6.1 INTRODUCTION

The familiar Voigt functions [74]

\[
K(x, y) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \exp \left( -yt - \frac{t^2}{4} \right) \cos(\pi t) \, dt \quad (6.1.1)
\]

\[
L(x, y) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \exp \left( -yt - \frac{t^2}{4} \right) \sin(\pi t) \, dt \quad (6.1.2)
\]

\[( -\infty < x < \infty ; y > 0 ),
\]

occur frequently in a wide variety of physical problems such as astrophysical spectroscopy and the theory of neutron reactions. Furthermore the function

\[
K(x, y) + iL(x, y) \quad (6.1.3)
\]
is, except for a numerical factor, identical to the so-called plasma dispersion function which is tabulated by Fried and Conte [34] and Fettis et al. [33]. In many given physical problems, a numerical or analytical evaluation of the Voigt functions is required. For an excellent review of various mathematical properties and computational methods concerning the Voigt functions see, for example, Armstrong and Nicholls [6] and Haubold and John [42].

On other hand, it is well known that, Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example, they arise in the theory of electromagnetism and in the study of free vibrations of a circular membrane [55].
Motivated by the relationships.

\[ J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos(z) \quad \text{and} \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin(z) \quad (6.1.4) \]

where

\[ J_{v}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{v+2m}}{m! \sqrt{v+m+1}} \quad \left| z \right| < \infty. \quad (6.1.5) \]

Srivastava and Miller [95] established a link of Bessel functions with the generalized Voigt function in the form

\[ V_{\mu,\nu}(x,y) = \sqrt{\frac{x}{2}} \int_{0}^{\infty} t^\mu \exp(-yt-zt^2) J_v(xt) \, dt \quad (6.1.6) \]

so that

\[ V_{\mu,-\nu}(x,y) = K(x,y) \quad \text{and} \quad V_{\nu,\nu}(x,y) = L(x,y). \quad (6.1.7) \]

Subsequently, following the work of Srivastava and Miller [95] closely, Klusch [52] proposed an integral representation of the Voigt functions in the form

\[ \Omega_{\mu,\nu}[x,y,z] = \sqrt{\frac{x}{2}} \int_{0}^{\infty} t^\mu \exp(-yt-zt^2) J_v(xt) \, dt \quad (6.1.8) \]

\[ \left( x, y, z \in \mathbb{R}^+ : \text{Re} (\mu+\nu) > -1 \right). \]

By comparing (6.1.6) and (6.1.8), we find that

\[ \Omega_{\mu,\nu}[x,y,z] = (2\sqrt{z})^{-\mu-\nu/2} V_{\mu,\nu} \left( \frac{x}{2\sqrt{z}}, \frac{y}{2\sqrt{z}} \right). \quad (6.1.9) \]

The relations (6.1.6) and (6.1.8) are, in fact, unification (and generalization) of the Voigt functions K(x,y) and L(x,y).
In an attempt to generalize the work of Srivastava and Miller [59], Siddiqui (cf. [80] and [89]) studied the following unification (and generalization) of the Voigt functions $K(x,y)$ and $L(x,y)$ in the form

$$
\Omega_{\eta,\nu,\lambda}^{\mu} [x,y] = \sqrt{ \frac{x}{2} } \int_0^\infty t^n \exp(-yt-\frac{1}{4}t^2) J_{\nu,\lambda}^{\mu}(xt) \, dt \quad (6.1.10)
$$

where

$$
J_{\nu,\lambda}^{\mu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{2\lambda+2m}}{(\lambda+m+1) (\nu+\lambda+m\mu+1)}. \quad (6.1.11)
$$

Srivastava, Pathan and Kamarujjama [96] have studied and investigated a slightly modified form of formula (6.1.10) in the form given below

$$
\Omega_{\eta,\nu,\lambda}^{\mu} [x,y,z] = \sqrt{ \frac{x}{2} } \int_0^\infty t^n \exp(-yt-zt^2) J_{\nu,\lambda}^{\mu}(xt) \, dt \quad (6.1.12)
$$

( $x, y, z, \mu \in \mathbb{R}^+; \text{Re}(\eta+\nu+2\lambda)>-1$).

For the purpose of our present study, we recall the definition of the generalized Bessel function $J_n(z,y;s)$ in the form [18,p.3(1.1)]

$$
J_n(z,y;s) = \sum_{m=-\infty}^{\infty} s^m J_{n-2m}(z) J_m(y), \quad (z, y) \in \mathbb{R} \quad (6.1.13)
$$

Function $J_n(z,y;s)$ has the following generating function [18,p.11(2.4.2)]

$$
\exp \left[ \frac{z}{2} (t - \frac{1}{t}) + \frac{y}{2} \left( st^2 - \frac{1}{st^2} \right) \right] = \sum_{n=-\infty}^{\infty} t^n J_n(z,y;s) \quad (6.1.14)
$$

( $s, t \neq 0$).
As far as the applications of $J_n(x,y;s)$ are concerned, they frequently arise in physical problems of quantum electrodynamics and optics, the emission of electromagnetic radiation, scattering of laser radiation from free or weakly bounded electrons, the generation of betatron harmonics and the mutual absorption of two nonparallel classical photon fields through the production of electron pairs (cf. [18],[19] and [20]). This chapter aims at some representation and unification of the Voigt functions $K(x,y)$ and $L(x,y)$ which play a rather important role in several diverse fields of physics. We derive in the present chapter several representations of these functions in terms of series and integrals which are specially useful in situations when the parameters and variables take on particular values. In section 6.2 of this chapter we will focus our attention on finding different expansions forms of an integral formula involving the product of three Bessel polynomials. In section 6.3 we will explore the possibility of considering bilinear representations of the generalized Voigt function $\Omega_{\eta,\lambda}^{\mu}[x,y,z]$ with the help of the generalized Bessel function $J_n(x,y;s)$.

In the remaining parts of this chapter we will establish a new partly bilateral and partly unilateral representation of the generalized Voigt functions $\Omega_{\eta,\lambda}^{\mu}[x,y,z]$ and some interesting generating relations by means of certain explicit representation of the generalized Voigt function $\Omega_{\mu,\nu}[x,y,z]$. 
6.2. **INTEGRALS**

The aim of this section is to obtain different expansions involving the confluent series \( \Psi_2^{(n)} \) [94,p.62], Kampe' de Fe'riet's function \( F^{(n)}_{A,B,B} \) of two variables [94,p.65], and Srivastava's function \( F^{(3)} \) [94,p.69].

Indeed, we first establish an integral involving the product of three Bessel polynomials in the form

\[
I_{\sigma, \delta, v, \mu}^{(3)}(z, y, x) = \int_0^{\infty} u^\eta \exp(-qu - y^2) \ J_\epsilon(zu) \ J_\delta(yu) \ J_\mu(xu) \ du \tag{6.2.1}
\]

\[
= z^\sigma y^\delta x^\mu K \sum_{l=0}^{\infty} \frac{(-x^2/4y)^l}{\Gamma(\lambda+l+1) \Gamma(v+\lambda+\mu+l+1)} \Psi_2^{(n)} \left[ \frac{\lambda}{2} \eta + \frac{\lambda}{2} + 2\lambda + v + \sigma + \delta + 1 \right; n+1, m+1, 1/2; \frac{-z}{4y}, \frac{-y}{4y}, \frac{q}{4y} \right]
\]

\[
- \frac{q}{\sqrt{y}} \left\{ \frac{(\lambda/2)(\eta+2\lambda+v+\sigma+\delta+1))}{\Psi_2^{(n)} \left[ \frac{\lambda}{2} \eta + \frac{\lambda}{2} + 2\lambda + v + \sigma + \delta + 2 \right; n+1, m+1, 1/2; \frac{-z}{4y}, \frac{-y}{4y}, \frac{q}{4y} \right] \right\} \tag{6.2.2}
\]

\[
= \left(1 - \frac{y^2}{z^2}\right)^{\sigma+\delta+1} z^\sigma y^\delta x^\mu K \sum_{l=0}^{\infty} \frac{(-x^2/4y)^l}{\Gamma(\lambda+l+1) \Gamma(v+\lambda+\mu+l+1)} \left\{ \frac{(\lambda/2)(\eta+2\lambda+v+\sigma+\delta+1))}{\Psi_2^{(n)} \left[ \frac{\lambda}{2} \eta + \frac{\lambda}{2} + 2\lambda + v + \sigma + \delta + 1 \right; n+1, m+1, 1/2; \frac{-z}{4y}, \frac{-y}{4y}, \frac{q}{4y} \right] \right\} \tag{6.2.3}
\]
where \( K = \frac{\gamma^{-\frac{1}{2}(\eta+2\lambda+\sigma+\nu+1)}}{\frac{1}{\sigma+1} \frac{1}{\delta+1} \frac{1}{\nu+1} 2^{2\lambda+\sigma+\nu+1}} \).

**Derivations of (6.2.2) to (6.2.4):**

To establish (6.2.2), expressing \( J_\alpha(zu) \), \( J_\gamma(yu) \) and \( J^\mu_{v\lambda}(xu) \) in series, expanding \( \exp(-qu) \) in the series form

\[
\sum_{k=0}^{\infty} \frac{(-qu)^k}{k!} \quad ,
\]

integrating term by term with the help of the result (cf. [96,p.58(3.4)])

\[
\int_0^\infty x^{s-1} \exp(-\alpha x^2) dx = \frac{1}{2} \alpha^{-\frac{s}{2}} \Gamma\left(\frac{1}{2}s\right) \quad ,
\]

\((\text{Re}(s)>0 \quad , \quad \text{Re}(\alpha)>0 \quad )\),

and then separating the k-series into its even and odd terms [94,p.200(3)], we arrive at (6.2.2). If we use the relation [23,p.114(7) and p.64(23)]:

\[
\text{I} \sigma,s,v,\mu \quad \xi \eta \gamma \mu \quad \lambda + \mu + 1 \quad \nu + \mu + 1 \quad \frac{(-x^2/4\gamma)^l}{(\lambda + l + 1)(\nu + l + 1)} \{ \frac{1}{2}(\eta + 2\lambda + \nu + \sigma + \delta + 1) \}
\]

\[
\text{F} \begin{cases} 1;2;0 \quad \left[ \frac{1}{2}(\eta + 2\lambda + \nu + \sigma + \delta + 1) : \frac{1}{2}(\sigma + \delta + 1), \frac{1}{2}(\sigma + \delta + 2) ; \quad \frac{y^2 q^2}{\gamma^2} \right] \\
0;3;1 \\
\end{cases}
\]

\[
\text{F} \begin{cases} 2;2;0 \quad \left[ \frac{1}{2}(\eta + 2\lambda + \nu + \sigma + \delta + 2) : \frac{1}{2}(\sigma + \delta + 1), \frac{1}{2}(\sigma + \delta + 2) ; \quad \frac{y^2 q^2}{\gamma^2} \right] \\
0;3;1 \\
\end{cases}
\]
(α+1) (δ+1) \( J_\sigma(zu) \) \( J_\delta(yu) = \left( \frac{zu}{2} \right)^\sigma \left( \frac{yu}{2} \right)^\delta \sum_{m=0}^{\infty} \frac{(-1)^m (zu/2)^{2m}}{m! (\sigma+1)_m} \left(1 - \frac{y^2}{z^2}\right)^{\sigma+\delta+2m+1} \)

\[ _2F_1 [\delta+m+1, \sigma+\delta+m+1; \delta+1; \frac{y^2}{z^2}]. \] (6.2.7)

replace \( J_{\nu,\lambda}^{\mu}(zu) \) by its series representation (6.1.11), expand \( \exp(-qu) \) as in (6.2.5), integrate term by term with the help of the result (6.2.6) and then separate the k-series into its even and odd terms, we get (6.2.3). On other hand, if in (6.2.1), we set \( z=y \), apply the relation [5,p.11(49)]

\[
J_\sigma(yu) J_\delta(yu) = \frac{(yu/s)^{\sigma+\delta}}{\Gamma(\sigma+1) \Gamma(\delta+1)} F_3 \left[ \frac{1}{2}(\sigma+\delta+1), \frac{1}{2}(\sigma+\delta+2), \sigma+1, \delta+1, \sigma+\delta+1; -\frac{y^2}{z^2} \right]. \] (6.2.8)

express \( J_{\nu,\lambda}^{\mu}(zu) \) in its series form (6.1.11), expand \( \exp(-qu) \) as in (6.2.5), integrate term by term with the help of the result (6.2.6) and then separate the k-series into its even and odd terms, we arrive at (6.2.4).

6.3. **BILINEAR REPRESENTATIONS FOR THE GENERALIZED VOIGT FUNCTIONS** \( \Omega_{\eta,\nu,\lambda}^{\mu} [x,y,z] \)

As an application of the results obtained in section 6.2, the definition of the generalized Bessel function \( J_n(z,y;s) \) (cf.(5.1.13)) and its generating relation (6.1.14), we will explore in this section the possibility of considering bilinear representations of the generalized Voigt functions \( \Omega_{\eta,\nu,\lambda}^{\mu} [x,y,z] \) in term of more familiar special functions.

On replacing the generalzied Bessel function \( J_n(z,y;s) \) in equation (6.1.14) by the series representation (6.1.13), we get
\[
\exp \left[ \frac{z}{2} \left( t - \frac{1}{t} \right) + \frac{y}{2} \left( s^2 - \frac{1}{st} \right) \right] = \sum_{n,m=-\infty}^{\infty} t^n s^m J_{n-2m}(z) J_m(y). \quad (6.3.1)
\]

Now, starting from (6.3.1), multiplying both sides by
\[
\sqrt{\frac{x}{2}} \int_0^\infty \exp(-qu^2) J_{\nu,\lambda}(xu) \, du, \quad (\gamma > 0),
\]
(6.3.2)
replacing \( z \) and \( y \) by \( zu \) and \( yu \) respectively, integrating the resulting expression with respect to \( u \) over the semi-infinite interval \((0,\infty)\), and using the integral representation (6.1.10), we obtain
\[
\Omega_{\eta,\nu,\lambda}^{\mu} \left[ x, q - \frac{z}{2} \left( t - \frac{1}{t} \right) - \frac{y}{2} \left( s^2 - \frac{1}{st} \right), \gamma \right]
\]
\[
= \sum_{n,m=-\infty}^{\infty} t^n s^m \int_0^\infty u^n \exp(-qu^2) J_{n-2m}(zu) J_m(yu) J_{\nu,\lambda}(xu) \, du, \quad (6.3.3)
\]
\((x, q, \gamma, \mu \in \mathbb{R}; R(\eta + \nu + 2\lambda + n - m) > -1; \text{Re}[q - \frac{z}{2} \left( t - \frac{1}{t} \right) - \frac{y}{2} \left( s^2 - \frac{1}{st} \right)] > 0).\)

Since the generalized Voigt functions \( \Omega_{\eta,\nu,\lambda}^{\mu}[x,y,z] \) can be expressed in terms of integral representation involving the Bessel function \( J_n(x,y,u) \), the properties of this last function assume noticeable importance. Indeed, each of these properties will naturally lead to various other needed properties of the generalized Voigt functions \( \Omega_{\eta,\nu,\lambda}^{\mu}[x,y,z] \).

By means of integral formulae (6.2.2), (6.2.3) and (6.2.4) one can obtain the following bilinear representations:
\[ \Omega_{\eta,v,\lambda}[x, \quad q - \frac{\Delta x}{2} (t - \frac{1}{t}) - \frac{\Delta y}{2} (st^2 - \frac{1}{st^2}), \quad \gamma] \]

\[ = W \sum_{n,m=-\infty}^\infty \left( \frac{zt/2\sqrt{\gamma}}{n+1} \right)^n \left( \frac{yt^2 s/2\sqrt{\gamma}}{m+1} \right)^m (-x^2/4\gamma)^l \sum_{l=0}^\infty \frac{1}{(\lambda+l+1)(v+\lambda+\mu+l+1)} \]

\[ \left\{ \left( \frac{1}{2}(\eta+v+2\lambda+n+m+2l+1) \right)^{(3)} \Psi_{2}^{(3)} \left[ \left( \frac{1}{2}(\eta+v+2\lambda+n+m+2l+1); n+1,m+1; \frac{z^2}{4\gamma}, \frac{y^2}{4\gamma}, \frac{q^2}{4\gamma} \right] \right\} \]

\[ = \psi_{\sqrt{\gamma}} \left( \frac{1}{2}(\eta+v+2\lambda+n+m+2l+2) \right)^{(3)} \Psi_{2}^{(3)} \left[ \left( \frac{1}{2}(\eta+v+2\lambda+n+m+2l+2); n+1,m+1,\frac{z^2}{4\gamma}, \frac{y^2}{4\gamma}, \frac{q^2}{4\gamma} \right] \right\} \]

\[ \Omega_{\eta,v,\lambda}[x, \quad q - \frac{\Delta x}{2} (t - \frac{1}{t}) - \frac{\Delta y}{2} (st^2 - \frac{1}{st^2}), \quad \gamma] \]

\[ = W \sum_{n,m=-\infty}^\infty \left( \frac{zt/2\sqrt{\gamma}}{n+1} \right)^n \left( \frac{yt^2 s/2\sqrt{\gamma}}{m+1} \right)^m (-x^2/4\gamma)^l \sum_{l=0}^\infty \frac{1}{(\lambda+l+1)(v+\lambda+\mu+l+1)} \]

\[ \left\{ \left( \frac{1}{2}(\eta+v+2\lambda+n+m+2l+1) \right)^{(3)} \Psi_{2}^{(3)} \left[ \left( \frac{1}{2}(\eta+v+2\lambda+n+m+2l+1); n+1,m+1,\frac{z^2}{4\gamma}, \frac{y^2}{4\gamma}, \frac{q^2}{4\gamma} \right] \right\} \]

\[ = \psi_{\sqrt{\gamma}} \left( \frac{1}{2}(\eta+v+2\lambda+n+m+2l+2) \right)^{(3)} \Psi_{2}^{(3)} \left[ \left( \frac{1}{2}(\eta+v+2\lambda+n+m+2l+2); n+1,m+1,\frac{z^2}{4\gamma}, \frac{y^2}{4\gamma}, \frac{q^2}{4\gamma} \right] \right\} \]

\[ \{ (6.3.4) \}

\[ \{ (6.3.5) \} \]
\[ \Omega_{n,v,\lambda}^{\mu} [x, q^{-1 \over 2} (t - 1 \over t) - y (st - 1 \over st^2), y] = W \sum_{n,m=-\infty}^{\infty} (yt/2\sqrt{\gamma})^n (yt^2/2\sqrt{\gamma})^m \]

\[ \sum_{l=0}^{\infty} \frac{(-x^2/4\gamma)^l}{\lambda+l+1 \mid \nu+\lambda+\mu+l+1} \left\{ \frac{1}{2}(\eta+\nu+2\lambda+n+m+2l+1) \right\} \]

\[ F \begin{cases} 1:2;0 \left[ \frac{1}{2}(\eta+\nu+1+n+m+2l+1); \frac{1}{2}(n+m+1), \frac{1}{2}(n+m+2); -\frac{y^2}{2\gamma}, \frac{q^2}{2\gamma} \right] \\ 0:3;1 \left[ \frac{1}{2}(\eta+\nu+2\lambda+n+m+2l+2); \frac{1}{2}(n+m+1), \frac{1}{2}(n+m+2); -\frac{y^2}{2\gamma}, \frac{q^2}{2\gamma} \right] \end{cases} \]

where \[ W = \frac{x^{\nu+2\lambda+1/2} \sqrt{\gamma}^{n+m+2l+1}}{2^{\nu+2\lambda+3/2}}. \]

In addition, setting \( \lambda = \mu - 1 = 0 \) in equations (6.3.4), (6.3.5) and (6.3.6), one gets the following bilinear representations of the generalized Voigt function \( \Omega_{\eta,\nu}^{\mu} [x,y,z] \) in the form

\[ \Omega_{\eta,\nu}^{\mu} [x,q^{-1 \over 2} (t - 1 \over t) - y (st - 1 \over st^2), y] = R \sum_{n,m=-\infty}^{\infty} (zt/2\sqrt{\gamma})^n (yt^2/s\sqrt{\gamma})^m \]

\[ \left\{ \frac{1}{2}(\eta+\nu+m+n+1); \frac{1}{2}(\eta+\nu+m+n+1); n+1, m+1, v+1, \frac{1}{2}; -\frac{z^2}{4\gamma}, -\frac{y^2}{4\gamma}, -\frac{x^2}{4\gamma} \right\} \left(4\gamma \right) \]

\[ - \frac{q}{\sqrt{\gamma}} \left\{ \frac{1}{2}(\eta+\nu+n+2); \frac{1}{2}(\eta+\nu+n+2); n+1, m+1, v+1, \frac{3}{2}; -\frac{z^2}{4\gamma}, -\frac{y^2}{4\gamma}, -\frac{x^2}{4\gamma} \right\} \left(4\gamma \right) \]

\( (6.3.7) \)
\[ \Omega_{\eta,v}[x,q-\frac{z}{2}(t-\frac{1}{t})-\frac{y}{2}(st^2-\frac{1}{st^2}),\gamma] \]

\[ = R \left(1-\frac{y^2}{z^2}\right) \sum_{n,m=-\infty}^{\infty} \frac{\left[\frac{zt}{2\sqrt{\gamma}} (1-\frac{y^2}{z^2})\right]^n \left[\frac{yt^2s}{2\sqrt{\gamma}} (1-\frac{y^2}{z^2})\right]^m}{|n+1| |m+1|} \left\{\frac{1}{2}(\eta+\nu+n+m+1)\right\} \]

\( F_p^{(4)} \left[ \begin{array}{ccc} z^2 & y^2 & y^2 & -x^2 & q^2 \\ -\frac{1}{z^2} & \frac{1}{z^2} & 4y & 4y & 4y \end{array} \right] \]

\( \frac{q}{\sqrt{\gamma}} \left[\frac{1}{2}(\eta+\nu+n+m+2)\right] \)

where \( F_p^{(4)} \) is Pathan's function defined by (1.9.1), and

\[ \Omega_{\eta,v}[x,q-\frac{y}{2}(t-\frac{1}{t})-\frac{y}{2}(st^2-\frac{1}{st^2}),\gamma] \]

\[ = R \sum_{n,m=-\infty}^{\infty} \frac{(zt/2\sqrt{\gamma})^n (yt^2s/2\sqrt{\gamma})^m}{|n+1| |m+1|} \left\{\frac{1}{2}(\eta+\nu+n+m+1)\right\} \]

\( F^{(3)} \left[ \begin{array}{ccc} \frac{1}{2}(\eta+\nu+m+n+1); & \cdots; & \frac{1}{2}(\eta+\nu+m+2); & \cdots; & -\frac{y^2}{\gamma} \frac{x^2}{4\gamma} \frac{q^2}{4\gamma} \\ \cdots; & \cdots; & n+1, m+1, n+m+1; v+1; \frac{1}{2}; \gamma, \frac{1}{4\gamma}, \frac{1}{4\gamma} \end{array} \right] \]

\( \frac{q}{\sqrt{\gamma}} \left[\frac{1}{2}(\eta+\nu+m+n+2)\right] \)
\[ F^{(3)} \left[ \begin{array}{c} \frac{1}{2}(n+v+m+n+2); -\cdots -; \frac{1}{2}(n+m+1), \frac{1}{2}(n+m+2); -\cdots; -\frac{y^2}{\gamma}, -\frac{x^2}{4y} \frac{q}{4y} \\ \frac{1}{2}(n+1), m+1, n+m+1; v+1, 3/2; \frac{\gamma}{\gamma} \end{array} \right] \] (6.3.9)

where \( R = (x^{\gamma} + \gamma^{\frac{1}{2}(n+v+1)}/(2^{\gamma+3/2} (v+1)) ). \)

When \( s = t = 1 \), it is not difficult to observe that

\[ \Omega_{n,v}[x,q,y] = \Omega_{n,v}[x,q,y] \] (6.3.10)

Moreover, putting \( s = t = 1 \), \( \gamma = \frac{1}{4} \) and \( \eta = v = \pm \frac{1}{2} \), equation (6.3.7) reduces to

\[ K(x,q) = \sum_{n,m=-\infty}^{\infty} \frac{z^n y^m}{(n+1,m+1)} \{ (\frac{1}{2}(n+m+1)) \}
\]

\[ \Psi_2^{(4)} \left[ \frac{1}{2}(n+m+1); n+1, m+1, \frac{1}{2}, \frac{1}{2}; -z^2, -y^2, -x^2, q^2 \right] \]

\[ -2q\left(\frac{1}{2}(n+m+2)\right) \Psi_2^{(4)} \left[ \frac{1}{2}(n+m+2); n+1, m+1, 3/2, \frac{1}{2}; -z^2, -y^2, -x^2, q^2 \right] \} = (6.3.11) \]

\[ L(x,q) = \sum_{n,m=-\infty}^{\infty} \frac{z^n y^m}{(n+1,m+1)} \{ (\frac{1}{2}(n+m+2)) \}
\]

\[ \Psi_2^{(4)} \left[ \frac{1}{2}(n+m+2); n+1, m+1, 3/2, \frac{1}{2}; -z^2, -y^2, -x^2, q^2 \right] \]

\[ -2q\left(\frac{1}{2}(n+m+3)\right) \Psi_2^{(4)} \left[ \frac{1}{2}(n+m+3); n+1, m+1, 3/2, \frac{1}{2}; -z^2, -y^2, -x^2, q^2 \right] \} = (6.3.12) \]

Similarly other representations of \( K(x,y) \) and \( L(x,y) \) can be obtained from equations (6.3.8) and (6.3.9). For \( z, y \to 0 \), equations \( \{ (6.3.4), (6.3.5) \) and (6.3.6) \} and \( \{ (6.3.7), (6.3.8) \) and (6.3.9) \} reduce to known results due to Srivastava et al. [96] and Klusch [52] respectively. It is also not difficult to verify that known results due to Srivastava and Miller [95, Equations (10) and (11)] and Exton [29, Equation (8) and (9)] are special cases of our results of this section.
6.4 PARTLY BILATERAL AND PARTLY UNILATERAL REPRESENTATIONS
FOR THE GENERALIZED VOIGT FUNCTIONS $\Omega_{n,\nu,\lambda}^\mu [x, y, z]$

We begin by recalling the modified result of Exton due to Pathan and Yasmeen ([68] and [69]) :

$$\exp(s+t-zt/s) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m! n!} \text{$_1F_1$} [-n; m+1; z], \quad (6.4.1)$$

where $m^* = \max(0, -m)$. From equation (5.4.2), we deduce the following formula :

$$\exp \left( (s+t)(1+\omega)-zt/s-y\omega \right) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{p=0}^{\infty} \frac{s^m t^n \omega^p}{m! n! p!} (1+m+n-p) \text{$_1F_1$} [-n; m+1; z] \text{$_1F_1$} [-p; 1+m+n-p; y], \quad |\omega|<1. \quad (6.4.2)$$

On replacing $s, t, z$ and $y$ by $su^2, tu^2, zu^2$ and $yu^2$ respectively, multiplying both sides of (6.4.2) by $u^n \exp(-qu-\gamma u^2) \text{$_1F_1$} [a; b; xu] \text{$_1F_1$} [c; d; yu]$, $(\gamma)>0$, (6.4.3)

integrating with respect to $u$ over the semi-infinite interval $(0, \infty)$, using the integral representation (6.1.10) and the series representation (6.1.11) and expanding exp(-qu) as in (6.2.5), we can integrate the resulting expression term by term by means of Millen transform (cf. [96,p.58 (3.4)].)

$$\int_0^{\infty} u^{-1} \exp(-\alpha u^2) \text{$_1F_1$} [a; b; xu^2] \text{$_1F_1$} [c; d; yu^2] = \frac{\gamma}{2} \alpha^{-\frac{1}{2}} \frac{x}{\alpha} \frac{y}{\alpha} \left[ \text{F}_2 \left[ \frac{1}{2}s; a, c; b, d; \frac{x}{\alpha}, \frac{y}{\alpha} \right] \right], \quad (6.4.4)$$

where $\text{F}_2$ is Appell function [94,p.53(5)].
We thus find that

\[
\Omega^\mu_{\eta,\nu,\lambda}[x, q, \gamma, (s+t)(\omega+1)+zt/s+y\omega] = \frac{x^{\nu+2\lambda+1}}{\gamma^{v+2\lambda+3/2}} \sum_{m=0}^{\infty} \sum_{n=m^*}^{\infty} \frac{(s/\gamma)^m}{m!} \frac{(t/\gamma)^n}{n!}
\]

\[
\sum_{l,k=0}^{\infty} \frac{(-x^2/4\gamma)^l}{\lambda+l+1} \frac{(-q/v^\gamma)^k}{v+\lambda+\mu+1} \sum_{p=0}^{\infty} \frac{\omega^p}{p!} (1+m+n-p) \text{F}_2 \left[ \begin{array}{c} \frac{1}{2} (\eta+\nu+2\lambda+2m+2\nu+2\nu+2\mu+2\nu+2\mu+2k+2) \\ 1 \end{array} ; -n, -n \right]
\]

\[
\frac{1+m+n-p, m+1;}{\frac{x}{\gamma}, \frac{z}{\gamma}} \left[ \left( \frac{1}{2} (\eta+\nu+2\lambda+2m+2\nu+2\nu+2\mu+2\nu+2\mu+2k+2) \right) \right]. \quad (6.4.5)
\]

In view of the result (cf. (4.2.8)):

\[
\sum_{k=0}^{\infty} \frac{(-n-m)_k}{k!} (-x)^k \text{F}_2 \left[ \begin{array}{c} \frac{1}{2} (\eta+\nu+2\lambda+2m+2\nu+2\nu+2\mu+2\nu+2\mu+2k+2) \\ 1 \end{array} ; -n, -n \right] = \text{F} \left[ \begin{array}{c} \frac{1}{2} (\eta+\nu+2\lambda+2m+2\nu+2\nu+2\mu+2\nu+2\mu+2k+2) \\ 1 \end{array} ; -n, -n \right]. \quad (6.4.6)
\]

we get

\[
\Omega^\mu_{\eta,\nu,\lambda}[x, q, \gamma, (s+t)(1+\omega)+zt/s+y\omega] = \frac{x^{\nu+2\lambda+1}}{\gamma^{v+2\lambda+3/2}} \sum_{m=0}^{\infty} \sum_{n=m^*}^{\infty} \frac{(s/\gamma)^m}{n!} \frac{(t/\gamma)^n}{m!}
\]

\[
\sum_{l,k=0}^{\infty} \frac{(-x^2/4\gamma)^l}{\lambda+l+1} \frac{(-q/v^\gamma)^k}{v+\lambda+\mu+1} \sum_{p=0}^{\infty} \frac{\omega^p}{p!} (1+m+n-p) \text{F}_2 \left[ \begin{array}{c} \frac{1}{2} (\eta+\nu+2\lambda+2m+2\nu+2\nu+2\mu+2\nu+2\mu+2k+2) \\ 1 \end{array} ; -n, -n \right]
\]

\[
\frac{1+m+n-p, m+1;}{\frac{z}{\gamma}, \frac{-\omega y}{\gamma}} \left[ \right]. \quad (6.4.7)
\]

Now, separating the k-series into its even and odd terms, we get
\[ \Omega_{n,v,\lambda}^{\mu}[x,q,\gamma-(s+t)(1+\omega)+zt/s+y\omega] = \frac{x^{\nu+2\lambda+1} y^{-2(\eta+\nu+2\lambda+1)}}{2^{\nu+2\lambda+3/2}} \]

\[
\sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(s/\gamma)^m (t/\gamma)^n}{n!m!} \sum_{l=0}^{\infty} \frac{(-x^2/4\gamma)^l}{[\lambda/l+1][\nu+\lambda+\mu/l+1]} \{[l/2(\eta+2\lambda+\nu+2m+2n+2l+1)]

F^{(3)} \left[ \begin{array}{c}
\frac{1}{2}(\eta+2\lambda+\nu+2m+2n+2l+1) \\
\frac{1}{2}(\eta+2\lambda+\nu+2m+2n+2l+2)
\end{array} \right] = \frac{(q^2)}{4\gamma} \]

When \( s=t= \frac{z+y\omega}{2(1+\omega)} \), it is not difficult to observe that:

\[ \Omega_{n,v,\lambda}^{\mu}[x,q,\gamma-(s+t)(1+\omega)+zt/s+y\omega] = \Omega_{n,v,\lambda}^{\mu}[x,q,\gamma] \quad (6.4.9) \]

Further, if in (6.4.8), we set \( \mu-1=\lambda=0, \gamma=1/4, s=t= \frac{z+y\omega}{2(\omega+1)}, \eta=1/2 \) and \( v=\pm1/2 \), we shall obtain representations of the Voigt functions \( K(x,y) \) and \( L(x,y) \).

Finally if in (6.4.8), we set \( y=\omega=0 \), it reduces to a known result due to Srivastava et al. [96,p.59(3.6)].
6.5 GENERATING RELATIONS OBTAINABLE BY MEANS OF EXPLICIT REPRESENTATION OF $\Omega_{\mu \nu} [x, q, \gamma]$

Klusch [52] introduced an explicit expression of the generalized Voigt function $\Omega_{\mu \nu} [x, q, \gamma]$ in the form

$$
\Omega_{\mu \nu} [x, q, \gamma] = \frac{x^{v + \frac{1}{2}}}{2^{v + 1} \Gamma(v + 1)} \left\{ \Psi_2 \left[ \frac{1}{2}(\mu + v + 1); \nu + 1, \frac{1}{2}; -\frac{x^2}{4\gamma}, \frac{q^2}{4\gamma} \right] - \frac{q}{\sqrt{\gamma}} \Psi_2 \left[ \frac{1}{2}(\mu + v + 2); \nu + 1, \frac{3}{2}; -\frac{x^2}{4\gamma}, \frac{q^2}{4\gamma} \right] \right\} \quad (6.5.1)
$$

where $\Psi_2$ denotes one of Humbert's confluent hypergeometric functions of two variables [94,p.59(42)]. On taking $\lambda = \mu - 1 = 0$ in (6.4.8) and expanding the left member of the resulting expression by means of the representation (6.5.1), we get the following generating relation

$$
\left( \frac{\gamma - (s + t)(1 + \omega) + zt/s + y\omega}{\gamma} \right)^\alpha \left\{ \frac{\Psi_2 \left[ \alpha + 1, \frac{1}{2}; -\frac{x^2}{4\gamma}, \frac{q^2}{4\gamma} \right]}{4(\gamma - (s + t)(1 + \omega) + zt/s + y\omega)} - \frac{q\alpha(\alpha + \frac{1}{2})}{\sqrt{\gamma}((s + t)(1 + \omega) + zt/s + y\omega)} \Psi_2 \left[ \alpha + \frac{1}{2}, \nu + 1, 3/2; -\frac{x^2}{4\gamma}, \frac{q^2}{4\gamma} \right] \right\} \quad (6.5.2)
$$
Some special cases of (6.5.2) are worthy of note. Indeed, for \( y=\omega=0 \), relation (6.5.2) reduces to result due to Srivastava et al. [96,p.62(4.1)]. For \( q=0 \), \( y=1 \) and \( x^2=-4u \), (6.5.2) evidently reduces to a generating function for \( F^{(3)} \) in terms of \( _1F_1 \), and we thus obtain

\[
(1-(s+t)(1+\omega)+zt/s+y\omega)^{-\alpha} _1F_1\left[\alpha;\nu+1; \frac{u}{1-(s+t)(1+\omega)+zt/s+y\omega}\right]
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!n!} (\alpha)_{m+n} \ F^{(3)}\left[\alpha+m+n;\nu-n;\nu;\nu;\nu+m+1;\nu+1; \frac{zt}{s}-\omega y,u\right] (6.5.3)
\]

For \( \nu=\alpha-1 \), (6.5.3) reduces to

\[
(1-(s+t)(1+\omega)+zt/s+y\omega)^{-\alpha} \exp\left(\frac{u}{1-(s+t)(1+\omega)+zt/s+y\omega}\right)
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!n!} (\alpha)_{m+n} \ F^{(3)}\left[\alpha+m+n;\nu-n;\nu;\nu;\nu+m+1;\nu-\alpha; \frac{zt}{s}-\omega y,u\right] (6.5.4)
\]

On letting \( u\rightarrow0 \), (6.5.3) reduces to

\[
(1-(s+t)(1+\omega)+zt/s+y\omega)^{-\alpha}
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!n!} (\alpha)_{m+n} \ F\left[1:1;0;\frac{zt}{s}-\omega y,u\right] (6.5.5)
\]

Now, taking \( \omega=0 \) in (6.5.5), we obtain
\[(1-s-t+zt/s)^\alpha = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!n!} (\alpha)_{m+n} \, _2F_1[\alpha+m+n, -n; m+1; z] \quad (6.5.6)\]

On other hand, if in (6.5.5), \(z \to 0\), we get

\[\Gamma_0[\alpha; -(s+t)(\omega+1)+y \omega] = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!n!} (\alpha)_{m+n} \, \Gamma_0[\alpha+m+n; -; -\omega y] \quad (6.5.7)\]

If in (6.5.7), we replace \(s\), \(t\) and \(y\) by \(s/u\), \(t/u\) and \(y/u\) respectively, multiply both the sides by \(u^{-\lambda}\) and then take the inverse Laplace transform \([94, p.219(7)]\) (cf. (4.2.6)), we get

\[\Gamma_1[\alpha; -(s+t)(\omega+1)+y \omega] = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n (\alpha)_{m+n}}{m!n!(\lambda)_{m+n}} \, \Gamma_1[\alpha+m+n; \lambda; -\omega y] \quad (6.5.8)\]

Further, if in (6.5.8), we replace \(s\), \(t\) and \(y\) by \(su\), \(tu\) and \(yu\) respectively, multiply both the sides by \(u^{-\lambda} e^{-\lambda}\) and take Laplace transform \([94, p.219(6)]\), (cf. (4.2.3)), equation (6.5.8) yields

\[\Psi_1[\alpha+m+n, \sigma; \lambda; -\omega y] \quad (6.5.9)\]

For \(\omega, y \to 0\), equation (6.5.3) evidently reduces to

\[\left(1-s-t+zt/s\right)^{\alpha} \, \Gamma_1[\alpha; v+1; \frac{u}{1-s-t+zt/s}] = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!n!} (\alpha)_{m+n} \, \Psi_1[\alpha+m+n, -n;m+1,v+1;z,u] \quad (6.5.10)\]
where $\Psi_1$ is a confluent hypergeometric function of two variables [94, p.59(41)].

When $z, w \to 0$, in (6.5.3), we obtain

$$(1-s-t)^{\alpha} \, _1F_1[\alpha;\nu+1; \frac{u}{(1-s-t)}]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m! \, n!} (\alpha)_m \, _1F_1[\alpha+m+n;\nu+1;u]. \quad (6.5.11)$$