CHAPTER-V

ON GENERALIZATION OF GENERATING FUNCTIONS INVOLVING

BESSEL AND LAGUERRE POLYNOMIALS

5.1 INTRODUCTION

In the theory of special functions, the Bessel and Laguerre polynomials play the same role as in a number of other branches of mathematics and physics. For example, the Bessel polynomials arise in the theory of electromagnetism and in the study of free vibrations of a circular membrane [55]. On other hand, it is well known that, Laguerre polynomials occur in problems involving the integration of Helmholtz's equation in parabolic coordinates, in the theory of the hydrogen atom etc. [57].

The purpose of this chapter is to establish Bessel polynomials in several variables, which provide multivariable generalization of known Bessel polynomials. Certain integral representations have been obtained for these multivariable Bessel polynomials. In section 5.3 our main generating functions for \( y(m_1,\ldots,m_n)(x_1,\ldots,x_n) \) which are linear and partly bilateral and partly unilateral are obtained with the help of a result of Exton [31].

Section 5.4 deals with a technique of integral operators for obtaining generating functions of Lauricella function \( F_A^{(n)} \) and Horn function \( H_4^{(n)} \) which are partly bilateral and partly unilateral.

The generalized Horn's function \( H_4^{(n)} \) of Exton [27] is a function which not only generalizes Horn's functions \( H_4, H_4^{(p)} \) [48] but also Lauricella's \( F_A^{(n)}, F_C^{(n)} \), Appell's \( F_2, F_4 \) and \( F_1 \).
Many known results of Al-Salam [3], Agrawal [1], Mumtaz and Khursheed [63], Pathan and Kamarujjama [65] and Pathan and Yasmeen [69] are shown as special cases of the results of this chapter.

5.2 DEFINITION AND INTEGRAL REPRESENTATIONS OF THE BESSEL POLYNOMIALS $y_{m_1, \ldots, m_n}^{(a_1, \ldots, a_n, b)}(x_1, \ldots, x_n)$

Exton [32, p.4(3.1)] has introduced a Bessel polynomial in several variables. This is defined as follows:

$$y_{m_1, \ldots, m_n}(x_1, \ldots, x_n; a) = \sum_{k_1=0}^{m_1} \ldots \sum_{k_n=0}^{m_n} (a-1+m_1+\ldots+m_n)_{k_1+\ldots+k_n} (-m)_{k_1} \ldots (-m)_{k_n}$$

$$\frac{(-x_1)^{k_1}}{k_1!} \ldots \frac{(-x_n)^{k_n}}{k_n!}.$$  \hspace{1cm} (5.2.1)

If all but one of the variables are suppressed, we recover the Bessel polynomial

$$y_m(a;x) = \mathcal{F}_0[-m, a-1+m; \ldots; -x],$$  \hspace{1cm} (5.2.2)

which on replacing $x$ by $x/b$, gives us the Bessel polynomials

$$y_m(a,b;x) = \mathcal{F}_0[-m, a-1+m; \ldots; -x/b].$$  \hspace{1cm} (5.2.3)

The Bessel polynomials (5.2.3) were introduced by Krall and Frink [53] in connection with solution of the wave equation in spherical coordinates.

Several other authors including Agarwal [1], Carlitz [11], Grosswald [37], Al-Salam [3], Chatterjea ([14] and [15]) and Mumtaz and Khursheed [63] have contributed to the study of Bessel polynomials.
The polynomial \( y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n)}(x_1,\ldots,x_n) \) is defined as follows:

\[
y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n)}(x_1,\ldots,x_n) = \sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} (1+\beta+\alpha_1 m_1+\ldots+\alpha_n m_n)_{k_1+\ldots+k_n} \prod_{j=1}^{n} \left\{ \binom{m_j}{k_j} z_j^{k_j} \right\}.
\]

If in (5.2.4), we set \( \alpha_j = 1, \beta = a-2 \), we shall readily obtain Exton's Bessel polynomial (5.2.1). On setting \( n=1 \) and replacing \( x \) by \( x/2 \), (5.2.4) reduces to

\[
y_{m_1}^{(a,b)}(x) = F_0 \left[ \begin{array}{c} -m, a+1 \end{array} \right] \left[ \begin{array}{c} -x/2 \end{array} \right],
\]

a polynomial introduced by Mumtaz and Khursheed [63, p.152(2.1)].

Also, Bessel polynomials due to Al-Salam [3, p.529(2.2)], Chatterjea [5] and Krall and Frink (5.2.3) are contained in (5.2.4).

Erde'lyi [21] defined the multivariable Laguerre polynomials by the relation

\[
L_{m_1,\ldots,m_n}^{(\alpha)}(x_1,\ldots,x_n) = \frac{(\alpha+1)_{m_1+\ldots+m_n}}{m_1!\ldots m_n!} \Phi_{2}^{(n)} [-m_1,\ldots,-m_n; \alpha+1; x_1,\ldots,x_n],
\]

where \( \Phi_{2}^{(n)} \) is a confluent hypergeometric function of \( n \)-variables [94, p.62(10)] defined by (1.10.6).

In view of the identities (cf. (1.1.13) and (1.1.15))

\[
(-n)_k = \frac{(-1)^k n!}{(n - k)!}, \quad (\lambda)^n_k = \frac{(-1)^k (\lambda)_n}{(1-\lambda-n)_k},
\]
one can rewrite (2.3) in the form

$$L_{m_1,\ldots,m_n}^{(\alpha)}(x_1,\ldots,x_n) = \prod_{j=1}^{n} \left\{ \frac{-x_j^{m_j}}{m_j} \right\} \sum_{k_i=0}^{m_i} \sum_{k_n=0}^{m_n} (-\alpha-m_1,\ldots,-m_n)_{k_1,\ldots,k_n} \prod_{j=1}^{n} \left\{ \binom{m_j}{k_j} \left( \frac{1}{z_j} \right)^{k_j} \right\}.$$  \hspace{1cm} (5.2.8)

From (5.2.4) and (5.2.8), we see that the Bessel polynomials are essentially Laguerre polynomials. In fact we have

$$y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n\beta)}(x_1,\ldots,x_n) = \prod_{j=1}^{n} \left\{ m_j! (-x_j)^{m_j} \right\} L_{m_1,\ldots,m_n}^{(-\alpha_1-m_1,\ldots,-\alpha_n-m_n\beta)}(1/x_1,\ldots,1/x_n).$$ \hspace{1cm} (5.2.9)

where throughout this chapter

$$\nu = \alpha_1 m_1 + \ldots + \alpha_n m_n, \quad m_j, \quad (j=1,2,\ldots,n) \text{ are integers.}$$ \hspace{1cm} (5.2.10)

For \( n=1, \alpha=1 \) and \( x \) replaced by \( x/2 \), (5.2.9) reduces to

$$y_m^{(1)}(x) = m!(-x/2)^m L_m^{(-2m-1)}(2/x),$$ \hspace{1cm} (5.2.11)

a result due to Al-Salam [3, p.530(2.5)].

Now we will present a number of integral representations for the generalized Bessel polynomials of several variables (5.2.4) in terms of Euler and Laplace integrals.

Indeed, it is easy to derive the following integral representations

$$y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta)}(x_1,\ldots,x_n) = \frac{s^{v+\beta+1}}{(v+\beta+1)!} \int_0^\infty t^{v+\beta} e^{-st} \prod_{j=1}^{n} \left\{ \left(1+x_j t\right)^{m_j} \right\} dt,$$ \hspace{1cm} (5.2.12)

$$F_{21\ldots1}^{10\ldots0} \left[ \begin{array}{c} 1+\beta+v, \quad \lambda; \quad -m_1,\ldots,-m_n; \\ \lambda+\mu; \quad -x_1,\ldots,-x_n \end{array} \right]$$
\[
\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} \int_0^1 t^{\lambda-1} (1-t)^{\mu-1} y^{(\alpha_1, ..., \alpha_n; \beta)}_{m_1, ..., m_n} (x_1, ..., x_n) \, dt,
\]

(5.2.13)

\[
\frac{\Gamma(\lambda+\mu)}{(\lambda+\mu)^n} \int_0^1 t^{\lambda-1} (1-t)^{\mu-1} y^{(\alpha_1, ..., \alpha_n; \beta)}_{m_1, ..., m_n} (x_1(1-t), ..., x_n(1-t)) \, dt,
\]

(5.2.14)

where \( F_{c,d; \cdots; d}^{(n)} \) is the generalized Kampe' de Fer'iet function of \( n \)-variables defined by (1.11.2).

\[
\frac{-\pi}{\sin \pi(u+u'+\beta+\gamma)} \frac{\sin \pi(u'+\gamma)}{\sin \pi(u+\beta)} \frac{\sin \pi(u+\gamma+1)}{\sin \pi(u'+1)}
\]

\[
y^{(\alpha; \beta; \gamma)}_{m_1+k_1, ..., m_n+k_n} (x)
\]

(5.2.15)

where \( u = \alpha(m_1+...+m_n) \) and \( u' = \alpha(k_1+...+k_n) \).

**Proofs of equations (5.2.12) to (5.2.15)**: To prove (5.2.12), denote, for convenience, the right-hand side by \( V \), then it is easily seen that

\[
V = \sum_{k_1, ..., k_n=0}^{\infty} \prod_{j=1}^{n} \left\{ \frac{-x_j^{k_j}}{k_j!} \right\} (-m_i)_{k_i} \int_0^{e^{st}} t^{v+\beta+k_1+...+\kappa_1+1-1} e^{-st} \, dt.
\]

Now, evaluating the integral and using (5.2.4), we arrive at the result (5.2.12). The proofs of equation (5.2.13) to (5.2.15) are similar to that of equation (5.2.12).
Some special cases of the results mentioned above are worthy of note. Indeed upon setting \( n=1 \) in (5.2.12), if we let \( s=1, \alpha=1, \beta=a-2 \) and replace \( x \) by \( x/b \), (5.2.12) readily yields

\[ y_n(a,b,x) = \frac{1}{[(\alpha+n-1)]} \int_0^\infty t^{a-2+n} \left(1+\frac{xt}{b}\right)^n e^{-t} \, dt . \quad (5.2.16) \]

a result due to Agarwal [1]. On putting \( \mu = \beta, \lambda=v+1 \) in (5.2.13), we get

\[ y^{(\alpha_1,\ldots,a_n;0)}_{m_1\ldots m_n}(x_1,\ldots,x_n) = \frac{[v+\beta+1]}{[v+1][\beta]} \int_0^1 t^\nu (1-t)^{\beta-1} y^{(\alpha_1,\ldots,a_n,\beta)}_{m_1\ldots m_n}(x_1,\ldots,x_n,t) \, dt . \quad (5.2.17) \]

On other hand, setting \( \lambda=m_1+\ldots+m_n+\beta+1, \mu=v-m_1-\ldots-m_n \), in (5.2.13), we obtain

\[ y^{(\alpha_1,\ldots,a_n\beta)}_{m_1\ldots m_n}(x_1,\ldots,x_n) = \frac{[v+\beta+1]}{[m_1+\ldots+m_n+\beta+1][v-m_1-\ldots-m_n]} \int_0^1 t^{m_1+\ldots+m_n+\beta+1} (1-t)^{v-m_1-\ldots-m_n-1} \]

\[ y^{(\alpha_1,\ldots,a_n,\beta)}_{m_1\ldots m_n}(x_1,\ldots,x_n,t) . \quad (5.2.18) \]

On replacing \( \lambda \) and \( \mu \) in (5.2.13) by \( v+\beta+\lambda+1 \) and \(-\lambda\) respectively, equation (5.2.13) reduces to

\[ y^{(\alpha_1,\ldots,a_n,\beta+1\lambda)}_{m_1\ldots m_n}(x_1,\ldots,x_n) = \frac{-\sin\pi(\lambda)}{\pi} \frac{[1+\lambda[1+v+\beta+1]}{[v+\beta+\lambda+1]} \int_0^1 t^{v+\beta+\lambda+1} (1-t)^{-(\lambda+1)} \]

\[ y^{(\alpha_1,\ldots,a_n,\beta)}_{m_1\ldots m_n}(x_1,\ldots,x_n,t) \, dt . \quad (5.2.19) \]

Further, if in (5.2.14), we replace \( \lambda \) and \( \mu \) by \( \mu \) and \( v+\beta-\mu+1 \) respectively, (5.2.14) yields
For \( \mu = 1 \), \( \lambda = (v+\beta) \), (5.2.13) reduces to

\[
y_{m_1, \ldots, m_n}^{(\alpha_1, \ldots, \alpha_n; \beta-\mu)}(x_1, \ldots, x_n) = (v+\beta) \int_0^1 t^v (1-t)^{v+\beta-\mu} y_{m_1, \ldots, m_n}^{(\alpha_1, \ldots, \alpha_n; \beta)}(x_1 t, \ldots, x_n(1-t)) \, dt. \tag{5.2.20}
\]

Finally, if in (5.2.13), we put \( \lambda = 1 \) and \( \mu = v+\beta \), (5.2.14) reduces to

\[
y_{m_1, \ldots, m_n}^{(\alpha_1, \ldots, \alpha_n; \beta-1)}(x_1, \ldots, x_n) = (v+\beta) \int_0^1 (1-t)^v (1-t)^{v+\beta-1} y_{m_1, \ldots, m_n}^{(\alpha_1, \ldots, \alpha_n; \beta)}(x_1(1-t), \ldots, x_n(1-t)) \, dt. \tag{5.2.22}
\]

Integral representations (5.2.12) to (5.2.15), (5.2.17) to (5.2.19), (5.2.21) and (5.2.22) are generalizations of known results obtained by Mumtaz and Khursheed [63, Equations (3.5), (3.3) to (3.4), (3.6) to (3.9), (3.14) and (3.15)] respectively.

5.3 GENERATING FUNCTIONS FOR THE BESSEL POLYNOMIALS

\[
y_{m_1, \ldots, m_n}^{(\alpha_1, \ldots, \alpha_n; \beta)}(x_1, \ldots, x_n)
\]

It is not difficult to derive the following basic generating relations:

\[
e^{t_1 + \cdots + t_n} (1-x_1 t \cdots x_n t_n)^{(\beta+1)} = \sum_{m_1, \ldots, m_n = 0}^{\infty} y_{m_1, \ldots, m_n}^{(\alpha_1, \ldots, \alpha_n; \beta-\nu)}(x_1, \ldots, x_n) \frac{t_1^{m_1}}{m_1!} \cdots \frac{t_n^{m_n}}{m_n!}, \tag{5.3.1}
\]
Proof of equation (5.3.1) and (5.3.2): To prove (5.3.1), we write the left-hand side in the form:

\[ V = \sum_{s_1, \ldots, s_n, k_1, \ldots, k_n = 0}^{\infty} (\beta+1)_{k_1+\ldots+k_n} \prod_{j=1}^{n} \left\{ \frac{-x_j^k}{k_j! s_j!} \right\}. \]

Replace \( s_1+k_1, \ldots, \) and \( s_n+k_n \) by \( m_1, \ldots, \) and \( m_n \) respectively. Then after rearrangement justified by absolute convergence of the above series and using the definition (5.2.4), we arrive at (5.3.1). The proof of (5.3.2) is similar to that of (5.3.1).

In formula (5.2.12), put \( s=1, v+\beta=k \), where \( v \) is defined by (5.2.10) and \( k \) is integer, multiply throughout by \((-\lambda)^k\) and then sum to get

\[ \sum_{k=0}^{\infty} y_{m_1, \ldots, m_n}^{(a_1, \ldots, a_n; k-v)} \left( \frac{x_1, \ldots, x_n}{1+\lambda} \right) = (1+\lambda) \sum_{k=0}^{\infty} y_{m_1, \ldots, m_n}^{(a_1, \ldots, a_n, k-v)} (x_1, \ldots, x_n)(-\lambda)^k. \]  

Similarly, we find that

\[ \sum_{k=0}^{\infty} y_{m_1, \ldots, m_n}^{(a_1, \ldots, a_n; k-v)} (x_1, \ldots, x_n) \frac{(-\lambda)^k}{k!} = \int_0^{\infty} \prod_{j=1}^{n} \left\{ (1+x_j^t)^{m_j} \right\} J_0(2\sqrt{\lambda} t) \, dt. \]

Evaluating the right-hand side, we get

\[ \sum_{k=0}^{\infty} y_{m_1, \ldots, m_n}^{(a_1, \ldots, a_n; k-v)} (x_1, \ldots, x_n) \frac{(-\lambda)^k}{k!} = e^{-\lambda} \sum_{k_1=0}^{\infty} \sum_{k_n=0}^{\infty} \prod_{j=1}^{n} \left\{ \frac{(-m_j^k(-x_j^k)}{k_j!} \right\} \]

\[ \frac{1}{(k_1+\ldots+k_n+1)} L_{k_1+\ldots+k_n}^n(\lambda), \]  

(5.3.5)
where \( L_n(x) \) is Laguerre polynomial [73, p.200(2)].

On setting \( n=1, \alpha=1 \) in (5.3.5) and replacing \( x \) by \( x/2 \), it reduces to a known result due to Al-Salam [3, p.536 (6.5)]. In the same manner one can derive the following formulae:

\[
\sum_{k=0}^{\infty} y^{(\alpha_1,\ldots,\alpha_n; 2k-v)}_{m_1,\ldots,m_n} (x_1,\ldots,x_n) (-\lambda^2)^k = \int_0^\infty e^{-t} \prod_{j=1}^n \{(1+x_j t)^m\} \cos(\lambda t) \, dt, \tag{5.3.6}
\]

\[
\sum_{k=0}^{\infty} y^{(\alpha_1,\ldots,\alpha_n; 2k-v+1)}_{m_1,\ldots,m_n} (x_1,\ldots,x_n) (-\lambda^2)^k = (1/\lambda) \int_0^\infty e^{-t} \prod_{j=1}^n \{(1+x_j t)^m\} \sin(\lambda t) \, dt, \tag{5.3.7}
\]

\[
\sum_{k_1,\ldots,k_n=0}^{\infty} y^{(\alpha_1,\ldots,\alpha_n; n-k_1,\ldots,-k_n)}_{m_1,\ldots,m_n} (x_1,\ldots,x_n) \prod_{j=1}^n \frac{u_j}{k_j} = \frac{1}{\Gamma(\beta+v+1)} \int_0^\infty e^{-t} \prod_{j=1}^n \{(1+x_j t)^m\} \, dt, \tag{5.3.8}
\]

Now, by using the definition of the generalized Bessel polynomials (5.2.4), it is easy to derive the following results:

\[
\sum_{m_1,\ldots,m_n=0}^{\infty} y^{(\alpha_1,\ldots,\alpha_n; [\nu,1-v])}_{m_1,\ldots,m_n} (x_1,\ldots,x_n) \prod_{j=1}^n \{t_j^m\} \equiv \prod_{j=1}^n \{(1+t_j)^{-1}\}, \tag{5.3.9}
\]

\[
\sum_{m_1,\ldots,m_n=0}^{\infty} y^{(\alpha_1/m_1,\ldots,\alpha_n/m_n; \beta)}_{m_1,\ldots,m_n} (x_1,\ldots,x_n) \prod_{j=1}^n \{t_j^m\} \equiv \prod_{j=1}^n \{(1+t_j)^{-1}\}, \tag{5.3.10}
\]

\[
F^{1:1;\ldots;1}_{0:0;\ldots;0} \left[ \begin{array}{c}
\beta+1:1;\ldots;1; \\
\vdots;\vdots;\vdots; \\
(1-t_1);\ldots;\ldots; \\
(1-t_n)
\end{array} \right],
\]

\[
F^{1:1;\ldots;1}_{0:0;\ldots;0} \left[ \begin{array}{c}
\alpha_1;\ldots;\alpha_n + \beta+1:1;\ldots;1; \\
\vdots;\vdots;\vdots; \\
(1-t_1);\ldots;\ldots; \\
(1-t_n)
\end{array} \right].
\]
The formulae (5.3.1) to (5.3.4) and (5.3.6) to (5.3.10) are generalizations of known results [63, Equations (4.1), (4.3), (4.6) to (4.10) and (4.4) to (4.5)] respectively.

Now, we derive a partly bilateral and partly unilateral generating relation for \( y_{m_1, \ldots, m_n}^{(\alpha_1, \ldots, \alpha_n; \beta)} \left( x_1, \ldots, x_n \right) \). We begin with the modified result of Exton [31], (cf. 2.1.3)).

\[
\exp(s+t-\frac{zt}{s}) = \sum_{m=-\infty}^{\infty} \sum_{p=m^*}^{\infty} \frac{s^m t^p}{(m+p)!} L_p^{(m)}(z). \tag{5.3.11}
\]

On putting \( s=(1-s-t+zt/s) \) in (5.2.12) and making use of (5.3.11), we find that:

\[
(1-s-t+zt/s)^{(v+\beta+1)} y_{m_1, \ldots, m_n}^{(\alpha_1, \ldots, \alpha_n; \beta)} \left( \frac{x_1}{1-s-t+zt/s}, \ldots, \frac{x_n}{1-s-t+zt/s} \right) = \frac{1}{(v+\beta+1)} \sum_{m=-\infty}^{\infty} \sum_{p=m^*}^{\infty} \frac{s^m t^p}{m! p!} \int_0^\infty u^{(v+\beta+m+j+1)} e^{-u} \prod_{j=1}^n \left\{ 1+x_j u \right\}^{(m)} L_p^{(m)}(zu) du. \tag{5.3.12}
\]

Evaluating the right-hand side, we get

\[
(1-s-t+zt/s)^{(v+\beta+1)} y_{m_1, \ldots, m_n}^{(\alpha_1, \ldots, \alpha_n; \beta)} \left( \frac{x_1}{1-s-t+zt/s}, \ldots, \frac{x_n}{1-s-t+zt/s} \right) = \sum_{m=-\infty}^{\infty} \sum_{p=m^*}^{\infty} \frac{s^m t^p}{m! p!} \binom{v+\beta+m+p+1}{m+p+1} \left[ \begin{array}{c} v+\beta+m+p+1; -p; \vdots; m; \vdots; n; \\ 0; 0; \vdots; 0 \end{array} \right] \left[ \begin{array}{c} z, x_1, \ldots, x_n \end{array} \right]. \tag{5.3.13}
\]

Equation (5.3.13) gives a number of generating relations as special
cases. We present some interesting spacial cases here. If in (5.3.13), we set \( n=1 \), replace \( x \) by \( x/2 \), it yields generating relation involving the Bessel polynomials \( y_n^{(a,b)} \) (cf. (5.2.5)):

\[
(1-s-t+z/t/s)^{-(an+\beta+1)} \, y_n^{(a,b)} \left( \frac{x}{2(1-s-t+z/t/s)} \right)
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{p=m^*}^{\infty} \frac{s^m t^p}{m! \, p!} \, (\alpha n+\beta+1)_{m+p} \, F_{01:0}^{11:1} \left[ \begin{array}{c} \alpha n+\beta+m+p+1; -p; -n; \\ m+1; -z, -x/2 \end{array} \right]
\]  

(5.3.14)

which for \( \beta=\alpha-2, \alpha=1 \) and \( x \) replaced by \( 2x/b \), reduces to

\[
(1-s-t+z/t/s)^{-(\alpha+\beta-1)} \, y_n^{(a,b)} \left( \frac{x}{1-s-t+z/t/s} \right)
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{p=m^*}^{\infty} \frac{s^m t^p}{m! \, p!} \, (\alpha n+\beta-1)_{m+p} \, F_{01:0}^{11:1} \left[ \begin{array}{c} \alpha n+m+p-1; -p; -n; \\ m+1; -z, -x/b \end{array} \right]
\]  

(5.3.15)

Next, for \( z \to 0 \), equations (5.3.13) to (5.3.15) reduce to the following elegant results:

\[
(1-s-t)^{(v+\beta+1)} \, y_{m_1, \ldots, m_n}^{(a_1, \ldots, a_n; \beta)} \left( \frac{x_1}{1-s-t}, \ldots, \frac{x_n}{1-s-t} \right)
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{p=m^*}^{\infty} \frac{s^m t^p}{m! \, p!} \, (v+\beta+1)_{m+p} \, y_{m_1, \ldots, m_n}^{(a_1, \ldots, a_n; \beta+m+p)} (x_1, \ldots, x_n),
\]  

(5.3.16)

\[
(1-s-t)^{(an+\beta+1)} \, y_n^{(a, \beta)} \left( \frac{x}{2(1-s-t)} \right)
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{p=m^*}^{\infty} \frac{s^m t^p}{m! \, p!} \, (\alpha n+\beta+1)_{m+p} \, y_n^{(a, \beta+m+p)} (x)
\]  

(5.3.17)
and
\[(1-s-t)^{(a+n-1)} y_n \left( a, b; \frac{x}{1-s-t} \right) \]
\[= \sum_{m=-\infty}^{\infty} \sum_{p=m^*}^{\infty} \frac{s^m t^p}{m! p!} (a+n-1)_{m+p} y_n \left( a+m+p, b; x \right), \quad (5.3.18)\]
respectively.

5.4 GENERATING RELATIONS FOR $F_{\alpha}^{(n)}$ AND $H_4^{(n)}$

We begin with the modified result of Exton [31] (cf. (5.3.12)). On replacing $s$ and $t$ by $s(1+z_{1}) \ldots s(1+z_{p})$ and $t(1+z_{1}) \ldots (1+z_{p})$ respectively in equation (5.3.12), multiplying both the sides by
\[\exp \left(-x_{1} z_{1} - \ldots - x_{p} z_{p}\right)\]
and using the result [94, p. 209 (9)]
\[\sum_{n=0}^{\infty} L_{n}^{(a-n)} (x) \ t^{n} = (1+t)^{u} e^{xt}\]
we get
\[\exp \left(((s+t) \prod_{i=1}^{p} \{(1+z_{i})\} - x_{1} z_{1} - \ldots - x_{p} z_{p}\}\right)\]
\[= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k_{1}, \ldots, k_{p}=0}^{p} \frac{s^m t^n \prod_{i=1}^{p} (z_{i})^{k_{i}}}{(m+n)!} L_{(m)}(x) L_{(m+n-k_{1})}(x_{i}) \ldots L_{(m+n-k_{p})}(x_{p}), \quad |z| < 1 \quad (5.4.1)\]
which is equivalent to
\[\exp[(s+t) \prod_{i=1}^{p} \{(1+z_{i})\} - x_{1} z_{1} - \ldots - x_{p} z_{p}]\]
\[= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k_{1}, \ldots, k_{p}=0}^{p} \frac{\prod_{i=1}^{p} \left(1+m+n-k_{i}\right) x_{i}^{k_{i}} k_{i}!}{m! n!} \frac{s^m t^n}{F_{1} [-n; m+1, x]} \ F_{1} [-k_{i}; 1+m+n-k_{i}, x_{i}] \ldots \ F_{1} [-k_{p}; 1+m+n-k_{p}, x_{p}] \quad (5.4.2)\]
Now by replacing \( x, x_1, \ldots, x_p, s \) and \( t \) in (5.4.1) by \( xu, x_1u, \ldots, x_pu, su \) and \( tu \) respectively, multiplying both sides by \( u^{c-1} \) and taking Laplace transforms with help of the results [25, p.137(1)].

\[
\int_0^\infty e^{-au}u^{c-1} \, du = \left[ c - a \right]^{-c}, \text{ Re}(a)>0, \text{ Re}(c)>0.
\] (5.4.3)

and [94, p. 260(2(ii)]

\[
\int_0^\infty e^{at}t^{-1} L^{(a)}(x_1 t) \cdots L^{(a_n)}(x_n t) = \left[ a \right]^{\alpha_1+1, \ldots, \alpha_n+1; x_1/p, \ldots, x_n/p}. \text{ Re}(a)>0, \text{ Re}(p)>0.
\] (5.4.4)

we get the following generating relation for Lauricella function \( F_A^{(n)} \):

\[
[1-(s+t)] \prod_{i=1}^p (1+z_i) + xt/s + x_1z_1 + \ldots + x_pz_p]^{-c}
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k_1 \ldots k_p=0}^{\infty} \prod_{i=1}^p \left\{ \binom{m+n}{k_i} z_i^{k_i} \right\} \frac{s^m t^n}{n! m!} (c)_{m+n}
\]

\[
F_A^{(p+1)} [m+n+c, -n, -k_1, \ldots, -k_p; m+1, 1+m+n-k_1, \ldots, 1+m+n-k_p; x, x_1, \ldots, x_p].
\] Re(c)>0, \(|x|+|x_1|+\ldots+|x_p|<1 \) and \(|z|<1\). (5.4.5)

For \( s=t=((x/2+x_1z_1/2+\ldots+x_pz_p/2)/\prod(1+z_i)), (5.4.5) \) reduces to

\[
1 = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k_1 \ldots k_p=0}^{\infty} \prod_{i=1}^p \left\{ \binom{m+n}{k_i} z_i^{k_i} \right\} ((x+x_1z_1+\ldots+x_pz_p)/2\prod(1+z_i))^{m+n} (c)_{m+n}
\]

\[
F_A^{(p+1)} [m+n+c, -n, -k_1, \ldots, -k_p; m+1, 1+m+n-k_1, \ldots, 1+m+n-k_p; x, x_1, \ldots, x_p].
\] (5.4.6)
Equation (5.4.5) establish an important formula whereby integral powers of 
\((x - x_1z_1 - \ldots - x_pz_p)\) may be expanded as multiple series of Lauricella function.

First we note

\[ V(x, x_1, \ldots, x_p, s, t, c) = \left[ 1-(s+t) \prod_{i=1}^{p} (1+z_i) + xt/s + x_1z_1 + \ldots + x_pz_p \right]^c \]

gives

\[ V(x, x_1, \ldots, x_p, (x/2 + x_1z_1/2 + \ldots + x_pz_p/2)/\prod_{i=1}^{p} (1+z_i), (x/2 + x_1z_1/2 + \ldots + x_pz_p/2)/\prod_{i=1}^{p} (1+z_i)) \]

and

\[ \frac{\partial^r V}{\partial t^r} = (c)_r \left((x/s-\Pi(1+z_i))^{r} \prod_{i=1}^{p} (1+z_i) + x_1z_1 + \ldots + x_pz_p \right)^{c-r} \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{n=m+1}^{\infty} \sum_{k_0}^{\infty} \prod_{i=1}^{p} \binom{m+n}{k_i} (-n)^{(c)} \frac{s^m t^{n-k}}{m! n!} \]

\[ F^{(p+1)}_A [m+n+c, -n, -k_1, \ldots, -k_p, m+1, 1+m+n-k_1, \ldots, 1+m+n-k_p; x, x_1, \ldots, x_p] \quad (5.4.7) \]

When \(s=t=((x/2 + x_1z_1/2 + \ldots + x_pz_p/2)/\prod_{i=1}^{p} (1+z_i))\), (5.4.7) yields

\[ (x-x_1z_1 - \ldots - x_pz_p)^r = (2^r(-n)_r) \sum_{m=-\infty}^{\infty} \sum_{n=m+1}^{\infty} \sum_{k_0}^{\infty} \prod_{i=1}^{p} \binom{m+n}{k_i} (-n)^{(c)} \frac{s^m t^{n-k}}{m! n!} \]

\[ F^{(p+1)}_A [m+n+c, -n, -k_1, \ldots, -k_p, m+1, 1+m+n-k_1, \ldots, 1+m+n-k_p; x, x_1, \ldots, x_p] \quad (5.4.8) \]

for \(r=0, 1, 2, \ldots, \text{Re}(c)>0\).

A second set of expansions also exist which may be obtained in a similar manner by taking successive partial derivatives with respect to \(s\) of the generating relation (5.4.5) and letting \(s=t=((x/2 + x_1z_1/2 + \ldots + x_pz_p/2)/\prod_{i=1}^{p} (1+z_i))\).
The general formula of these expansions has not, so far been obtained, and the expansions of powers of \( x \) up to \( x^2 \) are given below.

\[
\frac{c(3x+x_1Z_1+\ldots+x_pZ_p)}{2} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{k_0=0}^{p} \prod_{i=1}^{p} \left\{ \left( \begin{array}{c} m+n \\ k_i \end{array} \right) z_i^{k_i} \right\} m(c)_{m+n} \\
((x+x_1Z_1+\ldots+x_pZ_p) / 2 \prod_{i=1}^{p} (1+z_i))^{m+n} F_A^{(p+1)} [m+n+c,-n,-k_1,\ldots,-k_p; m+1,1+m+n-k_p,\ldots]\]

\[
1+m+n-k_p; x, x_1,\ldots,x_p \] , \hspace{1cm} (5.4.9)

\[
\frac{c(c+1)(3x+x_1Z_1+\ldots+x_pZ_p)^2}{4} = -2cx= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{k_0=0}^{p} \prod_{i=1}^{p} \left\{ \left( \begin{array}{c} m+n \\ k_i \end{array} \right) z_i^{k_i} \right\} m(m-1) (c)_{m+n} \\
((x+x_1Z_1+\ldots+x_pZ_p) / 2\prod_{i=1}^{p} (1+z_i))^{m+n} F_A^{(p+1)} [m+n+c,-n,-k_1,\ldots,-k_p; m+1,1+m+n-k_p,\ldots]\]

\[
1+m+n-k_p; x, x_1,\ldots,x_p \] . \hspace{1cm} (5.4.10)

We shall now generalize relation (5.4.5) and obtain generating function for Horn's function of \((n+p+1)\) variables \(^{(k)}H_4^{(n+p+1)}\). We recall \([27, p. 104 (3.5.4.5)]\)

\[
^{(k)}H_4^{(n)} [a,b_{k+1},\ldots, b_n; c_1,\ldots, c_n; x_1,\ldots, x_k, x_{k+1},\ldots, x_n] = \int_{0}^{\infty} \frac{u^{n-1}e^{pu}}{[a]} \prod_{i=1}^{k} F_1 [-; c_i, x_i; u] \prod_{i=k+1}^{n} F_1 [b_i; c_i; x_i; u] \ du . \hspace{1cm} (5.4.11)
\]

On replacing \( x, x_1,\ldots, x_p, s \) and \( t \) by \( xu, x_1u,\ldots, x_pu, su \) and \( tu \) respectively in (5.4.2), multiplyng both sides by

\[
u^{n-1} e^{pu} \prod_{i=1}^{k} F_1 [-; c_i, y_i; u^2] \prod_{i=k+1}^{n} F_1 [b_i; c_i; y_i; u] , \hspace{1cm} (5.4.12)
\]

integrating the multiple series with respect to 'u' between the limits zero and infinity,
using integral (5.4.11) and definition (1.12.1) and adjusting the parameters, we get

\[
\omega^{-c} (k) H^m[n] \left[ c, b_{k+1}, ..., b_n; c_1, ..., c_n; y_0, y_1, ..., y_k, y_{k+1}, ..., y_n \right] = \sum_{m=-\infty}^{0} \sum_{n=m^*}^{0} \sum_{k=0}^{p} \frac{(1+m+n-k)}{k!} z_i^{k_i} \frac{s^m t^n}{m! n!} (c)_{m+n} (k) H^m_{n+1} \left[ m+n+c, b_{k+1}, ..., b_n; c_1, ..., c_n; y_0, y_1, ..., y_k, y_{k+1}, ..., y_n \right] (5.4.13)
\]

where Re(\(\omega\))>0, Re(c)>0 and

\[
P = \left[ 1-(s+t)n(1+z) + xt/s + xz + ... + xz \right] (5.4.14)
\]

Equation (5.4.5) is an interesting generalization of known results of Pathan and Yasmeen [68, p.241(2.1), p. 242(2.2),(2.3)]. For example if in (5.5.8), we put \(x_i=k_i=z_i=0, i=1,2,...,p\), and use the results [27, p.215(5.9.2) and p.216(6.9.6)] then we arrive at the result [68, p. 241(2.1)].

\[
(1-s-t+xt/s)^c = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{0} \frac{(s^m t^n(1-x)^n}{(m+n)! n!} \left[ x_1, ..., x_p, z_1, ..., z_p \right] (5.4.15)
\]

Obviously the results [68, p.242 (2.2) and (2.3)] follows from (5.5.17). The results (5.4.9) and (5.4.10) include recent results due to Pathan and Yasmeen [68,p.242(2.6) and (2.7)] as special cases for \(x_i=k_i=z_i=0, i=1,2,...,p\).

Further, if we put \(x_i=k_i=z_i=0, i=1,2,...,p\), in (5.4.13), it reduces to a known result of Pathan and Kamarujjama [65, p.33(2.4)].

On setting \(k=0\) and using the relationship [65,p.32(1.7)]

\[
^{(0)}H^n[a, b_1, ..., b_n; c_1, ..., c_n; x_1, ..., x_n] = F^n_A[a, b_1, ..., b_n; c_1, ..., c_n; x_1, ..., x_n] (5.4.16)
\]
equation (5.4.13) reduces to

\[(\omega)^c F^{(n)}[c, b_1, \ldots, b_n; c_1, \ldots, c_n; y_1/\omega, \ldots, y_n/\omega]\]

\[
\sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k=0}^{p} \left\{ \frac{(1+m+n-k_i)}{k_i} \right\} \frac{s^m t^n}{m!n!} F_A^{(n+p+1)}[m+n+c, b_1, \ldots, b_n, -k_1, \ldots, -k_p, -n; c_1, \ldots, c_n, 1+m+n-k_1, \ldots, 1+m+n-k_p, m+1; y_1, \ldots, y_n, x_1, \ldots, x_p, x]\]  

(5.4.17)

where \(\omega\) is given by (5.4.14). Note that for \(x_i=k_i=z_i=0, i=1,2,\ldots,p\), (5.4.17) reduces to another result due to Pathan and Yasmeen [68,p.143(3.3)].

\[(\omega')^c F^{(n)}[c, b_1, \ldots, b_n; c_1, \ldots, c_n; y_1/\omega', \ldots, y_n/\omega']\]

\[
\sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!n!} F_A^{(n+1)}[m+n+c, b_1, \ldots, b_n, -n; c_1, \ldots, c_n, m+1; y_1, \ldots, y_n, x_1, \ldots, x_p, x]\]  

(5.4.18)

where \(\omega' = (1-s-t+xt/s)\).