Chapter-1
PRELIMINARIES

The present chapter is devoted to review some basic notions, important terminology and known results with a view to making our thesis as self contained as possible. Of course, the elementary knowledge of the algebraic concepts like groups, rings, ideals, fields, and homomorphisms etc. has been pre-assumed and no attempt has been made to discuss them here. Suitable examples and necessary remarks are given at proper places.

1.2 SOME RING THEORETIC CONCEPTS

In the present section, we give a brief exposition of some important terminology in Ring theory. Throughout, until otherwise specified $R$ represents an associative ring (may be without unity and need not be commutative). For any pair of elements $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$ and anti-commutator $xy + yx$ by $x \circ y$.

**Definition 1.2.1 (Characteristic of a Ring).** The least positive integer $n$ (if exists) such that $nx = 0$, for all $x \in R$ is called the characteristic of the ring $R$ which is generally expressed as $\text{char} R = n$. If no such positive integer exists, then $R$ is said to have characteristic zero.

**Remark 1.2.1.** Obviously, if $\text{char} R \neq m$, then for some $x \in R$, $mx = 0$ implies that $x = 0$.

**Definition 1.2.2 (Torsion Free Element).** An element $x \in R$ is said to be $n$-torsion free if $nx = 0$ implies that $x = 0$. If $nx = 0$ implies $x = 0$, for every
$x \in R$, then we say that $R$ is $n$-torsion free.

**Definition 1.2.3 (Idempotent Element).** An element $e$ of a ring $R$ is said to be idempotent if $e^2 = e$.

**Remark 1.2.2.** It is trivial that zero of a ring $R$ is an idempotent element. Moreover, if $R$ contains unity 1, then 1 is also idempotent. However, there may exist many idempotent elements in $R$ other than 0 and 1.

**Definition 1.2.4 (Nilpotent Element).** An element $x$ of a ring $R$ is said to be nilpotent if there exists a positive integer $n$ such that $x^n = 0$.

**Remark 1.2.3.** It is trivial that zero of a ring $R$ is nilpotent. Moreover, every nilpotent element is necessarily a divisor of zero. For if $x \neq 0$ and $n$ is the smallest positive integer such that $x^n = 0$, then $n > 1$ and $x(x^{n-1}) = 0$ with $x^{n-1} \neq 0$.

**Definition 1.2.5 (Centre of a Ring).** The centre $Z(R)$ of a ring $R$ is the collection of all those elements of $R$ which commute with each element of $R$, that is,
\[
Z(R) = \{ x \in R \mid xy = yx, \text{ for all } y \in R \}.
\]

**Definition 1.2.6 (Centralizer).** Let $S$ be a non-void subset of a ring $R$. Then the centralizer $C_R(S)$ of $S$ in $R$ is defined as
\[
C_R(S) = \{ x \in R \mid xs = sx, \text{ for all } s \in S \}.
\]

**Definition 1.2.7 (Nilpotent Ideal).** A right (left, two sided) ideal $I$ of a ring $R$ is said to be nilpotent if there exists a positive integer $n > 1$
such that $I^n = \{0\}$.

**Definition 1.2.8 (Nil Ideal).** A right (left, two sided) ideal $I$ of a ring $R$ is said to be nil if each of its element is nilpotent.

**Example 1.2.1.** Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} ; a, b, c \in \mathbb{Z} \right\}$. Let $I$ be an ideal of $R$ generated by $\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$. Then $I$ is nilpotent and also a nil ideal.

**Remark 1.2.4.**

(i) If every element of a ring $R$ is nilpotent, then $R$ itself is called a nil ring.

(ii) Every nilpotent ideal is nil but a nil ideal need not be necessarily nilpotent.

**Example 1.2.2.** Let $p$ be a fixed prime and for each positive integer $i$, let $R_i$ be the ideal in $\mathbb{Z}/(p^{i+1})$, consisting of all nilpotent elements of $\mathbb{Z}/(p^{i+1})$, that is, consisting of the residue classes modulo $p^{i+1}$ which contain multiples of $p$. Then $R_i^{i+1} = (0)$, whereas $R_k^k \neq (0)$, for $k < i + 1$. Now consider the discrete direct sum $T$ of the rings $R_i$ ($i=1, 2, 3, \ldots$). Since each element of $T$ differs from zero in only a finite number of components i.e., each element of $T$ is nilpotent. Then $T$ is a nil ideal in $T$ but not a nilpotent ideal.

**Definition 1.2.9 (Commutator Ideal).** The commutator ideal $C(R)$ of a ring $R$ is the ideal generated by all commutators $[x, y]$ with $x, y \in R$.

**Definition 1.2.10 (Prime Ideal).** An ideal $P$ of a ring $R$ is said to be a prime ideal if and only if it has the property that for any two ideals $A, B$ in $R$, whenever $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. 

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Remark 1.2.5. Equivalently, an ideal $P$ in a ring $R$ is prime if and only if any one of the following holds:

(i) If for any $a, b \in R$ such that $aRb \subseteq P$, then $a \in P$ or $b \in P$.

(ii) If $(a)$ and $(b)$ are principal ideals in $R$ such that $(a)(b) \subseteq P$, then $a \in P$ or $b \in P$.

(iii) If $R$ is a commutative ring such that for any $a, b \in R$, $ab \in P$, then $a \in P$ or $b \in P$.

(iv) If $U$ and $V$ are right (left) ideals in $R$ such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

Definition 1.2.11 (Semiprime Ideal). An ideal $I$ in a ring $R$ is said to be a semiprime ideal if for any ideal $A$ in $R$, whenever $A^2 \subseteq I$, then $A \subseteq I$.

Remark 1.2.6.

(i) A prime ideal is necessarily semiprime but the converse need not be true in general.

(ii) Intersection of prime (semiprime) ideals is semiprime. Thus in the ring $\mathbb{Z}$ of integers, ideal $(2) \cap (3) = (6)$ is semiprime which is not prime.

Definition 1.2.12 (Maximal Ideal). An ideal $M$ of a ring $R$ is called maximal, if

(i) $M \neq R$, 

(ii) there exists no ideal $J$ in $R$ such that $M \subseteq J \subseteq R$.

Remark 1.2.7.

(i) If $M \neq R$ is a maximal in $R$, then for any ideal $J$ of $R$, $M \subseteq J \subseteq R$ holds only when either $J = M$ or $J = R$. 


(ii) Every maximal ideal in a commutative ring is a prime ideal.

**Definition 1.2.13 (Jacobson Radical).** The Jacobson radical $J(R)$ of a ring $R$ is the intersection of all maximal left (right) ideals of $R$.

**Definition 1.2.14 (Annihilator).** If $M$ is a subset of a commutative ring $R$, then the annihilator of $M$, denoted by $Ann(M)$ is the set of all elements $r$ of $R$ such that $rm = 0$, for all $m \in M$. Thus

$$Ann(M) = \{r \in R \mid rm = 0, \text{ for all } m \in M\}.$$  

**Definition 1.2.15 (Prime Ring).** A ring $R$ is said to be prime if and only if zero ideal $(0)$ is prime ideal in $R$.

**Remark 1.2.8.** Equivalently, a ring $R$ is prime if and only if any one of the following holds:

(i) If $(a)$ and $(b)$ are principal ideals in $R$ such that $(a)(b) = 0$, then $a = 0$ or $b = 0$.

(ii) If $a, b \in R$ such that $aRb = (0)$, then $a = 0$ or $b = 0$.

**Definition 1.2.16 (Semiprime Ring).** A ring $R$ is said to be semiprime if it has no nonzero nilpotent ideals.

**Remark 1.2.9.** Equivalently, a ring $R$ is prime if and only if

(i) For any $x \in R$, whenever $xRx = \{0\}$, then $x = 0$.

(ii) The centre of a semiprime ring contains no nonzero nilpotent elements.

(iii) In a semiprime ring $R$, the centre of a nonzero one sided ideal is contained in the centre of $R$. In particular, any commutative one sided ideal is contained in the centre of $R$. 

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Definition 1.2.17 (Simple Ring). A ring $R$ with more than one element is said to be simple if its only ideals are the two trivial ideals, namely, $(0)$ and $R$ itself.

Definition 1.2.18 (Semisimple Ring). A ring $R$ is said to be semisimple if its Jacobson radical is zero.

Definition 1.2.19 (Boolean Ring). A ring $R$ is said to be Boolean if all of its elements are idempotent i.e., $x^2 = x$, for all $x \in R$.

Definition 1.2.20 (Direct Sum and Subdirect Sum of Rings). Let $S_i, i \in U$ be a family of rings indexed by the set $U$ and $S$ denote the set of all functions defined on the set $U$ such that for each $i \in U$, the value of function at $i$ is an element of $S_i$. If addition and multiplication in $S$ are defined as : $(a + b)(i) = a(i) + b(i)$, $(ab) = a(i)b(i)$, for all $a, b \in S$, then $S$ is a ring which is called the complete direct sum of the rings $S_i, i \in U$. The set of all functions in $S$ which take the value zero at all but at most a finite number of elements $i \in U$ is a subring of $S$ which is called the discrete direct sum of rings $S_i, i \in U$. However, if $U$ is a finite set, the complete (discrete) direct sum of rings $S_i, i \in U$, as defined above is called a direct sum of rings $S_i, i \in U$.

Let $T$ be a subring of the direct sum $S$ of rings $S_i$ and for each $i \in U$ let $\theta_i \in U$ be a homomorphism of $S$ onto $S_i$ defined by $a\theta_i = a(i)$, for $a \in S$. If $T\theta_i = S_i$ for every $i \in U$, then $T$ is said to be a subdirect sum of the ring $S_i, i \in U$.

Definition 1.2.21 (Lie and Jordan Structure). Let $R$ be an associative ring. We can induce on $R$ two new operations as follows :
(i) For all \(x, y \in R\), the Lie product \([x, y] = xy - yx\).

(ii) For all \(x, y \in R\), the Jordan product \(x \circ y = xy + yx\).

The additive group \((R, +)\) together with the Lie product (resp. Jordan product) is sometimes called Lie (resp. Jordan) ring.

**Remark 1.2.10.** For any \(x, y, z \in R\), the following identities hold:

(i) \([xy, z] = x[y, z] + [x, z]y\).

(ii) \([x, yz] = y[x, z] + [x, y]z\).

(iii) \([[x, y], z] + [[y, z], x] + [[z, x], y] = 0\). This identity is generally known as Jacobi identity.

(iv) \(x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z\).

(v) \((xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]\).

**Definition 1.2.22 (Lie (Jordan) Subring).** A nonvoid subset \(U\) of a ring \(R\) is Lie (resp. Jordan) subring of \(R\) if \(U\) is an additive subgroup of \(R\) and \(a, b \in U\) implies that \([a, b]\) (resp. \((a \circ b))\) is also in \(U\).

**Definition 1.2.23 (Lie (Jordan) Ideal).** An additive subgroup \(U \subset R\) is said to be a Lie (resp. Jordan) ideal of \(R\) if whenever \(u \in U\) and \(r \in R\), then \([u, r]\) (resp. \((u \circ r))\) is also in \(U\).

**Example 1.2.3.** Let \(R=\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}\). Then it can be easily seen that \(U=\left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}\) is a Lie ideal of \(R\) and \(U=\left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \mid b \in \mathbb{Z}_2 \right\}\) is a Jordan ideal of \(R\).
Definition 1.2.24 (Commuting Function). Let $S$ be a subset of $R$. A function $F : R \rightarrow R$ is said to be a commuting function on $S$ if $[F(x), x] = 0$, for all $x \in S$.

Definition 1.2.25 (Centralizing Function). Let $S$ be a subset of $R$. A function $F : R \rightarrow R$ is said to be a centralizing function on $S$ if $[F(x), x] \in Z(R)$, for all $x \in S$ i.e., $[[F(x), x], z] = 0$, for all $x \in S$ and $z \in R$.

1.3 NEAR RINGS

This section deals with some preliminary concepts and simple properties of near rings.

Definition 1.3.1 (Near Ring). A left near ring $R$ is a triple $(R, +, *)$ with two binary operations $+$ and $*$ such that

(i) $(R, +)$ is a group (not necessarily abelian).

(ii) $(R, *)$ is a semigroup.

(iii) $a * (b + c) = a * b + a * c$, for all $a, b, c \in R$.

Analogously, if instead of (iii), we have the right distributive law

(iii)$'$ $(a + b) * c = a * c + b * c$

holds, then $R$ is said to be a right near ring.

As in both the cases, the theory of near rings runs completely parallel, we may consider left near rings throughout and for simplicity call them as near rings.
Example 1.3.1. (i) The set of all identity preserving mappings acting on the right of an additive group \( G \) (not necessarily abelian) into itself with pointwise addition and composition of mappings as multiplication is the most natural example of a right near ring.

(ii) \( R = \{0,a\} \) with addition and multiplication defined as follows:

\[
\begin{array}{c|cc}
+ & 0 & a \\
\hline
0 & 0 & a \\
a & a & 0
\end{array}
\quad
\begin{array}{c|cc}
* & 0 & a \\
\hline
0 & 0 & a \\
a & a & 0
\end{array}
\]

It is easily checked that \( R \) is a left near ring.

(iii) For more examples one may consult [54].

Definition 1.3.2 (Distributive Element). An element \( x \) of a near ring \( R \) is said to be distributive if \((y + z)x = yx + zx\), for all \( y, z \in R \).

Definition 1.3.3 (Distributive Near Ring). A near ring \( R \) is said to be distributive if all of its elements are distributive.

Remark 1.3.1. In any near ring \( R \),

(i) \( x0 = 0 \), for all \( x \in R \), but not necessarily \( 0x = 0 \). However, if \( R \) is distributive, then \( 0x = 0 \).

(ii) \( x(-y) = -xy \), for all \( x, y \in R \), but not necessarily \( (-x)y = -xy \). However, if \( R \) is distributive, then \( (-x)y = -xy \).
Definition 1.3.4 (Additive Centre). An additive centre of a near ring \( R \) is the set of all those elements of \( R \) which commute with every element of \( R \) under addition.

Multiplicative centre of a near ring is defined in the same manner as we have defined in the case of rings (cf. Definition 1.2.5).

Definition 1.3.5 (Distributively Generated Near Ring). A near ring \( R \) is said to be distributively generated \((d - g)\), if it contains a multiplicative subsemigroup of distributive elements which generates the additive group \((R, +)\).

Example 1.3.2. The near ring generated additively by all the endomorphisms of a group \((G, +)\) (not necessarily abelian), is a distributively generated near ring.

Definition 1.3.6 (Ideal). An ideal of a near ring \( R \) is defined to be a normal subgroup \( I \) of \( R^+ \) such that

\[(i) \quad RI \subseteq I.\]

\[(ii) \quad (x + i)y - xy \in I, \text{ for all } x, y \in R \text{ and } i \in I.\]

Normal subgroup of \((R, +)\) satisfying (i) are called the left ideals and satisfying (ii) are called right ideals.

In case of a \( d - g \) near ring, the condition (ii) above may be replaced by

\[(ii)^* \quad IR \subseteq I.\]

Definition 1.3.7 (Near Ring Homomorphism). A mapping \( f : R \rightarrow R^* \) of a near ring \( R \) into another near ring \( R^* \) is called a near ring homomorphism
if \( f(x + y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x, y \in R \).

**Definition 1.3.8 (Zero-symmetric Near Ring).** A near-ring \( R \) is said to be zero-symmetric, if \( 0x = 0 \), for all \( x \in R \) (recall that left distributivity yields \( x0 = 0 \)).

**Example 1.3.3.** Let \( R = \{0, a, b, c\} \) with addition and multiplication tables defined as below:

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It can be easily verified that \( R \) is a zero-symmetric near ring.

**Remark 1.3.2.** A \( d - g \) near ring is necessarily zero-symmetric.

**Definition 1.3.9. (Zero-commutative Near Ring).** A near ring \( R \) is said to be zero-commutative, if \( xy = 0 \) implies that \( yx = 0 \), for all \( x, y \in R \).

**Example 1.3.4.** \( R = \{0, a, b, c\} \) with addition and multiplication tables defined as below:

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Then \( (R, +, *) \) is a zero-commutative near ring.
1.4 SOME WELL KNOWN RESULTS

In this section, we state some well-known results which will be frequently used in the development of the subsequent chapters.

**Theorem 1.4.1 (Daif and Bell [56]).** Let $R$ be a semiprime ring and $I$ be a nonzero ideal of $R$. If $a$ in $R$ centralizes the set $[I, I]$, then $a$ centralizes $I$.

**Theorem 1.4.2 (Jacobson [87]).** Let $R$ be a ring in which for every $x \in R$ there exists an integer $n = n(x) > 1$, depending on $x$ such that $x^{n(x)} = x$, then $R$ is commutative.

**Theorem 1.4.3 (Frohlic [65]).** A $(d - g)$ near ring $R$ with unity 1 is a ring if $(R, +)$ is abelian or $R$ is distributive.

**Theorem 1.4.4 (Bell [22]).** Let $R$ be a zero-symmetric near ring having no nonzero nilpotent elements. Then

- (i) every distributive idempotent is central;
- (ii) for every idempotent $e$ and every element $x \in R$, $ex^2 = (ex)^2$;
- (iii) if $R$ has a multiplicative identity element, then all idempotents are central.

**Theorem 1.4.5 (Neumann [108]).** The additive group of a division near ring is abelian.