1. Introduction: In this chapter, we introduce the concepts of "Compression series" and "Compression chains" of a group. We call a finite descending system of subgroups of a group which begins with the group itself and ends in the smallest c.s.c-subgroup of the group, a compression series of the group. We define the isomorphism of two compression series, refinement of a compression series and complete series, then study analogously to that of normal series of a group and obtain analogous of Schreier and Jordan-Holder theorem. We find that the compression series of a group $G$ and normal series of $G/\text{c}\cdot \text{s}\cdot \text{c}$ are in one-one correspondence, and the corresponding series have the same length. We also establish that any group $G$ having a basis, for which $r_0(G) + r_2(G)$ has a complete series of length $r_0(G) + r_2(G)$. Further, we define the concept of descending compression chain of a group. In the study of descending compression chains, the most important is the shortest compression chain which is the descending sequence $G \supseteq G^* \supseteq (G^*)^* \supseteq \ldots \ldots \ldots \ldots$ of subgroups of $G$. We show that the shortest compression chain of any subgroup of a group which occurs in some descending compression chain of the group breaks off, if the shortest compression chain of the group breaks off; and both the chains end in the same subgroup. The shortest compression chain of a finitely generated subgroup breaks off if and only if the group is periodic. The important fact to be noted here is that if this chain breaks off in identity subgroup, then all the
elements of the group have orders of powers of 2. The criterion for the shortest compression chain of a group $G$ having basis, for which $r_1(G) + r_2(G) < \infty$, to break off is that every descending compression chain of $G$ should break off.

Finally, it is important to observe that groups having all ascending and descending compression chains to be finite have complete series.

2. Concept Of Compression Series:

Def.7.1- A finite system of subgroups of a group $G$,

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \ldots \supset G_k = G^*$$

beginning with $G$ and ending with the smallest c.s.c-subgroup $G^*$ of $G$ is called a 'Compression series' of $G$.

It is evident that every $G_i$ in the series (i) is a c.s.c-subgroup of $G$. Any group $G$ for which $G \supset G^*$ has a compression series. If $H$ be any c.s.c-subgroup of $G$, distinct from $G$ and $G^*$, then

$$G \supset H \supset G^*$$

is a compression series. This means that for any given c.s.c-subgroup of $G$ other than $G$ and $G^*$, there exists compression series that passes through it.

The factor groups

$$G/G_1, G_1G_2, \ldots, G_{k-1}/G^*$$

are called the factors of the compression series (i). The number of factors is called the length of the series, the length of series (i) is, for example, $k$. 
3. Properties Of Compression Series:

Def. 7.2 - Any two compression series of a group are isomorphic if there exists a one-to-one correspondence between the factors of the series such that the paired factors be isomorphic.

Def. 7.3 - For any group G a compression series

\[ G \supset H_1 \supset H_2 \cdots \supset H_k = G^* \]

is called a 'refinement' of the compression series

\[ G \supset G_1 \supset G_2 \cdots \supset G_k = G^* \]

if every subgroup \( G_1 \) coincides with one of the subgroups \( H_j \).

Evidently, every compression series is a refinement of itself and the length of a compression series is less than or equal to the length of its refinement. We have, here, an analogue of Schreier theorem on isomorphic refinements.

Theorem 7.1 - Any two compression series of a group G have isomorphic refinements.

Proof. Let

\[ G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_l = G^* \quad (i) \]

\[ G = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_m = G^* \quad (ii) \]

be any two compression series of G. Consider
Evidently, $G_{ij}$ and $H_{ij}$ are c.s.c-subgroups of $G$, and for all $i = 1, 2, \ldots, \ell$, $j = 1, 2, \ldots, m$, we have

$$G_{i-1} = G_i \supseteq G_{i, j-1} \supseteq G_{ij} \supseteq G_{im} = G_1$$

$$H_{j-1} = H_j \supseteq H_{i-1, j} \supseteq H_{ij} \supseteq H_{\ell j} = H_j$$

Thus, we may now obtain refinements of (i) and (ii) by inserting $G_{ij}, j = 1, 2 \ldots m$ between $G_{i-1}$ and $G_i$ for all $i = 1, 2, \ldots, \ell$ and $H_{ij}, i = 1, 2, \ldots, \ell$ between $H_{j-1}$ and $H_j$ for all $j = 1, 2, \ldots, m$ respectively. These are in general compression series with repetitions. By Zassenhaus lemma, $G_i,j-1/G_{ij} \cong H_{i-1,j}/H_{ij}$, hence these refinements are isomorphic. Finally, it is also clear that whenever $G_i,j-1 = G_{ij}$ then also $H_i,j-1 = H_{ij}$ and therefore we can eliminate simultaneously the repetitions in these refinements of series (i) and (ii) without effecting isomorphism.

This completes the proof.

Theorem 7.2 - For any group $G$, there exists a 1-1 correspondence between compression series of $G$ and normal series of $G/G^*$, moreover the corresponding series have the same length.
Proof. Let
\[ G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_k = G^* \]
be a compression series of \( G \). If we put \( G/G^* = \overline{G} \) and \( G_1/G^* = \overline{G_1} \) for all \( i = 1, 2, \ldots, k \) then
\[ B/G^* = \overline{G} \supset \overline{G_1} \supset \overline{G_2} \supset \ldots \supset \overline{G_k} = E \in G/G^* \]
is a normal series of \( G/G^* \) since \( G \) is abelian. Conversely, if
\[ G/G^* = \overline{H_0} \supset \overline{H_1} \supset \ldots \supset \overline{H_l} = E \]
be any normal series of \( G/G^* \), let \( H_i \) denotes the inverse image of \( \overline{H_i} \) under the natural homomorphism of \( G \) onto \( G/G^* \) for all \( i = 0, 1, \ldots, l \), then
\[ G = H_0 \supset H_1 \supset H_2 \supset \ldots \supset H_l = G^* \]
is a compression series of \( G \). Thus we note that to each compression series of \( G \) there corresponds a normal series of \( G/G^* \) of the same length and vice versa, moreover, this correspondence is 1-1 since there exists a 1-1 correspondence between the subgroups of \( G \) containing \( G^* \) and subgroups of \( G/G^* \) under the natural homomorphism of \( G \) onto \( G/G^* \).

This completes the theorem.

Note: The above theorem suggests another proof of theorem 7.1, in view of Schreiers theorem.

Def. 7.4 - A compression series of a group \( G \) that has no refinement other than itself is called a complete series.
Let \( G = G_0 \supset G_1 \supset G_2 \supset \ldots \ldots \supset G_k = G^* \)

be a complete series of a group \( G \) then every \( G_i, \ i = 1, 2, \ldots k \) is a maximal c.s.c-subgroup of \( G \) in \( G_{i-1} \). All factors \( G_{i-1}/G_i \) of the series have identity subgroup to be the maximal c.s.c-subgroup other than itself i.e. all elements of \( G_{i-1}/G_i \) are of order 2 and moreover \( [G_{i-1}: G_i] = 2 \). Conversely, every compression series of \( G \) all of whose factors have identity subgroup to be the maximal c.s.c-subgroup of the group other than itself cannot be further refined i.e. such a series is a complete series. Evidently, every compression series isomorphic to a complete series is a complete series. This fact gives a theorem equivalent to Jordan Holder theorem for compression series.

Theorem 7.3 - If a group \( G \) has a complete series then any two complete series of \( G \) are isomorphic.

(Proof is immediate, from theorem 7.1)

Lemma 7.1 - Let \( G \) be a finitely generated group such that \( r_o(G) + r_2(G) = m \), then \( G \) has a complete series of length \( m \).

Proof. Let \( \{g_1, g_2, \ldots g_m\} \) denotes the set of all elements of infinite order and of order \( 2^k(|x|) \) in a basis of \( G \) containing elements of prime power and / or infinite order.

We define

\[
G_i = [G^*, g_1, g_2, \ldots, g_i], \quad \text{The subgroup generated by } G^*
\]

and \( g_1, g_2, \ldots g_i \).
for all \(1 \leq i \leq m\), then

\[
\left[ G_i : G^* \right] = 2^i \quad \text{(Theorem 3.6)}
\]

Now, it is evident that

\[
G = G \supset G_{m-1} \supset \ldots \supset G_2 \supset G_1 \supset G_0 = G^*
\]

is a compression series of \(G\) of length \(m\). This is a complete series of \(G\), since

\[
\left[ G_i : G_{i-1} \right] = 2 \quad \text{for all } i = 1, 2, \ldots, m.
\]

Hence the proof is complete.

Note: In view of the remark following the Theorem 3.6, the result proved above holds true for any group \(G\) having a basis for which \(r_0(G) + r_2(G) < \infty\).

Theorem 7.4 - Let \(G\) be a group having a basis and \(r_0(G) + r_2(G) < \infty\) then any complete series of \(G\) has length \(r_0(G) + r_2(G)\).

(Proof of theorem is immediate from the note following lemma 7.1 and theorem 7.3).

Theorem 7.5 - A group \(G\) has a complete series if and only if \(G/G^*\) has a composition series.

(Proof is evident from theorem 7.2).

Theorem 7.6 - Let

\[
G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_k = G^*
\]
be a complete series of a group $G$. Then every subgroup $H$ of $G$ such that $G^* \cap H = H^*$, has a compression series whose factors are isomorphic to subgroups of distinct factors of the complete series.

Proof. We define

$$H_i = H \cap G_i \quad \text{for all } i = 0, 1, 2, \ldots, k.$$ 

By corollary 3.1, every $H_i$ is a c.s.c-subgroup of $H$. Further, it is clear that $H_{i-1} \supseteq H_i$ for all $i = 1, 2, \ldots, k$. Now, since $H_{i-1} \supseteq H_i$, $G_{i-1} \supseteq G_i$ we have by Zassenhaus lemma

$$H_{i-1}/H_i \cong G_i H_{i-1}/H_i$$

Here $G_{i-1} \supseteq G_i H_{i-1} \supseteq G_i$, hence the factor group $H_{i-1}/H_i$ is isomorphic to a subgroup of the factor group $H_{i-1}/H_i$. Thus the series

$$H = H_0 \supseteq H_1 \supseteq H_2 \supseteq \ldots \supseteq H_k = H^*$$

after deletion of repetitions is the required compression series of $H$.

This completes the theorem.

4- Compression Chains:

Def. 7.5 - A descending sequence of subgroups of a group $G$

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots \ldots \ldots \quad (i)$$

is called a 'descending compression chain' of $G$ if every subgroup $G_n$, $n = 1, 2 \ldots$ is a proper c.s.c-subgroup of $G_{n-1}$. 
The descending compression chain (i) may be finite or countably infinite. In the former case, we say that the chain breaks off. If we define $(G^*)^* = G^{2*}$, $(G^{2*})^* = G^{3*}$ and in general $(G^{(n-1)*})^* = G^{n*}$, then we obtain a descending sequence of subgroups of $G$

$$G = G^0 \supseteq G^* \supseteq G^{2*} \supseteq \ldots \supseteq G^{n*} \supseteq \ldots$$

which is clearly a descending compression chain of $G$. This we call 'shortest compression chain' of $G$. This chain may be continued transfinitely, if we define for any limit ordinal number $\alpha$, $G^{\alpha*}$ to be the intersection of $G^{\beta*}$, $\beta < \alpha$ and $G^{\alpha*} = (G^{(\alpha-1)*})^*$ if $\alpha$ is not a limit ordinal number. Evidently, for some ordinal number $\gamma$ whose cardinal number is not greater than that of the power of the group itself, we have

$$G^{\gamma*} = G^{(\gamma+1)*}$$

These subgroups $G^{\alpha*}$ obtained thus are fully invariant in $G$, since obviously the smallest c.s.c-subgroup of a group is fully invariant and the property of being fully invariant is transitive and is also preserved under operation of intersection.

Theorem 7.7- If the shortest compression chain of a group $G$ breaks off, then the shortest compression chain of any of its subgroup $H$ that occurs in some compression chain of $G$ breaks off; and both the chains end with the same subgroup. The length of the shortest compression chain of $G$ is greater than or equal to the length of the corresponding chain for $H$. 
Proof. Let the shortest compression chain of $G$ breaks off at the $n$th stage, then
\[ G_{n*} = G_{j*} \quad \text{for all } j \geq n. \]

Now, let $H$ be any subgroup of $G$ that occurs in a compression chain of $G$ and let that chain be
\[ G = H_0 \supset H_1 \supset H_2 \supset \ldots \ldots \supset H_m \supset \ldots \ldots \]

Then, evidently
\[ H_i \supset G_{i*} \quad \text{for all } i \geq 1 \]

\[ \Rightarrow H_i \supset G_{n*} \quad \text{for all } i \geq 0 \]

\[ \Rightarrow H \supset G_{n*} \quad \text{since } H \text{ is some } H_i \]

\[ \Rightarrow H_{n*} \supset G_{n*} \]

But, it is clear that
\[ G_{n*} \supset H_{n*} \]

\[ \Rightarrow H_{n*} = G_{n*} \]

\[ \Rightarrow H_{n*} = H_{i*} \quad \text{for all } i \geq n. \]

\[ \Rightarrow \text{The shortest compression chain of } H \text{ breaks off with the last elements equal to } G_{n*} \]

This proves the theorem completely.

Theorem 7.8 - Let the shortest compression chain of a group $G$ breaks off at the $n$th stage. Then a descending compression chain
of $G$ breaks off if for a subgroup $G_k$ in (i), (1) $G_k^* = G_{m*}^*$ and $G_k$ has a complete series, or (2) $G_k = G_{m*}^*$.

Proof. (1) Evidently, in the descending compression chain (i), we have

$$G \supset G_1 \supset G_2 \supset \ldots \ldots \supset G_k \supset \ldots \ldots \supset G_n \supset \ldots \ldots$$

Now, if $k$ be the smallest integer for which $G_k$ has a complete series, and that

$$G_k^* = G_{m*}^*$$

$$\Rightarrow G_1 \supset G_{m*} \supset G_k^*$$ for all $i > k$, since $G_{m*}^* = G_{i*}^*$ for all $i > m$.

$$\Rightarrow G_k \supset G_{k+1} \supset \ldots \ldots \text{is a part of a compression series of } G_k^*$$

Hence, since $G_k$ has a complete series, the descending chain (i) breaks off. This proves the first part.

For the 2nd part, let us have

$$G_k = G_{m*}^*$$

for some $G_k$ in (i).

$$\Rightarrow G_i = G_{m*}^*$$ for all $i > k$, since $G_{m*}^* = G_{i*}^*$ for all $i > m$.

$$\Rightarrow G_k = G_i$$ for all $i > k$. 
Hence the descending chain (i) breaks off.

This proves the theorem completely.

Theorem 7.9 - Let $G$ be a finitely generated group, then the shortest compression chain of $G$ breaks off if, and only if, $G$ be periodic.

Proof. Let $B = \{g_1, g_2, \ldots, g_n\}$ be a basis of $G$. If $G$ be periodic, then every $g_i$ ($1 \leq i \leq n$) is of finite order. From corollary 3.4,

$$G^* = [g_1^2] \times [g_2^2] \times \ldots \times [g_n^2]$$

Furthem from theorem 3.4

$$G^{2*} = [g_1^4] \times [g_2^4] \times \ldots \times [g_n^4]$$

.........................

$$G^{m*} = [g_1^{2m}] \times [g_2^{2m}] \times \ldots \times [g_n^{2m}]$$

......................... and so on.

It is clear that for some number $k > 0$, $0(g_1^2) = \text{odd}$, for all $g_i$'s and consequently

$$g_i^{k*} = g_i^i* \text{ for all } i > k$$

Hence the shortest compression chain of $G$ breaks off. Conversely, let the shortest compression chain of $G$ breaks off. Now, if $G$ is not periodic, then for some $1 \leq m_1 \leq n$, we have $0(g_{m_1}) = \infty$. Here, it is immediate that

$$g_i^{m'*} = g_i^{n'*} \text{ for any } (m' \neq n')$$
because, \([g_{m1}^{m}] \neq [g_{n1}^{n}]\) for any two integers \(m\), \(n\) \(\geq 0(m \neq n)\).

A contradiction to supposition that the shortest compression chain of \(G\) breaks off. This completes the proof.

Note. The above theorem also holds true for any group \(G\) having a basis orders of whose elements admit an upper bound. It also follows clearly from the theorem that the shortest compression chain of a finitely generated group breaks off at identity if, and only if, every element of the group be of order \(2^n(n \geq 0)\).

Theorem 7.10 - Let \(G\) be a group having basis such that \(r_0(G) + r_2(G) < \infty\). The shortest compression chain of \(G\) breaks off if and only if every descending compression chain of \(G\) breaks off.

Proof. If every descending compression chain of \(G\) breaks off, then clearly the smallest compression chain of \(G\) breaks off, since it is an ascending compression chain of \(G\). To prove the converse, let

\[
G = H_0 \supset H_1 \supset H_2 \supset \ldots \ldots \supset H_n \supset \ldots \ldots \ldots \ldots \quad (i)
\]

be a descending compression chain of \(G\). Evidently

\[
H_1 \supseteq G^*, \ H_2 \supseteq G^{*k}, \ldots, H_n \supseteq G^{*n}, \ldots
\]

Now, if the shortest compression chain of \(G\) breaks off at \(m\)th stage, we have \(G^{*m} = G^{*m}\) for all \(i \geq m\), hence \(H_i \supseteq G^{*m}\) for all \(i \geq 0\). We know that \(r_0(G) + r_2(G) < \infty\), hence by the remark following theorem 3.6

\[
[G : G^*] = r_0(G) + r_2(G) < \infty
\]

Further, it is clear that any subgroup \(H\) of \(G\) has a basis and \(r_0(H) + r_2(H) \leq r_0(G) + r_2(G) < \infty\) hence

\[
[G^{i*} : G^{*(i+1)*}] = r_0(G^{i*}) + r_2(G^{i*}) < \infty, \ \text{for all} \ i = 1, 2, \ldots, m-1
\]
Consequently

\[ [G : G^{m^*}] < \infty \]

\[ \Rightarrow \text{The chain } (i) \text{ breaks off since } H_i \supseteq G^{m^*} \text{ for all } i \geq 0. \]

This completes the theorem.

It is to be noted that even if a group \( G \) has a complete series, a subgroup of \( G \) may or may not have it. However, the following theorem holds:

Theorem 7.11 - Let \( G \) be a group having basis such that \( G \) has a complete series, then every subgroup \( H \) of \( G \) has a complete series.

Proof. Since \( G \) has a basis, every subgroup \( H \) of \( G \) has a basis. Further, since \( G \) has a complete series, it follows, in view of the remark following theorem 3.6, that

\[ [G : G^*] = 2^{r_0(G) + r_2(G)} < \infty \]

\[ \Rightarrow r_0(G) + r_2(G) < \infty \]

\[ \Rightarrow r_0(H) + r_2(H) \leq r_0(G) + r_2(G) < \infty \]

Hence, from the remark following lemma 7.1, it follows that \( H \) has a complete series.

This proves the theorem.

Def. 7.6 - An ascending sequence of subgroups of a group \( G \)

\[ G_1 \subset G_2 \subset G_3 \subset \ldots \ldots \subset G_n \subset \ldots \ldots \]
is called an 'ascending compression chain' of G if every $G_i$, $i = 1, 2, \ldots$ is a proper c.s.c-subgroup of $G_{i+1}$ and all the subgroups $G_i$'s occur in some descending compression chain of G.

We note here, that in general if the shortest compression chain of a group G breaks off, then every ascending or descending compression chain of G need not break off. For example

Let $G = \prod_{i=1}^{\infty} [a_i]$ where $a_i^2 = e$ for all $i$

$$G_n = \prod_{i=1}^{\infty} [a_i + n], \quad H_n = \prod_{i=1}^{n} [a_i] \quad \forall \ n = 1, 2, \ldots$$

Then, evidently

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \ldots \supset G_n \supset \ldots \ldots \quad (i)$$

and

$$H_1 \subset H_2 \subset H_3 \subset \ldots \ldots \subset H_n \subset \ldots \ldots \quad (ii)$$

are the descending and ascending compression chains of G respectively. These are infinite; but clearly the shortest compression chain of G breaks off at the 1st stage itself. However, the following result holds:

Theorem 7.12 - A group G has a complete series if all its ascending and descending compression chains break off.
Proof. The condition that every ascending compression chain of $G$ breaks off implies that every c.s.c-subgroup $H$ of $G$ contains a c.s.c-subgroup $H'$ of $G$ such that $H' \subset H$ is maximal subgroup of this type. In particular, $G = H_0$ has a maximal proper c.s.c-subgroup. Let, subgroups.

$$G = H_0, H_1, H_2, \ldots, H_n$$

have been selected such that every $H_i$ is a proper c.s.c-subgroup of $G$ and $H_{i+1} \subset H_i$ is a maximal proper subgroup of this type. In case $H_n \neq G^*$, we can choose a c.s.c-subgroup $H_{n+1}$ of $G$ such that $H_{n+1} \subset H_n$ be maximal. Since every descending compression chain of $G$ breaks off, we must arrive after a finite number of steps at $G^*$, thus we obtain a complete series of $G$.

This proves the theorem.
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