H)) \phi \subseteq C_{G'}(H \phi)
\]

Conversely, let \( x' \in C_{G'}(H \phi) \) be arbitrary, then

\[
x'(H \phi)x' = H \phi
\]

\[\implies (x \phi)(H \phi)(x \phi) = H \phi \quad \text{for some} \ x \in G\]

\[\implies (xHx)\phi = H \phi\]

\[\implies xHx = H \quad \text{since} \ H \supseteq K\]

\[\implies x \in C_G(H)\]

\[\implies x \phi = x' \in (C_G(H)) \phi\]

\[\implies C_{G'}(H \phi) = (C_G(H)) \phi\]

Hence

\[
C_{G'}(H \phi) = (C_G(H)) \phi
\]

This completes the proof.
8. Direct Products And Compressors:

We note that in a direct product, the direct product of the compressors of subsets of direct factors in the corresponding direct factors is the compressor of the product of these subsets in the direct product group.

Theorem 4.18 - Let a group $G$ be a direct product of its subgroups $H_i$, $i = 1, 2, \ldots, n$. If $S_i$ denotes subsets of $H_i$ for all $i$, then

$$C_G\left(\prod_{i=1}^{n} S_i\right) = \prod_{i=1}^{n} C_{H_i}(S_i)$$

Proof. We shall first prove that

$$C_G\left(\prod_{i=1}^{n} S_i\right) = \prod_{i=1}^{n} C_{H_i}(S_i)$$

Evidently,

$$\prod_{i=1}^{n} C_{H_i}(S_i) \subseteq \prod_{i=1}^{n} C_G(S_i) \subseteq C_G\left(\prod_{i=1}^{n} S_i\right) \quad \text{(Theorem 4.13)}$$

On the other hand, if $x \in C_G\left(\prod_{i=1}^{n} S_i\right)$ and $s_i \in S_i$, for $i = 1, 2, \ldots, n$, then

$$x(s_1 \cdot s_2 \cdots s_1 \cdots s_n) x = s'_1 \cdot s'_2 \cdots s'_1 \cdots s'_n \in \prod_{i=1}^{n} S_i$$

or for $x = h_1 \cdot h_2 \cdots h_i \cdots h_n$, $h_i \in H_i$ in $G = \prod_{i=1}^{n} H_i$,

$$(h_1 \cdot h_2 \cdots h_i \cdots h_n)(s_1 \cdot s_2 \cdots s_1 \cdots s_n)(h_1 \cdot h_2 \cdots h_i \cdots h_n) = s'_1 \cdot s'_2 \cdots s'_n$$

$$=> (h_1 \cdot s_i \cdot h_1)(h_2 \cdot s_i \cdot h_2) \cdots (h_i \cdot s_i \cdot h_i) \cdots (h_n \cdot s_i \cdot h_n) = s'_1 \cdot s'_2 \cdots s'_i \cdots s'_n$$
\[ h_i s_i h_i = s_i' \in S_i \quad \text{by uniqueness of representation} \]

\[ \text{im } G = \bigcap_{i=1}^{n} H_i \]

\[ \Rightarrow h_i s_i h_i \subseteq S_i \quad \text{for all } i \]

Again, since \( x^{-1} = h_1^{-1} \cdot h_2^{-1} \cdots h_i^{-1} \cdots h_n^{-1} \) also belongs to \( C_G(\bigcap_{i=1}^{n} S_i) \), we have as above

\[ h_1^{-1} s_i h_1^{-1} \subseteq S_i \quad \text{for all } i \]

\[ \Rightarrow s_i \subseteq h_1 s_i h_1 \]

Hence

\[ S_i = h_1 s_i h_1 \]

\[ \Rightarrow h_1 \in C_{H_1}(S_i) \]

\[ \Rightarrow x = h_1 \cdot h_2 \cdots h_i \cdots h_n \in \bigcap_{i=1}^{n} C_{H_1}(S_i) \]

\[ \Rightarrow C_G(\bigcap_{i=1}^{n} S_i) \subseteq \bigcap_{i=1}^{n} C_{H_1}(S_i) \]

Consequently

\[ C_G(\bigcap_{i=1}^{n} S_i) = \bigcap_{i=1}^{n} C_{H_1}(S_i) \]

Finally, since \( C_{H_1}(S_i) \subseteq H_1 \) and \( H_i \cap H_j = e \) for \( i \neq j \), it is evident that

\[ C_G(\bigcap_{i=1}^{n} S_i) = \bigcap_{i=1}^{n} C_{H_1}(S_i) \]

This completes the proof.
Cor. 4.9 - Let a group $G$ be a direct product of its subgroups $H_i$, $i = 1, 2, \ldots, n$, then if $H_i'$ denotes the subgroup of $H_i$ for all $i$, we have

$$C_G(\prod_{i=1}^{n} H_i') = \prod_{i=1}^{n} C_{H_i}(H_i')$$

Theorem 4.19 - Let $G_i$ be arbitrary groups and $S_i$ denote subsets of $G_i$ for all $i = 1, 2, \ldots, n$, then

$$\prod_{i=1}^{n} C_{G_i}(S_i) = C_{\prod_{i=1}^{n} G_i}(\prod_{i=1}^{n} S_i)$$

Proof. Let $(g_i) = (g_1, g_2, \ldots, g_i, \ldots, g_n) \in \prod_{i=1}^{n} C_{G_i}(S_i)$ be arbitrary, then

$$(g_i)(\prod_{i=1}^{n} S_i)(g_i) = \prod_{i=1}^{n} (g_iS_i)g_i$$

$$= \prod_{i=1}^{n} S_i$$

since $g_i \in C_{G_i}(S_i)$

$$(\Rightarrow) (g_i) \in C_{\prod_{i=1}^{n} G_i}(\prod_{i=1}^{n} S_i)$$

$$(\Rightarrow) \prod_{i=1}^{n} C_{G_i}(S_i) \subseteq C_{\prod_{i=1}^{n} G_i}(\prod_{i=1}^{n} S_i)$$

On the other hand, for any element $(g'_i) = (g'_1, g'_2, \ldots, g'_i, \ldots, g'_n)$ in $C_{\prod_{i=1}^{n} G_i}(\prod_{i=1}^{n} S_i)$,

$$(g'_i)(\prod_{i=1}^{n} S_i)(g'_i) = \prod_{i=1}^{n} S_i$$
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\[ \prod_{i=1}^{n} (g_i' S_i g_i') = \prod_{i=1}^{n} S_i \]

\[ \Rightarrow g_i' S_i g_i' = S_i \text{ for all } i \]

\[ \Rightarrow g_i' \in C_{G_i}(S_i) \]

\[ \Rightarrow (g_i') \in \prod_{i=1}^{n} C_{G_i}(S_i) \]

\[ \Rightarrow C_n \left( \prod_{i=1}^{n} S_i \right) \subseteq \prod_{i=1}^{n} C_{G_i}(S_i) \]

Hence

\[ \prod_{i=1}^{n} C_{G_i}(S_i) = C_n \left( \prod_{i=1}^{n} S_i \right) \]

Thus the proof is complete.

Remark. The above theorem also holds true for complete direct products of groups which can be similarly verified.

9. Compressor And Smallest C.S.C-Subgroup:

We find that the compressor of a subgroup contains the compressor of its smallest c.s.c-subgroup, but it is not without interest to note that the smallest c.s.c-subgroup of the compressor of a subgroup is contained in the compressor of its smallest c.s.c-subgroup. Further if the smallest c.s.c-subgroup of the compressor of a subgroup be contained in another subgroup then the compressor of the first is contained in the compressor of the other.
Theorem 4.20 - Let $H, H_1$ be any two subgroups of a group $G$, then

(i) $(C_G(H))^* \subseteq C_G(H^*) \subseteq C_G(H)$

(ii) $(C_G(H))^* \subseteq H_1 \implies C_G(H) \subseteq C_G(H_1)$.

Proof. (i) Since $H^* \subseteq H$, the 2nd part of (i) is obvious and

$$C_G(H^*) \subseteq C_G(H) \quad \text{(Theorem 4.8)}$$

Also, for any $x \in C_G(H)$ and $h \in H$,

$$x^2 h^2 x^2 = x h x \cdot x h x$$

$$= h_1 h_1 \quad \text{where } h_1 \in H$$

$$= h_1^2 \in H^*$$

$$\implies x^2 H^* x^2 \subseteq H^*$$

$$\implies H^* = x^2 H^* x^2 \quad \text{for all } x \in C_G(H) \quad \text{(Theorem 1.9)}$$

$$\implies (C_G(H))^* \subseteq C_G(H^*)$$

This proves (i)

For (ii),

$$(C_G(H))^* \subseteq H_1$$

$$\implies \text{For any } x \in C_G(H), x^2 \in H_1$$

$$\implies H_1 = x^2 H_1$$

$$= x H_1 x \quad \text{for all } x \in C_G(H)$$
\[ \implies C_G(H) \subseteq C_G(H_1) \]

This completes the theorem.

Cor. 4.10 - If for two subgroups \( H_1, H_2 \) of a group \( G \),

\[ C_G(H_2) \subseteq C_G(H_1) \text{ then } C_G(H_1) = C_G(H_2) \text{ if and only if } (C_G(H_1))^* \subseteq H_2. \]

(The sufficiency is immediate from theorem 4.20 (ii) and for necessary part we remark that by theorem 4.6,

\[
H_2 = \bigcup_{h_i^2 \in H_2} \left( (C_G(H_1))^* \implies (C_G(H_1))^* \subseteq H_2 \right)
\]

10. Compressor And Anticenter :

We prove that the intersection of a subgroup with the anticenter of the compressor of the subgroup is contained in the compressor of anticenter of the subgroup. We also note that in a cyclic group the compressor of the anticenter of a subgroup is anticenter of the compressor of the subgroup, but not in general.

Theorem 4.21 - Let \( H \) be a subgroup of a group \( G \), then

\[ H \cap AC(C_G(H)) \subseteq AC(H) \subseteq C_G(AC(H)). \]

Proof. Since \( H \subseteq C_G(H) \), we have by theorem 2 in [6],

\[ AC(C_G(H)) \cap H \subseteq AC(H) \subseteq C_G(AC(H)) \text{ since } AC(H) \text{ is a subgroup.} \]

This establishes the result.
Theorem 4.22 - For any subgroup \( H \) of a cyclic group \( G \),

\[
AC(C_G(H)) = C_G(AC(H)) \quad \text{but it does not hold in general.}
\]

Proof. Since \( G \) is cyclic, the subgroups \( H \) and \( C_G(H) \) are also cyclic, hence by theorem 2 in [5]

\[
AC(C_G(H)) = C_G(H) = C_G(AC(H))
\]

To show, that it does not hold in general

Consider \( G = \{a, b, c, e\} \); \( a^2 = b^2 = c^2 = e, ab = ba = c, \)
\( bc = cb = a, ac = ca = b \)

Evidently,

\[
AC(G) = e
\]

Now if we put \( H = G \), then

\[
AC(C_G(H)) = AC(G) = e
\]

But

\[
C_G(AC(H)) = C_G(e) = G
\]

Thus

\[
AC(C_G(H)) \neq C_G(AC(H))
\]

This completes the proof.
11. Chain Of Compressors:

For any subgroup $H$ of a group $G$,

$$H \subseteq C_G(H) \subseteq C_G(C_G(H)) \subseteq \ldots$$ \hspace{1cm} (1)

is an ascending chain of subgroups of $G$. We write $C_G(H) = C_G^1(H)$,

$$C_G(C_G^1(H)) = C_G^2(H), \ C_G(C_G^2(H)) = C_G^3(H) \text{ and so on in general}$$

$$C_G^n(H) = C_G(C_G^{n-1}(H)). \text{ Evidently}$$

- $C_G^1(H) = \{g \mid g \in G, \ g^{2} \in H\}$
- $C_G^2(H) = \{g \mid g \in G, \ g^{2} \in H\}$
- $C_G^3(H) = \{g \mid g \in G, \ g^{3} \in H\}$

\ldots

$$C_G^n(H) = \{g \mid g \in G, \ g^{2^n} \in H\} \text{ and so on. If an}$$

element $g \in C_G^n(H)$ and $g \not\in C_G^{n-1}(H)$ then $g^{2^n} \in H$, $g^{2^{n-1}} \not\in H$. We now prove the following important theorem which gives a criterion for the chain of compressors to be finite.

Theorem 4.23 - Let $H$ be a subgroup of a group $G$. The chain

$$C_G^1(H) \subseteq C_G^2(H) \subseteq C_G^3(H) \ldots \subseteq C_G^n(H) \subseteq \ldots$$ \hspace{1cm} (1')
breaks off at \( n \)th stage \((n > 1)\) if and only if least upper bound of \( c \)-powers in \( H \) of elements in \( G - H \) be \( n \). The chain breaks off at the 1st stage if \( c \)-powers in \( H \) of all elements in \( G - H \) be < 2. (For \( c \)-power, see Def. 4.2).

Proof. Let, the chain \((1')\) breaks off at the \( n \)th \((n > 1)\) stage, then

\[
C_G^n(H) = C_G^k(H) \quad \text{for all } k > n.
\]

\( \Rightarrow \) For all \( g \in G - H \), either \( g \in C_G^n(H) \) or \( g \not\in C_G^1(H) \) for all \( i \geq 1 \).

\( \Rightarrow \) Either \( g^{2^i} \in H \) or \( g^{2^i} \not\in H \) for all integers \( i \geq 1 \).

\( \Rightarrow \) \( C \)-power of every \( g \in G - H \) is \( \leq n \) since the \( C \)-power in the last case is \( -\infty \).

Now, as \( C_G^n(H) \supset C_G^{n-1}(H) \) there exists a \( g' \in G - H \) such that

\[
g' \in C_G^n(H), \quad g' \not\in C_G^{n-1}(H)
\]

or \( g' \in C_G^n(H), \quad g' \not\in C_G^k(H) \) for \((1 \leq k \leq n-1)\)

\( \Rightarrow \) \( g'^{2^i} \in H, \quad g'^{2^k} \not\in H \) for \((1 \leq k \leq n - 1)\)

\( \Rightarrow \) \( C \)-power of \( g' \in G - H \) is \( n \) in \( H \).

Hence, least upper bound of \( c \)-powers in \( H \) of all elements, in \( G - H \) is \( n \).
Conversely, if least upper bound of c-powers in $H$ of all elements in $G - H$ be $n$, then for all $g \in G - H$, c-power in $H$ is $\leq n$.

$$\Rightarrow g^2 \in H \text{ for some } 1 \leq i \leq n \text{ or } g^2 \notin H \text{ for all integers } j \geq 1$$

$$\Rightarrow C_G^n(H) = C_G^k(H) \text{ for all } k \geq n$$

Again, due to property of least upper bound, there exists a $g^1 \in G - H$ such that c-power of $g^1$ be $n$ in $H$. Thus

$$g^{2^i} \in H, \quad g^{2^i} \notin H \text{ for all } 1 \leq i \leq n - 1$$

$$\Rightarrow g^1 \in C_G^n(H), \quad g^1 \notin C_G^{n-1}(H)$$

$$\Rightarrow C_G^n(H) \supset C_G^{n-1}(H))$$

$$\Rightarrow \text{The chain (1')} \text{ breaks off at } n^{th} \text{ stage.}$$

Finally, if c-powers in $H$ of all elements in $G - H$ be $< 2$, then clearly the chain (1') breaks off at the 1st stage.

This completes the theorem.

Remark: The chain in the above theorem breaks off always at

$C_G^n(H) = G$ if c-power of every element in $G - H$ is $i$ such that $1 \leq i \leq n$ otherwise $C_G^n(H)$ is a proper subgroup of $G$.

12. Results True For Non-Abelian Groups:

We have proved all the results given above for abelian groups only, however, it is easy to verify that the theorems 4.1, 4.2, 4.4, 4.5, 4.12, 4.14, 4.16, 4.17 and 4.19 also hold for non-abelian groups with no change in the proofs already supplied.
Bibliography:

   (Translated from The Russian And Edited By K.A. Hirsh)
   Chelsea Publishing Company New York, N.Y.

   D. Van Nostrand Company, INC.

3. L. Fuchs: Abelian Groups, Publishing House Of The
   Hungarian Academy Of Sciences Budapest, 1958.


   pp.61-63.

   pp. 469-472.


8. R. Bear: Norm And Hypercenter, Pub. Mathematicae
   Vol.4, 1955-56 pp.347-


10. M.A. Kazim: Notes On Some Problems In Group Theory
    (Unpublished).

    Student (To appear)