CHAPTER - THREE

SMALLEST C.S.C - SUBGROUP AND C -SIMPLE GROUPS

1. Introduction: This chapter is devoted to the study of smallest c.s.c-subgroup of a group denoted as $G^*$ and to groups which contain no c.s.c-subgroup other than itself designated as C-simple groups. We find that the smallest c.s.c-subgroup of a group $G$ is just the subgroup generated by squares of elements of $G$ and is isomorphic to the factor group of the group with respect to the subgroup of all elements of order two in the group. The smallest c.s.c-subgroup of a direct product comes out to be the direct product of smallest c.s.c-subgroups of the direct factors. We investigate that the index of the smallest c.s.c-subgroup in a group, having basis, is a power of 2, actually $2^{r_0(G) + r_2(G)}$, where $r_0(G)$ is the torsion free rank and $r_2(G)$ is 2-rank of $G$. We prove that the formation of smallest c.s.c-subgroup i.e. the star operation is a homomorphism of the semigroup of all subgroups of a group onto the semigroup of all subgroups of its smallest c.s.c-subgroup, and introduce a notion of essential equivalence between the subgroups of a group. This notion illuminates the study of smallest c.s.c-subgroup in an interesting manner. We use it to show that any two subgroups of a group have the same smallest c.s.c-subgroup if and only if they are essentially equal. We have investigated conditions in which the smallest c.s.c-subgroup of a group becomes cyclic and have pointed out that a group whose smallest c.s.c-subgroup is cyclic, is also cyclic if the smallest c.s.c-subgroup contains the subgroup of all elements
of order 2 in the group. It is interesting to note that the isomorphic groups have isomorphic smallest c.s.c-subgroups but not conversely in general, however, if a group $G$ is isomorphic to a subgroup of a group $G_1$ then $G^*$ is isomorphic to $G_1^*$ under the same isomorphism, if and only if the isomorphic image of $G$ be essentially equal to $G_1$. Further, we discuss a condition under which intersection of a subgroup of a group with its smallest c.s.c-subgroup coincides with the smallest c.s.c-subgroup of the subgroup; and find out that smallest c.s.c-subgroup of the anticenter of a group is the anticenter of the smallest c.s.c-subgroup of the group.

Finally, we prove an important theorem on the relation between ranks of a group and of its smallest c.s.c-subgroup showing that they differ by their 2-ranks, and further evaluate the difference in terms of power of elements of order 2. We conclude our discussions with the properties of groups having identical smallest c.s.c-subgroups which we call G-simple groups. Here we show that this class is a subclass of periodic groups, whose elements are of odd order only.

2. Structure Of Smallest C.S.C-Subgroup :

We now prove some theorems which enlighten us about the structure of this important subgroup. We first observe that $G^*$, the smallest c.s.c-subgroup of a group $G$ is the collection of squares of all the elements of $G$, and point out that this is a homomorphic image of $G$ with kernel the subgroup of all
elements of order two in \( G \). Also, further, every subgroup of the subgroup \( G^* \) is actually the smallest c.s.c-subgroup of some subgroup of \( G \). Finally it is important to note that the smallest c.s.c-subgroup of a direct product is the direct product of the smallest c.s.c-subgroups of its direct factors.

Theorem 3.1 - Let \( G \) be any group then the set \( G^* = \{g^2 \mid g \in G\} \) is a c.s.c-subgroup of \( G \). The subgroup \( G^* \) is the smallest subgroup of this type and is unique.

Proof. Evidently, \( G^* \) is a subgroup of \( G \) and further from cor.2.2, \( G^* \) is a c.s.c-subgroup of \( G \). Also \( G^* \) is a smallest c.s.c-subgroup of \( G \), for let \( H \) be any c.s.c-subgroup of \( G \), we have again from cor.2.3

\[
g^2 \in H \quad \text{for every} \ g \in G
\]

\[
\implies G^* \subseteq H
\]

For uniqueness, let \( H' \) be a smallest c.s.c-subgroup of \( G \), then since \( H' \) is a c.s.c-subgroup of \( G \), we have

\[
G^* \subseteq H'
\]

\[
\implies H' = G^*
\]

This shows that \( G^* \) is unique and the theorem is completely proved.
Cor. 3.1 - A subgroup $H$ of a group $G$ is completely self compressed if and only if $H \supseteq G^*$.

(Proof is immediate in view of Cor. 2.8)

Theorem 3.2 - The smallest c.s.c-subgroup $G^*$ of a group $G$ is a homomorphic image of the group $G$ with kernel of homomorphism being $O_2$, the subgroup of all elements of order 2 in $G$.

Proof. Define a mapping

$$\phi : g \rightarrow g^2$$

of $G$ onto $G^*$.

Evidently, $\phi$ is single valued, $\phi$ is a homomorphism, since for any $g_1, g_2 \in G$.

$$(g_1g_2)\phi = (g_1g_2)^2$$

$$= g_1^2 g_2^2$$

$$= (g_1\phi)(g_2\phi)$$

The kernel is the set of all elements $g \in G$ for which

$$(g)\phi = e$$

i.e.

$$g^2 = e$$

$$\implies \text{Kernel of } \phi \text{ is } O_2$$

This proves the theorem.
Cor. 3.2 - A group $G$ is isomorphic to $G^*$ if and only if there is no element ($\neq e$) of order 2 in $G$.

The corollary implies that if $G$ is torsion free, $G^*$ is isomorphic to $G$. In particular, if $G$ be finite group, we have

Cor. 3.2 - A finite group $G$ contains $G^*$ as a proper c.s.c-subgroup if and only if order of $G$ is even but will coincide with $G$ if order of $G$ be odd.

Theorem 3.3 - If $G$ be any group then to every subgroup $H'$ of its smallest c.s.c-subgroup $G^*$, there exists a subgroup $H$ of $G$ such that $H^* = H'$.

Proof : Given any subgroup $H'$ of $G^*$, let us define a set

$$H = \{ h | h \in G \text{ such that } h^2 \in H' \}$$

Since $H'$ is a subgroup, we have for any $h_1, h_2 \in H$

$$(h_1 h_2^{-1})^2 = h_1^2 (h_2^{-1})^2$$

$$= h_1^2 (h_2^{-1})^{-1} \in H'$$

$$\implies h_1 h_2^{-1} \in H$$

Hence $H$ is a subgroup of $G$.

Now, evidently

$$H^* \subseteq H'$$

Also since $H' \subseteq G^*$, to every element $h' \in H'$, there exists
an element \( g \in H \subseteq G \) such that
\[
h' = g^2 \]
\[\Rightarrow H' \subseteq H^*\]
Consequently,
\[H^* = H^*\]
This proves the theorem.

Note: It can be easily seen that in the above theorem the subgroup \( H \) of \( G \) is the largest subgroup for which \( H^* = H^* \).

Theorem 3.4 - If \( G \) be any cyclic group generated by an element \( a \), then its smallest c.s.c-subgroup \( G^* \) is the subgroup generated by \( a^2 \).
Proof. Firstly, it is clear that
\[a^2 \in G^*\]
\[\Rightarrow [a^2] \subseteq G^*\.
Secondly, if \( a^n \) be any element of \( G \),
\[(a^n)^2 = (a^2)^n \in [a^2]\]
\[\Rightarrow G^* \subseteq [a^2]\]
Hence,
\[G^* = [a^2]\]

Theorem 3.5 - If a group \( G \) be a direct product \( \prod_{i=1}^{n} G_i \) of its subgroups \( G_i \)'s, then
\[G^* = G_1^* \times G_2^* \times \ldots \times G_n^*\]
Proof. Evidently,
\[ G^*_1 \subseteq G^* \quad \text{for all } i \]
\[ \implies G^*_1 \times G^*_2 \times \ldots \times G^*_n \subseteq G^*. \]

on the other hand, for any \( g \in G \), let
\[ g = g_1 \cdot g_2 \ldots \cdot g_n \quad \text{where } g_i \in G_i, \quad i = 1, 2, \ldots, n. \]
\[ \implies g^2 = (g_1 \cdot g_2 \ldots \cdot g_n)^2 \]
\[ = g_1^2 \cdot g_2^2 \ldots \cdot g_n^2 \in G^*_1 \times G^*_2 \times \ldots \times G^*_n \]
\[ \implies G^* \subseteq G^*_1 \times G^*_2 \times \ldots \times G^*_n \]

Hence,
\[ G^* = G^*_1 \times G^*_2 \times \ldots \times G^*_n \]

This completes the theorem.

Cor. 3.4 - If a group \( G \) has a basis \( \{a_i\}_{\alpha \in \mathcal{A}} \) where \( \mathcal{A} \) is an index set then \( G^* = \bigtimes_{\alpha \in \mathcal{A}} [a_\alpha^2] \).

(Proof is immediate in view of theorems 3.4 and 3.5)

3. Index of \( G^* \) in \( G \):

In the following, we determine \([ G : G^* ]\) the index in \( G \) of the smallest c.s.c-subgroup \( G^* \) of a group \( G \) having a basis and formulate it in terms of torsion free rank and 2-rank of the group.
Def. 3.1 - The non-identity elements \( a_1, a_2, \ldots, a_k \) of the group \( G \) are called linearly independent, or briefly, independent, if any relation

\[
a_1^{n_1} a_2^{n_2} \ldots a_k^{n_k} = e \quad (n_i \in \mathbb{I})
\]

implies

\[
a_1^{n_1} = a_2^{n_2} \ldots a_k^{n_k} = e
\]

i.e. \( n_1 = 0 \) if \( 0(a_1) = \infty \), and \( 0(a_1) \mid n_1 \) if \( 0(a_1) < \infty \). In the contrary case, they are called dependent.

Def. 3.2 - The cardinal number of a maximal independent set in a group \( G \) containing merely elements of order \( \infty \) is the torsion free rank \( r_0(G) \) of \( G \). For any prime \( p \), the p-rank \( r_p(G) \) of \( G \) is the cardinal number of a maximal independent set in \( G \) containing only the elements of orders of powers of \( p \).

Now, we prove an important theorem to achieve our end:

Theorem 3.6 - Let \( G \) be a finitely generated group and \( L = \{ a_1, a_2, \ldots, a_m \} \) be the set of all elements of order infinity and of \( 2^k \) \((k \geq 1)\) in a basis of \( G \) containing elements of infinite and prime power order, then if \( L \neq \emptyset \), \( G^* \) is properly contained in \( G \). Further if \( H = [ G^*, a_1, a_2, \ldots, a_l ] \), the subgroup generated by \( G^* \) and \( a_1 \)'s such that \( a_i \in L \) where \( 0 \leq l \leq m \). Then

\[
[H : G^*] = 2^l.
\]
Proof. Let $B = \{a_1, a_2, \ldots, a_n\}$ be a basis of $G$ under consideration then, by a well known theorem

$$G = [a_1] \times [a_2] \times \cdots \times [a_n]$$

where $[a_1]$ is the cyclic group generated by $a_1$.

Also, by Cor 3.4, we have

$$G^* = [a_1^2] \times [a_2^2] \times \cdots \times [a_n^2]$$

Evidently $[a_i^2] = [a_i]$ only if $O(a_i) = p^k (k \geq 1)$ where $p$ be an odd prime, hence if $L \neq \emptyset$ it is clear that

$$G^* \subset G$$

Now we assert that for any $g \in G$ and $a_i \in L$, $g^2 \neq a_i$

Since, otherwise

$$g = a_1^\alpha_1 \cdot a_2^\alpha_2 \cdots \cdot a_n^\alpha_n$$

$$\implies g^2 = a_1^{2\alpha_1} \cdot a_2^{2\alpha_2} \cdots \cdot a_n^{2\alpha_n} = a_1$$

$$\implies a_1^\alpha_1 \cdot a_2^{2\alpha_2} \cdots \cdot a_n^{2\alpha_n} = e$$

$$\implies a_1^{2\alpha_1} = a_2^{2\alpha_2} = \cdots = a_1^{2\alpha_i} = \cdots = a_n^{2\alpha_n} = e$$

since $B$ is a linearly independent set.

$$\implies 2^{\alpha_i - 1}, \text{ the power of } a_i, \text{ is either zero or divisible by } 2^k \text{ for some } k \geq 1.$$
A contradiction that \( \alpha_i \) is integral, hence our assertion follows. It is therefore clear that,

\[
g^2 \neq a_1 \cdot a_2 \cdot \ldots \cdot a_h \quad \text{for any } g \in G \text{ and } a_i \in L
\]

We now put \( L' = \{a_1, a_2, \ldots, a_l\} \) and define

\[
K_1 = \text{set of all cosets } a_i G^* \text{, } a_i \in L'
\]

\[
K_2 = \text{set of all cosets } a_i a_j G^* \text{, } a_i, a_j \in L'
\]

\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 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where $a_i$'s are all distinct elements of $L'$

\[ \implies H \subseteq G^* \cup K_1 \cup K_2 \cup \ldots \cup K_l \]

Consequently,

\[ H = G^* \cup K_1 \cup K_2 \cup \ldots \cup K_l \]

\[ \implies [H : G^*] \leq (c_0 + c_1 + \ldots + c_l) = 2^l \]

Finally, if for any two different subsets $S' = \{a_1', a_2', \ldots, a_i'\}$ and $S'' = \{a_1'', a_2'', \ldots, a_j''\}$ in $L'$ we get

\[ a_1' a_2' \ldots a_i' G^* = a_1'' a_2'' \ldots a_j'' G^* \]

\[ \implies a_1' a_2' \ldots a_i' = a_1'' a_2'' \ldots a_j'' g^2 \text{ where } g \in G \]

Now if $g = a_{1}^{\beta_1} a_{2}^{\beta_2} \ldots a_{n}^{\beta_n}$,

\[ a_1' a_2' \ldots a_i' = a_1'' a_2'' \ldots a_j'' a_1^{2\beta_1} a_2^{2\beta_2} \ldots a_n^{2\beta_n} \]

\[ \implies a_{1}^{\beta_1} a_{2}^{\beta_2} \ldots a_{h'}^{\beta_{h'}-1} \ldots a_{n}^{\beta_n} = e \text{ if } a_{h'} \in S' \text{ and } S'' \]

\[ \implies a_{h'}^{\beta_{h'}-1} = e \text{ since } B \text{ is a linearly independent set.} \]

A contradiction that $\beta_{h'}$ is integral.

Therefore, the two cosets in question cannot be equal and
hence are distinct.

\[ \implies [H : G^*] \leq 2^l. \]

Thus we prove

\[ [H : G^*] = 2^l. \]

Cor. 3.5 - If \( G \) be a finitely generated group then,

\[ [G : G^*] = 2^{r_0(G)} + 2^{r_2(G)} \]

where \( r_0(G) \) = torsion free rank of \( G \), \( r_2(G) = 2 - \text{rank of } G. \)

(The proof follows immediately if \( L' = L \) since then \( [G:G^*] = 2^m \))

Remark: The theorem is proved above for finitely generated groups, but if \( G \) be any group having a basis, the result holds true. In case, \( L \) is infinite, \( [G : G^*] \) is infinite. In particular, if \( G = [a] \) be a cyclic group, then \( [G : G^*] = 2 \), if \( G \) be of infinite or even order and is equal to 1 if otherwise.

4. Rank Of A Group And Of Its Smallest C.S.C-Subgroup:

We shall show that ranks of a group and of its smallest c.s.c-subgroup differ by the difference of the 2-ranks of the group and of its smallest c.s.c-subgroup. To get at this, we shall first prove some lemmas.

Lemma 1- In a group \( G \), if a set of elements \( \{g_i\}_{i=1}^n \), containing no element of order 2 is linearly independent implies \( \{g_i^2\}_{i=1}^n \) is linearly independent in \( G^* \).
Proof. Let for a system of integers \( \{\alpha_i\}_{i=1}^n \),

\[
\begin{pmatrix}
(g_1^{\alpha_1})^1 & (g_1^{\alpha_2})^2 & \cdots & (g_1^{\alpha_n})^n \\
(g_2^{\alpha_1})^1 & (g_2^{\alpha_2})^2 & \cdots & (g_2^{\alpha_n})^n \\
\vdots & \vdots & \ddots & \vdots \\
(g_n^{\alpha_1})^1 & (g_n^{\alpha_2})^2 & \cdots & (g_n^{\alpha_n})^n
\end{pmatrix} = e
\]

\[\Rightarrow g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_n^{\alpha_n} = e\]

\[\Rightarrow g_1^{\alpha_i} = e \text{ since } \{g_1\}_{i=1}^n \text{ is linearly independent.}\]

\[\Rightarrow (g_1^{\alpha_i})^i = e \text{ for all } i = 1, 2, \ldots, n.\]

Hence, since \( \{g_1\}_{i=1}^n \) is a system of elements \((\neq e)\), it is linearly independent.

Lemma 2 - In a group \( G \), \( r_0(G) = r_0(G^*) \) and also for any prime \( p \neq 2 \), \( r_p(G) = r_p(G^*) \).

Proof. For the first part \( r_0(G) = r_0(G^*) \), let \( \{g_\alpha\}_{\alpha \in \mathbb{A}} \) be any maximal linearly independent system of elements of infinite order in \( G \). Then \( \{g_\alpha^2\}_{\alpha \in \mathbb{A}} \) is a system of elements of order infinity in \( G^* \). Since linear independence is a property of finite character, it follows from lemma 1, that \( \{g_\alpha^2\}_{\alpha \in \mathbb{A}} \) is a linearly independent system of elements in \( G^* \). Hence

\[ r_0(G) \leq r_0(G^*) \]

But clearly, since \( G^* \subseteq G \),

\[ r_0(G^*) \leq r_0(G) \]
Consequently
\[ r_o(G) = r_o(G^*) \]

For the second part \( r_p(G) = r_p(G^*) \) for all primes \( p(\neq 2) \), let \( \{g^p\}_{\mu \in \mathbb{C}M} \) be a maximal linearly independent system of elements in \( G \) containing elements of orders of powers of a prime \( p(\neq 2) \). Again, it is evident from lemma 1, that \( \{g^2\}_{\mu \in \mathbb{C}M} \) is a linearly independent system of elements in \( G^* \) and also it can be checked that orders of elements in it are powers of same prime \( p \).

Hence
\[ r_p(G) \leq r_p(G^*) \]

Also, it is clear that
\[ r_p(G^*) \leq r_p(G) \]

Thus
\[ r_p(G) = r_p(G^*) \]

This completely proves the lemma.

Theorem 3.7 - For any group \( G \), \( r(G) + r_2(G^*) = r(G^*) + r_2(G) \).

Proof. We know from [3] theorem 8.2, that
\[
\begin{align*}
r(G^*) &= r_o(G^*) + \sum_{p=2,3,5,\ldots} r_p(G^*) \\
&= r_o(G) + \sum_{p=3,5,\ldots} r(G) + r_2(G^*) \text{ by lemma 2.}
\end{align*}
\]
\[ \Rightarrow r(G^*) = r(G) - r_2(G) + r_2(G^*) \]

\[ \Rightarrow r(G) + r_2(G^*) = r(G^*) + r_2(G) \]

This completes the theorem.

**Theorem 3.8** - For any group \( G \),

(i) \[ r(G^*) = r(G) - \log_2 \frac{|G[2]|}{|G^*[2]|} \] if \( r_2(G) < \infty \).

(ii) \[ r(G) - r(G^*) = |G[2]| - |G^*[2]| \] if \( r_2(G) = \infty \).

**Proof.** (1) We know from theorem 3.7, that

\[ r(G) - r(G^*) = r_2(G) - r_2(G^*) \]

Now

\[ r_2(G) = r(G_2) \text{ where } G_2 \text{ is the } 2\text{-component of } \]

the maximal torsion subgroup of \( G \).

\[ = r(S(G_2)) \]

\[ = r(g_2[2]) \]

\[ = r(G[2]) \]

From [3] p. 33

\[ |S(G_2)| = 2^k \text{ where } K = r(S(G_2)) \]

\[ \Rightarrow |G[2]| = 2^{r_2(G)} \]

\[ \Rightarrow r_2(G) = \log_2 |G[2]| \]
Similarly
\[ r_2(G^*) = \log_2 |G^*[2]| \]

Hence
\[ r(G) - r(G^*) = \log_2 \frac{|G[2]|}{|G^*[2]|} \]

This proves (i)

(ii) If \( r_2(G) = \infty \),
\[ r(G_2) = r(S(G_2)) = \infty \]
\[ \Rightarrow r(S(G_2)) = |S(G_2)| \quad \text{by [3] p. 33} \]
\[ \Rightarrow r_2(G) = |G[2]| \]

Similarly
\[ r_2(G^*) = |G^*[2]| \]
\[ \Rightarrow r(G) - r(G^*) = |G[2]| - |G^*[2]| \]

This completes the proof.

Cor. 3.6 - In a group \( G \), \( r(G) = r(G^*) \) if and only if \( |G[2]| = |G^*[2]| \)

5. Smallest C.S.C-Subgroup And Fundamental Mappings:

We observe that any homomorphism of a group onto another group induces a homomorphism of the smallest c.s.c-subgroup of the group onto the smallest c.s.c-subgroup of the image group, and further establish that the formation of smallest c.s.c-subgroup is a homomorphism of the semigroup of all subgroups of a group \( G \) onto the semigroup of all subgroups of \( G^* \).

Theorem 3.9 - If \( \phi \) be a homomorphism of a group \( G \) onto a group \( G^* \), then the smallest c.s.c-subgroup of \( G \) is the homomorphic image of the smallest c.s.c-subgroup of \( G \) under \( \phi \).

Proof. From Theorem 3.1, we know that \( G^* \) is a c.s.c-subgroup of \( G \) and hence from cor. 2.9, \((G^*) \phi \) is c.s.c-subgroup of \( G^* \). Thus by Cor. 3.1,
Again from theorem 3.1, for any \( g \in G \)
\[
(g^2)\phi = (g \phi)^2 \in G_1^*
\]

\( \Rightarrow (G^*)\phi \subseteq G_1^* \)

Hence,
\[
(G^*)\phi = G_1^*
\]

This proves the theorem.

Theorem 3.10 - If \( H_1, H_2 \) be any two subgroups of a group \( G \) then,

\[
[H_1 \cdot H_2]^* = H_1^* \cdot H_2^*
\]

Proof. Clearly
\[
H_i^* \subseteq [H_1 \cdot H_2]^* \quad \text{for } i = 1, 2
\]

\( \Rightarrow H_1^* \cdot H_2^* \subseteq [H_1 \cdot H_2]^* \).

On the other hand, if \( h_i \in H_i, \ i = 1, 2 \) then \( (h_1h_2)^2 \in [H_1 \cdot H_2]^* \).

Now
\[
(h_1h_2)^2 = h_1^2 h_2^2 \in H_1^* \cdot H_2^*
\]

\( \Rightarrow [H_1 \cdot H_2]^* \subseteq H_1^* \cdot H_2^* \)

Consequently
\[
[H_1 \cdot H_2]^* = H_1^* \cdot H_2^*
\]

This proves the theorem.

Theorem 3.11 - If \( \{H_i\} \) denotes the semigroup of all subgroups of a group \( G \) then,

\[
H_i \longrightarrow H_i^*
\]

is a homomorphism of \( \{H_i\} \) onto the semigroup of all subgroups of \( G^* \).

Proof. (It immediately follows from theorems 3.3 and 3.10).
6. Essential Equality In Subgroups And Their Smallest C.S.C-Subgroups:

We define here a new concept of essential equality between subgroups of a group and establish that any two subgroups of a group have the same smallest c.s.c-subgroup if and only if they are essentially equal and note that the 'essential equality' between subgroups of a group is an equivalence relation. This fact enables us to partition the family of all subgroups of a group into disjoint subclasses such that all subgroup in one class have the same smallest c.s.c-subgroup. Further, we also observe that product of any two such partition classes is in a partition class. We show also that any two subgroups of a group have the same smallest c.s.c-subgroup if and only if so do their isomorphic images; in case of homomorphism however, the c.s.c-subgroups of given subgroups should contain the kernel.

Def. 3.3 - Any two subgroups $H_1, H_2$ of a group $G$ are said to be essentially equal if

$$H_1 O_2 = H_2 O_2$$

where $O_2$ is the subgroup of all elements of order 2 in $G$, in symbols, $H_1 \mathop{\longrightarrow}^\sim H_2$.

The following observations can be easily checked:

(i) If $H_1, H_2$ be any two subgroups of a group $G$ such that $H_i \supseteq O_2$ for $i = 1, 2$ then,

$$H_1 \mathop{\longrightarrow}^\sim H_2 \text{ if and only if } H_1 = H_2$$

(ii) For every subgroup $H$ of a group $G, H \mathop{\longrightarrow}^\sim H O_2^f$ where $O_2^f$ is a subgroup of $O_2$. 
(iii) In a group $G$, $O_2$ and any of its subgroup are essentially equal.

(iv) Essential equality is an equivalence relation in the family of all subgroups of a group.

Theorem 3.12 - Let $H_1$, $H_2$ be any two subgroups of a group $G$ then,

$$H_1^* = H_2^* \text{ if and only if } H_1 \leftrightarrow H_2.$$ 

Proof. Let $H_1^* = H_2^*$, then to any $h_1 \in H_1$ there exists $h_2 \in H_2$ such that

$$h_1^2 = h_2^2$$

$$\implies h_1 h_2^{-1} \in O_2 \text{ where } O_2 \text{ is the subgroup of all elements of order } 2 \text{ in } G.$$ 

$$\implies h_1 \in h_2 O_2$$

$$\implies H_1 O_2 = H_2 O_2$$

$$\implies H_1 \leftrightarrow H_2$$

Conversely, let

$$H_1 \leftrightarrow H_2$$

$$\implies H_1 O_2 = H_2 O_2$$

$$\implies (H_1 O_2)^* = (H_2 O_2)^*$$

$$\implies H_1^* O_2^* = H_2^* O_2^* \quad [\text{Theorem 3.10}]]$$

$$\implies H_1^* = H_2^* \text{ since } O_2^* = e$$
\[ \implies H_1^* = H_1^* \]
\[ \implies H_1^* \cdot H_2^* = H_1^* \cdot H_2^* \]
\[ \implies [H_1 \cdot H_2]^* = [H_1^* \cdot H_2^*]^* \quad \text{(Theorem 3.10)} \]
\[ \implies H_1 \cdot H_2 \iff H_1^* \cdot H_2^* \text{ since } [H_1 \cdot H_2] = H_1 H_2, \]
\[ [H_1^* \cdot H_2^*] = H_1^* \cdot H_2^* \]

This completes the proof.

**Theorem 3.14** - If \( H_1, H_2 \) be two subgroups of a group \( G \) and \( \phi \) be an isomorphism of \( G \) onto a group \( G' \), then

\[ H_1 \iff H_2 \text{ if and only if } (H_1 \phi)^* \iff (H_2 \phi)^* \]

**Proof.** By theorem 3.12,

\[ H_1 \iff H_2 \]
\[ \iff H_1^* = H_2^* \]
\[ \iff (H_1^* \phi)^* = (H_2^* \phi)^* \quad \text{(Theorem 3.9)} \]
\[ \iff (H_1 \phi)^* \iff (H_2 \phi)^* \quad \text{(Theorem 3.12)} \]

Hence the theorem is complete.

**Cor. 3.9** - If \( H_1, H_2 \) be any two subgroups of a group \( G \) and \( \phi \) be an isomorphism of \( G \) onto a group \( G' \), then

\[ H_1^* = H_2^* \text{ if and only if } (H_1 \phi)^* = (H_2 \phi)^* \]
Theorem 3.15 - If $H_1, H_2$ be any two subgroups of a group $G$ and $\phi$ be a homomorphism of $G$ onto a group $G'$ with kernel $K$, then if $H_1^*$ and $H_2^*$ contain $K$.

$$H_1 \cong H_2 \text{ if and only if } (H_1^*)^\phi \cong (H_2^*)^\phi$$

Proof. If $H_1 \cong H_2 \Rightarrow (H_1^*)^\phi \cong (H_2^*)^\phi$ follows as in theorem 3.14.

For converse, if

$$(H_1^*)^\phi \cong (H_2^*)^\phi$$

$$\Rightarrow (H_1^*)^\phi = (H_2^*)^\phi$$

$$\Rightarrow (H_1^*)^\phi = (H_2^*)^\phi$$

$$\Rightarrow H_1^* = H_2^* \text{ since } K \subseteq H_1^*$$

$$\Rightarrow H_1 \cong H_2$$

Hence the theorem is complete.

Cor. 3.10 - If $H_1, H_2$ be any two subgroups of a group $G$ and $\phi$ be a homomorphism of $G$ onto $G'$, then

$$H_1^* = H_2^* \text{ if and only if } (H_1^\phi)^* = (H_2^\phi)^* \text{ where } H_1^* \supseteq \text{Ker. } \phi.$$

7. $G^*$ As A Cyclic Subgroup:

We investigate below the circumstances in which the smallest c.s.c-subgroup of a group becomes cyclic and find out a condition under which a group is cyclic if its smallest c.s.c-subgroup is cyclic.
Theorem 3.16 - If a group $G$ is cyclic, then $G^*$ is cyclic but conversely $G$ is cyclic if $G^*$ is cyclic and contains $O_2$, the subgroup of all elements of order 2 in $G$.

Proof. Firstly, if $G$ be cyclic, then clearly $G^*$ is cyclic. Conversely if $G^*$ be cyclic, let

$$G^* = [g_1^2] \text{ where } g_1 \in G$$

Then for any $g \in G$, since $g^2 \in G^*$,

$$g^2 = (g_1^2)^i \text{ where } i \in I$$

$$\Rightarrow (g g_1^{-1})^2 = e$$

$$\Rightarrow g g_1^{-1} \in O_2 \subseteq G^*$$

$$\Rightarrow g \in g_1 [g_1^2] \subseteq [g_1]$$

$$\Rightarrow G \subseteq [g_1] \subseteq G$$

Hence $G = [g_1]$

This completes the theorem.

Cor 3.11 - If $G$ be a group in which $O_2 = e$, then $G$ is cyclic if and only if $G^*$ is cyclic.

Theorem 3.17 - If $G$ be any finitely generated group, then $G^*(\neq e)$ is cyclic if any basis of $G$ contains only one element of order other than 2.

Proof. (It immediately follows from Cor.3.4, since

$$G^* = [a_1^2] \times \ldots \times [a_n^2]$$

where $\{a_1, a_2, \ldots, a_n\}$ be a basis of $G$)
Note: The theorem 3.17, also holds for any group G having basis.

8. Relations In G/H, (G/H)* And G*/H*:

We show now an interesting situation where a factor group, its smallest c.s.c-subgroup and the factor group of the corresponding smallest c.s.c-subgroups are all isomorphic.

Theorem 3.18. If H be any subgroup of a group G, then

\[(G/H)^* \cong G^*/H^*\]

Proof. Since every element of \((G/H)^*\) is of the form \(g^2 H\).

We define a mapping \(\phi\) of \((G/H)^*\) onto \(G^*/H^*\) as

\[\phi : g^2 H \longrightarrow g^2 H^*\]

Evidently \(\phi\) is single valued. If \(g_1^2 H, g_2^2 H \in (G/H)^*\), then

\[\((g_1^2 H)(g_2^2 H))\phi = (g_1^2 g_2^2 H)\phi\]

\[= ((g_1 g_2) g_2 H)\phi\]

\[= (g_1 g_2)^2 H^*\]

\[= (g_1^2 H^*)(g_2^2 H^*)\]

\[= (g_1^2 H)\phi (g_2^2 H)\phi\]

\[\implies \phi\ is\ a\ homomorphism\]

Finally, if

\[(g_1^2 H)\phi = (g_2^2 H)\phi\]
Hence \( \phi \) is an isomorphism

This proves the theorem.

Theorem 3.19 - If \( H \) be any subgroup of a group \( G \), then

\[
\frac{G}{H} \cong \frac{G^*/H^*}{G/H}
\]

and the kernel of the homomorphism is \( H O_2^*/H \) where \( O_2 \) is the subgroup of all elements of order 2 in \( G \).

Proof. We define a mapping \( \eta \) of \( G/H \) onto \( G^*/H^* \) as

\[
\eta : gH \longrightarrow g_1^2 H^*
\]

Clearly \( \eta \) is single valued. Further for \( g_1 H, g_2 H \in G/H \)

\[
((g_1 H)(g_2 H))\eta = (g_1 g_2 H)\eta
\]

\[
= (g_1 g_2)^2 H^*
\]

\[
= (g_1^2 H^*)(g_2^2 H^*)
\]

\[
= (g_1 H)\eta (g_2 H)\eta
\]

\[\Rightarrow \eta \text{ is a homomorphism.}\]
Finally, if
\[(gH)^n = H^*\]
\[<=> \quad g^2 H^* = H^*\]
\[<=> \quad g^2 \in H^*\]
\[<=> \quad g^2 = H^2 \quad \text{where } h \in H\]
\[<=> \quad gH^1 \in O_2\]
\[<=> \quad g \in hO_2\]
\[<=> \quad g \in H^2\]

Hence the theorem is complete.

Cor. 3.12 - If \( H \supseteq O_2 \), \( G/H \cong G^*/H^* \).

(Proof follows immediately, since by fundamental theorem of homomorphism \( G/H \cong G^*/H^* \)).

Cor. 3.13 - For any subgroup \( H \) of a group \( G \), if \( H \supseteq O_2 \)
\[ G/H \cong (G/H)^* \cong G^*/H^* \].

Proof (It immediately follows from theorem 3.18 and Cor. 3.12)

9. Non Isomorphic Groups Having Isomorphic Smallest C.S.C. Subgroups:

Theorem 3.20 - Any two isomorphic groups have isomorphic smallest c.s.c. subgroups but not conversely.

Proof. The direct part of the theorem is obvious in view of theorem 3.9. For the converse, consider
\[ G = [a, b]; \, a^3 = e, \, b^2 = e, \, ab = ba \]

\[ G_1 = [a'], \text{ the cyclic group of order } 3. \]

which give

\[ G^* = \{a, a^2, e\}, \, G_1^* = \{a', a'^2, e\} \]

We observe that \( G^* \) and \( G_1^* \) being cyclic groups of the same order are isomorphic but \( G \) is not isomorphic to \( G_1 \) since \( O(G) > O(G_1) \).

Thus the converse does not hold.

Theorem 3.21 - If \( \phi \) be an isomorphism of a group \( G \) into a group \( G_1 \), then \( G^* \cong G_1^* \) if and only if \( (G) \phi \xrightarrow{\sim} G_1 \).

Proof. (The proof is immediate in view of theorem 3.9 and 3.12)

Cor. 3.14 - Any two groups whose any two bases under an isomorphism of one group into the other differ only by elements of order 2, have isomorphic smallest c.s.c-subgroups under the same isomorphism.

Proof. Let, for two groups \( G, G_1 \) and for an isomorphism \( \phi \) of \( G \) into \( G_1 \) the condition of the corollary be satisfied, then it is evident that \( (G) \phi \xrightarrow{\sim} G_1 \), and hence, from the above theorem, the proof follows immediately.

10. \( G_1 \cap G^* = G_1^* \) For An Arbitrary Subgroup \( G_1 \).

We determine, in this section, a condition for the smallest c.s.c-subgroup of an arbitrary subgroup of a group to be equal to the intersection of the smallest c.s.c-subgroup of the group with the arbitrary subgroup, and further find out under the same
condition, the existence of a c.s.c-subgroup of the group corresponding to each c.s.c-subgroup of an arbitrary subgroup.

Theorem 3.22 - If \( G \) be any subgroup of a group \( G \), then

\[ G \cap G^* = G^* \text{ if and only if } (G - G^*) \cap G^* = \emptyset \]

Proof. Let

\[ (G - G^*) \cap G^* = \emptyset \]

\[ \implies G^* \subseteq (G - (G - G^*)) = G^* \]

But clearly

\[ G^* \subseteq G \]

\[ \implies G_1^* = G^* \]

Conversely, let \( G \cap G^* = G^* \). Now suppose \( g_1 \in (G_1 - G_1^*) \) such that

\[ g_1^2 \in G^* \]

\[ \implies g_1 \in G_1 \cap G^* = G^* \]

A contradiction, hence

\[ (G - G_1^*) \cap G^* = \emptyset \]

Thus the proof is complete.

Theorem 3.23 - If for a subgroup \( G_1 \) of a group \( G \), \( G \cap G^* = G^* \) then for any c.s.c-subgroup \( \overline{G} \) of \( G \) there exists a c.s.c-subgroup \( \overline{G_1} \) of \( G \) such that \( \overline{G_1} = \overline{G} \cap G_1 \).
Proof. We define
\[ \overline{G} = [\overline{G}_1, G^*] \]

Evidently, $G$ is a c.s.c-subgroup of $G$ since $\overline{G} \supseteq G^*$. Also
\[ \overline{G}_1 \subseteq \overline{G} \cap G_1 \]

Now, if $g^* \in \overline{G} \cap G_1$, we have
\[ g^* = g_1 = g^* \overline{g} \text{ where } g_1 \in G_1, g^* \in G^*, \overline{g} \in \overline{G}_1 \]
\[ \Rightarrow g^* = g_1 (\overline{g})^{-1} \in G_1 \]
\[ \Rightarrow g^* \in G^* \text{ since } G_1 \cap G^* = G_1^* \]
\[ \Rightarrow g^* = g^* \overline{g} \in \overline{G}_1 \text{ since } G_1^* \subseteq \overline{G}_1 \]
\[ \Rightarrow \overline{G} \cap G_1 \subseteq \overline{G}_1 \]

Consequently
\[ \overline{G}_1 = \overline{G} \cap G_1 \]

Thus the result is established.

11. Groups With Identical Smallest C.S.C-Subgroups:

Def. 3.4 - A group $G$ is called C-simple if $G = G^*$.

Evidently the following are some of the classes of C-simple groups.

(i) Finite groups of odd order

and

(ii) Groups having a basis each of whose element is of odd order.

Theorem 3.24 - The union of an ascending sequence of C-simple groups is itself a C-simple group.

Proof. Let a group $G$ be union of an ascending sequence
of C-simple proper subgroups of $G$. If $G$ is not C-simple, then there exists a proper c.s.c-subgroup $H$ of $G$. Now, since $H \subseteq G$, for some index $k$,

$$G_k \cap H = G_k$$

$$\implies G_k^* \neq G_k$$

since $G_k = G_k^*$ and $H \supseteq G^* \supseteq G_k^*$

A contradiction, that $G_k$ is C-simple. Hence the result follows.

Theorem 3.25 - If every subgroup of a group is G-simple then the group is C-simple but not conversely.

Proof. Let

$$G = \left\{ (r_1, r_2, \ldots, r_n, \ldots) \middle| r_i, r_j \in \mathbb{R} \right\}$$

be the group of all sequences of rational numbers with respect to addition and suppose

$$H = \left\{ (i_1, i_2, \ldots, i_n, \ldots) \middle| i_i, i_j \in I \right\}$$

be the subgroup of $G$ consisting of all sequences of integers, then it is evident that

$$G^* = G$$

but

$$H \neq H^*$$

Conversely, if the condition be satisfied, the proof is trivial.

This completes the theorem.
12. Anticenter And Smallest C.S.C-Subgroup:

This section is of more or less academic interest. Here, we establish that anticenter of the smallest c.s.c-subgroup of a group is the smallest c.s.c-subgroup of the anticenter of the group. To begin with we give the following definitions in case of an arbitrary (abelian or non-abelian) group.

Def.3.5 - In a group G, the set \( R(G) = \{ g \mid gh = hg \text{ for any } h \in G \implies g = k^i, h = k^j \text{ for some } k \in G, i,j \in \mathbb{I} \} \) is called rim of G.

Def.3.6 - If G be a group then the subgroup generated by \( R(G) \), the rim of G is called 'anticenter' of G and is denoted by \( AC(G) \).

Theorem 3.26 - Let G be any group in which \( O_2 \), the subgroup of all elements or order 2 in G be identity, then

\[ AC(G^*) = (AC(G))^* \]

Proof. Let \( g \in AC(G) \) then for any \( h \in G \)

\[ g^2 h^2 = h^2 g^2 \]

\[ \implies g h = h g \]

\[ \implies g = g_1^i, h = g_1^j, i,j \in \mathbb{I}, g_1 \in G \]

\[ \implies g^2 = (g_1^i)^1, h^2 = (g_1^j)^j \text{ where } g_1^2 \in G^* \]

\[ \implies g^2 \in R(G^*) \]

\[ \implies g^2 \in AC(G^*) \]
On the other hand, if \( g^2 \in \text{AC}(G^*) \) then for any \( h \in G \), we have

\[
gh = hg
\]

\[
\Rightarrow g^2h^2 = h^2g^2
\]

\[
\Rightarrow g^2 = (g^2)^1, h^2 = (g^2)^j, i, j \in I, g^2 \in G^*
\]

\[
\Rightarrow (gg^t)^{-1} = e, (hg^t)^{-1} = e
\]

\[
\Rightarrow gg^t = e, hg^t = e \text{ since } O_2 = e
\]

\[
\Rightarrow g = g^t, h = g^t \text{ where } g^t \in G
\]

\[
\Rightarrow g \in \text{AC}(G)
\]

\[
\Rightarrow g^2 \in (\text{AC}(G))^*
\]

Hence,

\[
\text{AC}(G^*) \subseteq (\text{AC}(G))^*
\]

Consequently,

\[
\text{AC}(G^*) = (\text{AC}(G))^*
\]

This completes the theorem.

Cor. 3.15 - In any group \( G \), if \( r_2(G) = 0 \), \( \text{AC}(G^*) = (\text{AC}(G))^* \)

where \( r_2(G) \) is the 2-rang of \( G \). In particular if \( G \) be torsion free the result holds.
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