APPENDIX

A REMARK ON THE COMMUTATIVITY OF CERTAIN RINGS

In a recent paper [9] R.N. Gupta proved that a division ring satisfying the polynomial identity \( xy^2x = yx^2y \) is commutative. His proof is based on the Cartan-Brauer-Hasse theorem.

In this appendix our aim is to prove the following:

Theorem. Let \( R \) be a semiprime ring with its center \( Z \). Suppose that \( xy^2x - yx^2y \in Z \) for all \( x, y \) in \( R \). Then \( R \) is commutative.

This result can easily be proved by Posner's theorem [20] on polynomial identity when \( R \) has characteristic different from 2. Here we have given an elementary proof of this result.

We begin with the following:

Lemma 1. Let \( R \) be a prime ring with \( xy^2x - yx^2y \in Z \), the center of \( R \), for all \( x, y \) in \( R \). Then \( Z \) is non-zero.

Proof. Suppose on the contrary \( Z = \{ 0 \} \). Then by hypothesis

\[
xy^2x = yx^2y \text{ for all } x, y \in R \tag{1}
\]

In (1) replace \( y \) by \( y + y^2 \), then

\[
xy^2x + xy^3x + xy^3x + xy^4x = yx^2y + y^2x^2y + y^2x^2y + y^2x^2y^2
\]
In view of (1), the last equation gives
\[ 2xy^3 x = y^2 x^2 y + xy^2 y^2 \] for all \( x, y \in R \) \hspace{1cm} (2)

**Case I.** \( \text{Char. } R = 2. \) Then (2) gives
\[ y^2 x^2 y + xy^2 y^2 = 0 \]

Since \( xy^2 y = xy^2 x \) for all \( x, y \in R. \) Therefore we get
\[ yxy^2 x + xy^2 y^2 = 0 \]
i.e.
\[ yx(y^2x + xy^2) = 0 \] for all \( x, y \in R \) \hspace{1cm} (3)

Replace \( x \) by \( x+y \) in (2), we get
\[ y(x+y) \left\{ y^2(x+y) + (x+y)y^2 \right\} = 0 \]
i.e.
\[ (yx + y^2)(y^2x + xy^2) = 0 \]

By (3) the last equation yields
\[ y^2 (y^2x + xy^2) = 0 \] for all \( x, y \in R \) \hspace{1cm} (4)

In (4) replace \( x \) by \( rx \) when \( r \in R, \) from
\[ y^2rx + rxy^2 = \{ y^2rx + rxy^2 \} + \{ rxy^2 + rxy^2 \} \]
we get
\[ y^2 r (y^2x + xy^2) = 0 \] for all \( x, y \in R. \)

Since \( R \) is prime, either \( y^2 = 0 \) or \( y^2 \in Z \) for all \( y \in Z. \)
As \( Z = \langle n(0) \rangle \), \( y^2 = 0 \) for all \( y \in \mathbb{R} \).

Case II. \( \operatorname{Char.} \mathbb{R} = 2 \).

Replace \( y \) by \( y + y \) in (1) to obtain

\[
xy^2x + xy^4x + xy^4x + xy^6x = yx^2y + y^3x^2y + yx^2y^3 + y^3x^2
\]

In view of (1), the last equation gives

\[
xy^4x + xy^4x = yx^2y + yx^2y^3
\]

Again by (1) we get

\[
y^2x^2y^2 + y^2x^2y^2 = y^2 xy^2x + xy^2x y^2
\]

i.e., \( y^2x(xy^2 - y^2x) = (xy^2 - y^2x) y^2 \) for all \( x, y \in \mathbb{R} \) \( \quad (5) \)

In (5) replace \( x \) by \( x+y \), then

\[
y^2(x+y) \left\{ (x+y)y^2 - y^2(x+y) \right\} = \left\{ (x+y)y^2 - y^2(x+y) \right\} (x+y)y^2
\]

i.e., \( (y^2x + y^3)(xy^2 - y^2x) = (xy^2 - y^2x)(xy^2 + y^3) \)

In view of (5) the last equation yields

\[
y^3(xy^2 - y^2x) = (xy^2 - y^2x)y^3
\]

i.e., \( y^3 I y^2 (x) = 0 \) for all \( x, y \in \mathbb{R} \).

By Lemma 2.3 either \( y^3 \in \mathbb{Z} \) or \( y^2 \in \mathbb{Z} \) for all \( y \in \mathbb{R} \). Since \( Z = \langle n(0) \rangle \), \( y^2 = 0 \) or \( y^3 = 0 \) for all \( y \in \mathbb{R} \). If for some \( y \) in \( \mathbb{R} \), \( y^2 \neq 0 \), then for that \( y, y^3 = 0 \), hence by (2), we get
\[ y^2 x^2 y + y x^2 y^2 = 0 \text{ for all } x \in R \] (6)

Replace \( x \) by \( xy \) in (6), then

\[ y^2(x^2+xy+yz+y^2)y + y(x^2+xy+yx+y^2)y^2 = 0 \]

i.e., \( y^2x^2y + y^2xy^2 + yx^2y^2 + y^2xy^2 = 0 \), since \( y^3 = 0 \)

By (6), the last equation gives

\[ 2y^2xy^2 = 0, \quad y^2xy^2 = 0 \text{ for all } x \in R \]

Since \( R \) is prime, \( y^2 = 0 \). Hence \( y^2 = 0 \) for all \( y \in R \).

So in both cases, \( y^2 = 0 \) for all \( y \in R \). Therefore,

\[ 0 = (x+y)^2 = x^2 + xy + yx + y^2 = xy + yx \]

Multiply by \( x \) on the right to obtain \( xyx = 0 \) for all \( x,y \) in \( R \). Since \( R \) is prime, \( x = 0 \) for all \( x \in R \), a contradiction.

Hence the conclusion that \( Z \neq (0) \) holds.

**Lemma 2.** If \( R \) is a prime ring with \( xy^2z - yz^2y \in Z \) for all \( x, y \in R \), then \( R \) is commutative.

**Proof.** By Lemma 1, \( Z \neq (0) \). Let \( a (\neq 0) \in Z \).

Replace \( x \) by \( x+z \) in the relation \( xy^2z - yz^2y \in Z \), we get

\[ (x+z)y^2(x+z) - y(x^2 + xz + sx + s^2) \quad y \in Z \]

i.e., \( (xy^2z - yz^2y) + z (y^2x - 2yxy + xy^2) \in Z \).
But, \( xy^2z - yz^2y \in Z \). Therefore

\[
s(2y^2x - 2z^2y + xy^2) \in Z \text{ for all } x, y \in R.
\]

Since \( z \neq 0 \), by Lemma 1.3,

\[
(y^2x - 2z^2y + xy^2) \in Z \text{ for all } x, y \in R \quad (7)
\]

Replace \( x \) by \( xy \) in (7), thus

\[
(y^2x - 2z^2y + xy^2) y \in Z \text{ for all } x, y \in R \quad (8)
\]

If for some \( y \) in \( R \), \( y^2x - 2z^2y \neq 0 \), then by (7), (8) and Lemma 1.3, \( y \in Z \), a contradiction. Hence

\[
y^2x - 2z^2y + xy^2 = 0 \text{ for all } x, y \in R \quad (9)
\]

Case I. Char. \( R \neq 2 \). Equation (9) can be written as

\[
y(xy - y) = (yx - xy) y \text{ for all } x, y \in R.
\]

By Lemma 2.1 \( y \in Z \) for all \( y \in R \). Hence \( R \) is commutative.

Case II. Char. \( R = 2 \). Then (9) gives \( y^2 \in Z \) for all \( y \in R \).

In particular, \( (x+y)^2 = x^2 + xy + yx + y^2 \in Z \)

Hence \( xy + yx \in Z \). Replace \( x \) by \( xy \), then \( (xy + yx) y \in Z \).

If for some \( y \) in \( R \), \( xy + yx \neq 0 \) for some \( x \in R \), then by Lemma 1.3 \( y \in Z \), a contradiction. Thus \( xy + yx = 0 \) for all \( x, y \in R \). Hence \( R \) is commutative.

Let \( R \) be a semiprime ring in which \( xy^2x - yz^2y \) is central of \( R \) for all \( x, y \) in \( R \). Since \( R \) is semiprime it is isomorphic
to a subdirect sum of prime rings $R_\kappa$ each of which, as a homomorphic image of $R$, enjoys the hypothesis placed on $R$.

By Lemma 2, $R_\kappa$ are commutative, since subdirect sum of commutative rings is commutative, $R$ is commutative. Thus we have proved the following:

**Theorem 1.** If $R$ is a semi prime ring with $xy^2x - yx^2y \in Z$, the center of $R$, for all $x, y \in R$, then $R$ is commutative.

Indeed, rings of $3 \times 3$ strictly upper triangular matrices of any ring satisfy the hypothesis placed on $R$ in Theorem 1, but these rings may not be commutative.