5.2 SOME PRELIMINARY DEFINITIONS AND RESULTS

We begin with some preliminary definitions and results in order to make our subject matter as much self contained as possible.

Perhaps, motivated by two basic properties of a differential operator, say $D$ (namely, (i) $D(f_1 + f_2) = D(f_1) + D(f_2)$, (ii) $D(f_1 f_2) = D(f_1) f_2 + f_1 D(f_2)$), the notion of derivations was introduced in rings.

**Definition 5.2.1 (Derivation).** A mapping $d : R_1 \rightarrow R_2$ from a ring $R_1$ to a ring $R_2$ is said to be a derivation if for all $x, y \in R_1$, the following hold:

(i) $d(x_1 + x_2) = d(x_1) + d(x_2)$,

(ii) $d(x_1 x_2) = d(x_1)x_2 + x_1 d(x_2)$

**Example 5.2.1.** Let $R = \mathbb{R}[X]$ be the ring of polynomials over the field $\mathbb{R}$ of real numbers and

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \ a_i \in \mathbb{R}$$

be an arbitrary element. Set

$$d(p(x)) = a_1 + 2a_2 x + \ldots + na_n x^{n-1}.$$ 

It can be readily verified that $d$ is a derivation on $R$.

**Example 5.2.2.** Let $R$ be a ring and $a$ be a fixed element of $R$. Define the mapping $\delta : R \rightarrow R$ by $\delta(x) = [x, a] = xa - ax$, for all $x \in R$, then $\delta$ is a derivation on $R$ which is usually called the inner derivation of $R$ and generally denoted by $I_\theta$. 

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Example 5.2.3. Let $R$ be the ring of $2 \times 2$ matrices over $GF(2)$. Define $d : R \rightarrow R$ by $d \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 0 & -b \\ c & 0 \end{array} \right)$; $a, b, c, d \in GF(2)$. It is easy to see that $d$ is a derivation on $R$.

We shall make frequent use of the following well-known results which may be found in [40], [104], [13] and [116] respectively. However, in order to make our subsequent text as much self contained as possible, we are giving the sketches of their proofs that are at times simpler and straightforward.

Proposition 5.2.1. Let $I$ be a nonzero left (right) ideal of a prime ring $R$. If $d$ is a nonzero derivation of $R$, then $d$ is nonzero on $I$.

Proof. Let $d(x) = 0$, for all $x \in I$. Replacing $x$ by $rx$ in the above relation and using it, we get $d(r)x = 0$, for all $x \in I$ and $r \in R$. Now replace $x$ by $sx$, to get $d(r)sx = 0$, for all $x \in I$ and $r, s \in R$ i.e., $d(r)R I = \{0\}$, for all $r \in R$. Since $I$ is nonzero, the primeness of $R$ yields that $d(r) = 0$, for all $r \in R$.

Proposition 5.2.2. If the prime ring $R$ contains a nonzero commutative right ideal $A$, then $R$ is commutative.

Proof. Since $I$ is commutative, $I_x(A) = [x, A] = \{0\}$, for all $x \in A$. By the above proposition, $I_x = 0$ on $R$ and $x$ is in the centre. Thus $[x, R] = \{0\}$, for every $x \in A$. Hence $I_a(A) = \{0\}$, for all $a \in R$. Again using the above proposition, we obtain $I_a = 0$ and $a$ is in the centre for all $a \in R$. Therefore, $R$ is commutative.

Proposition 5.2.3. Let $R$ be a 2-torsion free prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a derivation $d$ with $d^2(x) = 0$, for all $x \in I$, then $d = 0$. 

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Proof. we have \( d^2(x) = 0 \), for all \( x \in I \). Replacing \( x \) by \( xy \), we get \( d^2(xy) = 0 \), for all \( x, y \in I \) i.e.,

\[
d^2(x)y + 2d(x)d(y) + xd^2(y) = 0, \quad \text{for all } x, y \in I.
\]

But \( d^2(x) = 0 = d^2(y) = 0 \) by the hypothesis, the above relation implies that \( 2d(x)d(y) = 0 \), for all \( x, y \in I \). Since \( R \) is 2-torsion free, we find that \( d(x)d(y) = 0 \), for all \( x, y \in I \). Now for any \( r \in R \), replace \( y \) by \( yr \), to get \( d(x)yd(r) = 0 \), for all \( x, y \in I \) and hence \( d(x)IRd(r) = 0 \), for all \( x, r \in I \) and \( r \in R \). Thus primeness of \( R \) yields that either \( d(r) = 0 \) or \( d(x)I = \{0\} \). If \( d(x)I = \{0\} \), for all \( x \in I \), then \( d(x)RI = \{0\} \), for all \( x \in I \). Since \( R \) is prime and \( I \neq \{0\} \), we find that \( d(x) = 0 \), for all \( x \in I \) and by Proposition 5.2.1, we get the required result. □

Proposition 5.2.4. Let \( R \) be a 2-torsion free prime ring and \( U \) be a nonzero Lie ideal of \( R \). If \( U \) is a commutative Lie ideal of \( R \), then \( U \subseteq Z(R) \), the centre of \( R \).

Proof. Since \( U \) is a commutative Lie ideal of \( R \), \( [u, v] = 0 \), for all \( u, v \in U \). Replacing \( v \) by \( [u, r] \) in the above relation, we get \( [u, [u, r]] = 0 \), for all \( u \in U \) and \( r \in R \). Again replace \( r \) by \( rs \), to get \( [u, [u, rs]] = 0 \), for all \( u \in U \) and \( r, s \in R \) that is,

\[
[u, [u, r]]s + r[u, [u, s]] + 2[u, r][u, s] = 0, \quad \text{for all } u \in U \text{ and } r, s \in R.
\]

This implies that \( 2[u, r][u, s] = 0 \), for all \( u \in U \) and \( r, s \in R \). Since \( R \) is 2-torsion free, we find that \( [u, r][u, s] = 0 \), for all \( u \in U \) and \( r, s \in R \). Replacing \( s \) by \( sr \), we get \( [u, r][u, r] = 0 \), for all \( u \in U \) and \( r, s \in R \) i.e., \( [u, r]R[u, r] = \{0\} \), for all \( u \in U \) and \( r \in R \). Thus primeness of \( R \) implies that \( [u, r] = 0 \), for all \( u \in U \).
\[ r \in R \] and hence \( U \subseteq Z(R) \). \( \square \)

**Remark 5.3.1.** The above result can be extended to semiprime rings also.

### 5.3 IDEALS AND GENERALIZED DERIVATIONS IN PRIME RINGS

The notion of derivation in rings has been generalized in several directions such as left derivation, Jordan derivation, semi-derivation; to mention a few. In the theory of operator algebras, an additive mapping \( D_{a,b} : A \rightarrow A \) on an algebra \( A \) defined by \( D_{a,b}(x) = ax + xb \), for fixed \( a,b \in A \) plays an important role. One can notice that such a map can be considered as a generalization of inner derivation \( I_a : x \rightarrow ax - xa \) and is naturally termed as *generalized inner derivation* or alternatively *inner generalized derivation*.

Further, for any \( x, y \in A \)

\[
D_{a,b}(xy) = \alpha(xy) + (xy)b
\]

\[
= (ax + xb)y - xby + (xy)b
\]

\[
= D_{a,b}(x)y + x(yb - by)
\]

\[
= D_{a,b}(x)y + xI_b(y).
\]

This prompts us to formulate the following definition.

**Definition 5.3.2 (Generalized Derivation).** Let \( S \) be a nonempty subset of \( R \). An additive mapping \( F : R \rightarrow R \) is said to be a generalized derivation on \( S \) if there exists a derivation \( d : R \rightarrow R \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in S \).
Example 5.3.4. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$. Define $F : R \rightarrow R$ by $F(x) = 2cx - xc$, where $c = e_{12} + e_{21}$. Then $F$ is a generalized derivation with associated derivation $d$ given by $d(x) = cx - xc$.

As observed in [86] that the concept of generalized derivation includes both the concept of derivation as well as that of inner generalized derivation. Further, with $d = 0$, generalized derivation leads to the concept of left multiplier.

Remark 5.3.2. The following example is sufficient to show that a generalized derivation need not be a derivation in general.

Example 5.3.5. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$. Define a map $F : R \rightarrow R$ by $F \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ and a derivation $d : R \rightarrow R$ by $d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Then it can be easily verified that $F$ is a generalized derivation on $R$ but not a derivation on $R$.

Over the past few years there has been an ongoing interest in studying the relationship between the commutativity of a ring and the existence of certain specific types of derivations in the ring. In the present section, we discuss the commutativity of prime rings admitting a generalized derivation which satisfies certain functional identities.

Theorem 5.3.1. Let $R$ be a 2-torsion free prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ with associated derivation $d$ such that $[d(x), F(y)] = [x, y]$, for all $x, y \in I$, then either $d = 0$ or $R$ is commutative.
For developing the proof of the above theorem, we require the following lemma which can be found in [41].

**Lemma 5.3.1.** Let $R$ be a prime ring and $I$ be a nonzero left ideal of $R$. If $R$ admits a nonzero derivation $d$ such that $[x, d(x)]$ is central for all $x \in I$, then $R$ is commutative.

**Proof of Theorem 5.3.1.** If $F = 0$, then $[x, y] = 0$, for all $x, y \in I$ and so by Proposition 5.2.2, $R$ is commutative. Hence onward, we assume that $F \neq 0$.

We have

\[
[d(x), F(y)] = [x, y], \quad \text{for all } x, y \in I.
\]

Replacing $y$ by $yz$ in (5.3.1) and using (5.3.1), we have

\[
F(y)[d(x), y] + [d(x), d(z)] + [d(x), y]d(z) = y[x, z], \quad \text{for all } x, y, z \in I.
\]

Again replacing $z$ by $zd(x)$ in (5.3.2) and using (5.3.2), we obtain

\[
y[d(x), z]d^2(x) + yz[d(x), d^2(x)] + [d(x), y]zd^2(x) = yz[x, d(x)].
\]

Now replace $y$ by $ry$ in (5.3.3), to get

\[
ryz[d(x), d^2(x)] + ry[d(x), z]d^2(x) + r[d(x), y]zd^2(x) + [d(x), r]yzd^2(x) = ryz[x, d(x)].
\]

Then (5.3.3) yields that $[d(x), r]yzd^2(x) = 0$, for all $x, y, z \in I$, $r \in R$ i.e., $[d(x), r]IRd^2(x) = \{0\}$, for all $x \in I$ and $r \in R$. The primeness of $R$ forces that for each fixed $x \in I$, either $[d(x), r]I = \{0\}$ or $d^2(x) = 0$. Now let $I_1 = \{x \in I| d^2(x) = 0\}$ and $I_2 = \{x \in I| [d(x), r]I = \{0\}, \quad \text{for all } r \in R\}$. Then $I_1$ and $I_2$ are additive subgroups of $I$ whose union is $I$. But a group can
not be union of two of its proper subgroups and hence $I = I_1$ or $I = I_2$. If $I = I_1$, then $d^2(x) = 0$, for all $x \in I$. Thus by Proposition 5.2.3, we get $d = 0$. On the other hand, if $I = I_2$, then $[d(x), r]I = \{0\}$, for all $x \in I$, $r \in R$ and hence $[d(x), r]RI = \{0\}$, for all $x \in I$, $r \in R$. Since $R$ is prime and $I \neq \{0\}$, it follows that $[d(x), r] = 0$, for all $x \in I$, $r \in R$. In particular, $[d(x), x] = 0$, for all $x \in I$. Hence $R$ is commutative by Lemma 5.3.1. □

Proceeding on the same lines with necessary variations, we can prove the following:

**Theorem 5.3.2.** Let $R$ be a 2-torsion free prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ with associated derivation $d$ such that $[d(x), F(y)] + [x, y] = 0$, for all $x, y \in I$, then either $d = 0$ or $R$ is commutative.

**Theorem 5.3.3.** Let $R$ be a 2-torsion free prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ with associated derivation $d$ such that $[d(x), F(y)] = 0$, for all $x, y \in I$, then either $d = 0$ or $R$ is commutative.

**Proof.** We have

$$(5.3.4) \quad [d(x), F(y)] = 0, \text{ for all } x, y \in I.$$  

Replacing $y$ by $yz$ in $(5.3.4)$ and using $(5.3.4)$, we get

$$(5.3.5) \quad F(y)[d(x), z] + y[d(x), d(z)] + [d(x), y]d(z) = 0, \text{ for all } x, y, z \in I.$$  

Now replacing $z$ by $zd(x)$ in $(5.3.5)$ and using $(5.3.5)$, we obtain
(5.3.6) \[ yz[d(x), d^2(x)] + y[d(x), z]d^2(x) + [d(x), y]zd^2(x) = 0, \] for all \( x, y, z \in I. \]

Again replace \( y \) by \( ry \) in (5.3.6) and use (5.3.6), to get \( [d(x), r]yzd^2(x) = 0, \) for all \( x, y, z \in I \) and \( r \in R \) i.e., \([d(x), r]IRd^2(x) = \{0\},\) for all \( x \in I, \ r \in R. \) Now using the similar arguments as we have used in the proof of Theorem 5.3.1, we get the required result. \( \Box \)

**Theorem 5.3.4.** Let \( R \) be a 2-torsion free prime ring and \( I \) be a nonzero ideal of \( R. \) If \( R \) admits a generalized derivation \( F \) with associated derivation \( d \) such that \( d(x) \circ F(y) = 0, \) for all \( x, y \in I, \) then either \( d = 0 \) or \( R \) is commutative.

**Proof.** We have \( d(x) \circ F(y) = 0, \) for all \( x, y \in I. \) Replacing \( y \) by \( yr, \) we get \( d(x) \circ F(yr) = 0, \) for all \( x, y \in I \) and \( r \in R \) and hence we find that

\[
(d(x) \circ y)d(r) - y[d(x), d(r)] + (d(x) \circ F(y))r - F(y)[d(x), r] = 0.
\]

Now using our hypotheses, the above relation yields that

\[
(d(x) \circ y)d(r) - y[d(x), d(r)] - F(y)[d(x), r] = 0, \quad \text{for all } x, y \in I \text{ and } r \in R.
\]

Again replace \( r \) by \( d(x), \) to get

(5.3.7) \( (d(x) \circ y)d^2(x) - y[d(x), d^2(x)] = 0, \) for all \( x, y \in I \) and \( r \in R. \)

Now replacing \( y \) by \( zy \) in (5.3.7), we obtain

\[
(d(x) \circ zy)d^2(x) - zy[d(x), d^2(x)] = 0, \quad \text{for all } x, y, z \in I.
\]

This implies that

\[
z(d(x) \circ y)d^2(x) + [d(x), z]yd^2(x) - zy[d(x), d^2(x)] = 0, \quad \text{for all } x, y, z \in I.
\]
In view of (5.3.7), the above expression yields that \([d(x), z]yd^2(x) = 0\), for all \(x, y, z \in I\) and hence \([d(x), z]IRd^2(x) = \{0\}\), for all \(x, z \in I\). Further, application of similar arguments as used in the end of the proof of Theorem 5.3.1, we get the required result. □

**Theorem 5.3.5.** Let \(R\) be a 2-torsion free prime ring and \(I\) be a nonzero ideal of \(R\). If \(R\) admits a generalized derivation \(F\) with associated derivation \(d\) such that \(d(x) \circ F(y) = x \circ y\), for all \(x, y \in I\), then either \(d = 0\) or \(R\) is commutative.

**Proof.** Given that \(d(x) \circ F(y) = x \circ y\), for all \(x, y \in I\). If \(F = 0\), then \(x \circ y = 0\), for all \(x, y \in I\). Replacing \(y\) by \(yz\) and using the fact that \(x \circ y = 0\), we obtain \(y[x, z] = 0\), for all \(x, z \in I\). In particular, \(IR[x, z] = \{0\}\), for all \(x, z \in I\) and the primeness of \(R\) forces that \([x, z] = 0\), for all \(x, z \in I\). Hence \(R\) is commutative by Proposition 5.2.2.

Therefore, now onward we assume that \(F \neq 0\). For any \(x, y \in I\), we have \(d(x) \circ F(y) = x \circ y\). Replacing \(y\) by \(yr\), we get

\[
(d(x) \circ y)d(r) - y[d(x), d(r)] + (d(x) \circ F(y))r - F(y)[d(x), r] = (x \circ y)r - y[x, r].
\]

Using our hypotheses, we get

\[
(d(x) \circ y)d(r) - y[d(x), d(r)] - F(y)[d(x), r] + y[x, r] = 0.
\]

In the above expression replacing \(r\) by \(d(x)\), we obtain

\[
(5.3.8) \quad (d(x) \circ y)d^2(x) - y[d(x), d^2(x)] + y[x, d(x)] = 0, \text{ for all } x, y \in I.
\]

Now replacing \(y\) by \(zy\) in (4.2.8), we get

\[
(z(d(x) \circ y) + [d(x), z]yd^2(x) - zy[d(x), d^2(x)] + zy[x, d(x)] = 0.
\]
In view of (5.3.8), we find that \([d(x),z]yd^2(x) = 0\), for all \(x,y,z \in I\) and hence \([d(x),z]IRd^2(x) = \{0\}\), for all \(x,z \in I\). Thus primeness of \(R\) forces that either \(d^2(x) = 0\) or \([d(x),z]I = \{0\}\). Arguing in the similar manner as we have done in the proof of Theorem 5.3.1, we find the required result. □

Using the similar arguments as we have used in the above theorem, we can prove the following which generalizes Theorem 4.5 of [13].

**Theorem 5.3.6.** Let \(R\) be a 2-torsion free prime ring and \(I\) be a nonzero ideal of \(R\). If \(R\) admits a generalized derivation \(F\) with associated derivation \(d\) such that \(d(x) \circ F(y) + x \circ y = 0\), for all \(x,y \in I\), then either \(d = 0\) or \(R\) is commutative.

**Theorem 5.3.7.** Let \(R\) be a prime ring and \(I\) be a nonzero ideal of \(R\). If \(R\) admits a generalized derivation \(F\) with associated derivation \(d\) such that \(d(x)F(y) - xy \in Z(R)\), for all \(x,y \in I\), then either \(d = 0\) or \(R\) is commutative.

**Proof.** For any \(x,y \in I\), we have \(d(x)F(y) - xy \in Z(R)\). If \(F = 0\), then \(xy \in Z(R)\), for all \(x,y \in I\). In particular, \([xy,x] = 0\), for all \(x,y \in I\) and hence \(x[y,x] = 0\). Replacing \(y\) by \(yz\), we get \(xy[z,x] = 0\), for all \(x,y,z \in I\). Further, replace \(y\) by \(ry\), to get \(xry[z,x] = 0\), for all \(x,y,z \in I\), \(r \in R\) and hence \(xRI[z,x] = \{0\}\). Thus primeness of \(R\) forces that for each \(x \in I\), either \(x = 0\) or \(I[z,x] = \{0\}\). But \(x = 0\) also implies that \(I[z,x] = \{0\}\) and hence \(IR[z,x] = \{0\}\), for all \(x,z \in I\). Since \(I \neq \{0\}\) and \(R\) is prime, the above relation yields that \([z,x] = 0\), for all \(x,z \in I\). Hence by Proposition 5.2.2, \(R\) is commutative.

Now, we assume that \(F \neq 0\). For any \(x,y \in I\), we have
Replacing \( y \) by \( yr \), we get \( d(x)F(yr) - xyr \in Z(R) \), for all \( x, y \in I \) and \( r \in R \) i.e.,

\[
(d(x)F(y) - xy)r + d(x)yr \in Z(R), \quad \text{for all } x, y \in I \text{ and } r \in R.
\]

This implies that

\[
[d(x)yr, r] = 0, \quad \text{for all } x, y \in I \text{ and } r \in R.
\]

Hence it follows that

\[
d(x)[yd(r), r] + [d(x), r]yd(r) = 0, \quad \text{for all } x, y \in I \text{ and } r \in R.
\]

Now replacing \( y \) by \( d(x)y \), we get \([d(x), r]d(x)yd(r) = 0\), for all \( x, y \in I \) and \( r \in R \). This implies that \([d(x), r]d(x)r_1yd(r) = 0\), for all \( x, y \in I \), \( r, r_1 \in R \) and hence \([d(x), r]d(x)RId(r) = \{0\}\), for all \( x \in I \), \( r \in R \). Thus for each \( r \in R \), primeness of \( R \) forces that either \([d(x), r]d(x) = 0\) or \( Id(r) = \{0\}\). Now if \( A = \{r \in R \mid [d(x), r]d(x) = 0\}, \) \( B = \{r \in R \mid Id(r) = \{0\}\} \), then by using Braur’s trick, we find that either \([d(x), r]d(x) = 0\) or \( Id(r) = \{0\}\). If \( Id(r) = \{0\}\), for all \( r \in R \), then \( IRd(r) = \{0\}, \) for all \( r \in R \). Since \( R \) is prime and \( I \neq \{0\} \), the above relation yields that \( d = 0 \). On the other hand, assume the remaining possibility that \([d(x), r]d(x) = 0\), for all \( x \in I \) and \( r \in R \). For any \( s \in R \), replace \( r \) by \( rs \), to get \([d(x), r]sd(x) = 0\), for all \( x \in I \), \( r \in R \) and hence \([d(x), r]Rd(x) = \{0\}\), for all \( x \in I \), \( r \in R \). The primeness of \( R \) implies that for each \( x \in I \), either \( d(x) = 0 \) or \([d(x), r] = 0\). But \( d(x) = 0 \) also implies that \([d(x), r] = 0\). Hence \([d(x), r] = 0\), for all \( x \in I \), \( r \in R \) and by Lemma 5.3.1, \( R \) is commutative. \( \Box \)

Proceeding on the same lines with necessary variations, we can prove the following theorem which includes Theorem 2.5 of [14].

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**Theorem 5.3.8.** Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ with associated derivation $d$ such that $d(x)F(y) + xy \in Z(R)$, for all $x, y \in I$, then either $d = 0$ or $R$ is commutative.

**Theorem 5.3.9.** Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. Then the followings are equivalent:

(i) $R$ is 2-torsion free and $R$ admits a generalized derivation $F$ with associated derivation $d \neq 0$ such that $[d(x), F(y)] - [x, y] = 0$ or $[d(x), F(y)] + [x, y] = 0$, for all $x, y \in I$.

(ii) $R$ admits a generalized derivation $F$ with associated derivation $d \neq 0$ such that $d(x)F(y) - xy \in Z(R)$ or $d(x)F(y) + xy \in Z(R)$, for all $x, y \in I$.

(iii) $R$ is commutative.

**Proof.** Obviously, $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$.

$(i) \Rightarrow (iii)$. For each fixed $x \in I$, we put $I_1 = \{y \in I \mid [d(x), F(y)] - [x, y] = 0\}$ and $I_2 = \{y \in I \mid [d(x), F(y)] + [x, y] = 0\}$. Then it can be easily seen that $I_1$ and $I_2$ both are additive subgroups of $I$ whose union is $I$. Then by Braur's trick, either $I_1 = I$ or $I_2 = I$. Further using similar arguments as above, we find that $I = \{x \in I \mid I_1 = I\}$ or $I = \{x \in I \mid I_2 = I\}$. Therefore, $R$ is commutative by Theorems 5.3.1 and 5.3.2.

$(ii) \Rightarrow (iii)$. For each fixed $x \in I$, we put $I_1 = \{y \in I \mid d(x)F(y) - xy \in Z(R)\}$ and $I_2 = \{y \in I \mid d(x)F(y) + xy \in Z(R)\}$. Using the similar arguments as above and using Theorems 5.3.7 and 5.3.8, we get the required result.
Remark 5.3.3. The following example demonstrates that the above results are not true in case of arbitrary rings.

Example 5.3.6. Consider $S$ as any ring. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$ and let $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S \right\}$ be an ideal of $R$. Define $F : R \rightarrow R$ by $F(x) = 2e_{11}x - xe_{11}$, where $e_{ij}$ denotes an identity matrix. Then $F$ is a generalized derivation with associated derivation $d$ given by $d(x) = e_{11}x - xe_{11}$.

It can be easily seen that $R$ satisfies the properties: (i) $d(x) \circ F(y) = 0$, (ii) $[d(x), F(y)] = 0$, (iii) $d(x) \circ F(y) = x \circ y$, (iv) $d(x) \circ F(y) + x \circ y = 0$, (v) $d(x)F(y) - xy \in Z(R)$, (vi) $d(x)F(y) + xy \in Z(R)$, (vii) $[d(x), F(y)] = [x, y]$ and (viii) $[d(x), F(y)] + [x, y] = 0$, for all $x, y \in I$. However, $R$ is not commutative.

5.4 LIE IDEALS AND GENERALIZED DERIVATIONS IN PRIME RINGS

In [14], Ashraf and Nadeem established that a prime ring $R$ with a nonzero ideal $I$ must be commutative, if it admits a derivation $d$ satisfying either of the properties: (i) $d(xy) + xy \in Z(R)$ and (ii) $d(xy) - xy \in Z(R)$, for all $x, y \in I$.

Inspired by this result, we have proved the following:

Theorem 5.4.10. Let $R$ be a $2$-torsion free prime ring and $U$ be a nonzero Lie ideal of $R$ with $u^2 \in U$, for all $u \in U$. If $R$ admits a generalized derivation $F$ with associated derivation $d \neq 0$ such that $F(uv) - uv \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$. 

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For developing the proof of the above theorem, we require the following lemmas which are essentially proved in [45].

**Lemma 5.4.2.** Let $R$ be a 2-torsion free prime ring and $U$ be a Lie ideal of $R$. If $U \not\subseteq Z(R)$, then $C_R(U) = Z(R)$.

**Lemma 5.4.3.** If $U \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring $R$ and $a, b \in R$ such that $aUb = \{0\}$, then $a = 0$ or $b = 0$.

The following lemma is in fact, an extension of a result [104, Lemma 2(a)] due to J. H. Mayne.

**Lemma 5.4.4.** Let $R$ be a 2-torsion free prime ring and $U$ be a Lie ideal of $R$ such that $U \not\subseteq Z(R)$. If $R$ admits a derivation $d$ which is zero on $U$, then $d$ is zero on $R$.

**Proof.** By our hypotheses, we have

\[(5.4.1) \quad d(u) = 0, \text{ for all } u \in U.\]

Replacing $u$ by $[u, r]$ in (5.4.1), we find that $d([u, r]) = ud(r) - d(r)u = 0$ and hence $[u, d(r)] = 0$, for all $u \in U, r \in R$. This yields that $d(r) \in C_R(U)$. Thus, the application of Lemma 5.4.2 gives $d(r) \in Z(R)$. Hence $[d(r), s] = 0$, for all $r, s \in R$. Replacing $r$ by $rr_1$ in the latter relation and using it, we obtain $d(r)[r_1, s] + [r, s]d(r_1) = 0$, for all $r, r_1, s \in R$. Now replace $r_1$ by $d(r)$, to get $[r, s]d^2(r) = 0$, for all $r, s \in R$. Again replacing $s$ by $us$, we find that $[r, u]sd^2(r) = 0$, for all $u \in U$ and $r, s \in R$ i.e., $[r, u]Rd^2(r) = \{0\}$, for all $u \in U, r \in R$. Thus primeness of $R$ implies that either $[r, u] = 0$ or $d^2(r) = 0$.  

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Since $U \not\subset Z(R)$, we have $d^2(r) = 0$, for all $r \in R$. Replace $r$ by $rs$, in the above relation, to get $2d(r)d(s) = 0$, for all $r, s \in R$. Since $R$ is 2-torsion free, the latter relation yields that $d(r)d(s) = 0$, for all $r, s \in R$. We conclude that $d(r)d(sr) = \{0\}$, for all $r, s \in R$. Thus $d(r)d(s) = \{0\}$, for all $r \in R$. The primeness of $R$ forces that $d = 0$. □

**Proof of Theorem 5.4.10.** If $F = 0$, then $uv \in Z(R)$, for all $u, v \in U$. Hence $[uv, r] = 0$, for all $u, v \in U$ and $r \in R$. This gives that $u[v, r] + [u, r]v = 0$, for all $u, v \in U$. Replacing $u$ by $2wu$ and using the fact that characteristic $R = 2$, we get $[w, r]uv = 0$, for all $u, v, w \in U$ and $r \in R$. Replace $r$ by $rs$, to get $[w, r]suw = 0$, for all $u, v, w \in U$ and $r, s \in R$ i.e., $[w, r]Ruw = \{0\}$, for all $u, v, w \in U$, $r \in R$. Thus primeness of $R$ implies that either $[w, r] = 0$ or $uv = 0$. If $uv = 0$, for all $u, v \in U$, then replacing $v$ by $[v, r]$, we get $uv = 0$, for all $u, v \in U$ and $r \in R$. Hence $uUv = \{0\}$, for all $u, v \in U$. Thus primeness of $R$ forces that $U = \{0\}$, which is not possible. Hence we have $[w, r] = 0$, for all $w \in U$ and $r \in R$ i.e., $U \subset Z(R)$.

Hence onward, we assume that $F \neq 0$. Suppose on contrary that $U \not\subset Z(R)$. Since we have $F(uv) - uv \in Z(R)$, for all $u, v \in U$, $[F(uv) - uv, w] = 0$, for all $u, v, w \in U$. Replacing $v$ by $2vw$ and using the fact that characteristic $R \neq 2$, we get $[(F(uv) - uv)w + uvd(w), w] = 0$, for all $u, v, w \in U$. Hence $[uvd(w), w] = 0$, for all $u, v, w \in U$ i.e.,

$$ (5.4.2) \quad u[\{d(w), w\} + u[v, w]d(w) + [u, w]vd(w) = 0, \text{ for all } u, v, w \in U. $$

Replace $u$ by $2u_1u$ in (5.4.2) and use (5.4.2), to obtain $[u_1, w]uvd(w) = 0$, for all $u, u_1, v, w \in U$. Hence $[u_1, w]Uvd(w) = \{0\}$, for all $u_1, v, w \in U$. Thus by Lemma 5.4.3, for each $w \in U$ either $[u_1, w] = 0$ or $vd(w) = 0$. Now, let
$U_1 = \{ w \in U \mid v d(w) = 0, \text{ for all } v \in U \}$ and $U_2 = \{ w \in U \mid [u_1, w] = 0, \text{ for all } u_1 \in U \}$. Then $U_1$ and $U_2$ both are additive subgroups of $U$ and $U_1 \cup U_2 = U$.

Thus either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $vd(w) = 0$, for all $v, w \in U$.

Replacing $v$ by $[v, r]$ in the above relation and using it, we get $urd(w) = 0$, for all $v, w \in U$ and $r \in R$, i.e. $URd(w) = \{ 0 \}$, for all $w \in U$. Since $R$ is prime and $U$ is nonzero, we conclude that $d(w) = 0$, for all $w \in U$. Hence by Lemma 5.4.4, we get $d = 0$, a contradiction. On the other hand, if $U_2 = U$, then $[u_1, w] = 0$, for all $u_1, w \in U$. Thus by Proposition 5.2.4, we get $U \subseteq Z(R)$, again a contradiction. This completes the proof of the theorem. □

Using the similar arguments, we get the following:

**Theorem 5.4.11.** Let $R$ be a 2-torsion free prime ring and $U$ be a nonzero Lie ideal of $R$ with $u^2 \in U$, for all $u \in U$. If $R$ admits a generalized derivation $F$ with associated derivation $d \neq 0$ such that $F(uv) + uv \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.

Following is the immediate consequence of Theorem 5.4.10.

**Corollary 5.4.1.** Let $R$ be a prime ring. If $R$ admits a generalized derivation $F$ with associated derivation $d \neq 0$ such that $F(xy) - xy \in Z(R)$, for all $x, y \in R$, then $R$ is commutative.

**Remark 5.4.4.** Since every ideal in a ring $R$ is a Lie ideal of $R$, conclusion of the above theorem holds even if $U$ is assumed to be an ideal of $R$. Though the assumption that $u^2 \in U$, for all $u \in U$ seems close to assuming that $U$ is an ideal of the ring, but there exist Lie ideals with this property which are not ideals.
Example 5.4.7. Let \( R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\} \). Then it can be easily seen that \( U = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in \mathbb{Z} \right\} \) is a Lie ideal of \( R \) satisfying \( u^2 \in U \), for all \( u \in U \). However, \( U \) is not an ideal of \( R \).

Remark 5.4.5. In conclusion, it is tempting to conjecture as follows:

**Conjecture 5.4.1.** Let \( R \) be a 2-torsion free prime ring and \( U \) be a nonzero Lie ideal of \( R \). If \( R \) admits a generalized derivation \( F \) with associated derivation \( d \neq 0 \) such that \( F(uv) - uv \in Z(R) \) or \( F(uv) + uv \in Z(R) \), for all \( u, v \in U \), then \( U \subseteq Z(R) \).

**Theorem 5.4.12.** Let \( R \) be a 2-torsion free prime ring and \( U \) be a nonzero Lie ideal of \( R \) with \( u^2 \in U \), for all \( u \in U \). If \( R \) admits a generalized derivation \( F \) with associated derivation \( d \neq 0 \) such that \( F(uv) - vu \in Z(R) \), for all \( u, v \in U \), then \( U \subseteq Z(R) \).

**Proof.** If \( F = 0 \), then \( vu \in Z(R) \), for all \( u, v \in U \). Using the same arguments as we have used in the beginning of the proof of Theorem 5.4.10, we get the required result.

Hence, onward we assume that \( F \neq 0 \). Suppose on contrary that \( U \not\subseteq Z(R) \). Since for any \( u, v \in U \), we have \( F(uv) - vu \in Z(R) \), \( [F(uv) - vu, v] = 0 \), for all \( u, v \in U \). Replacing \( u \) by \( 2uv \) and using the fact that \( \text{char} R \neq 2 \), we get \( [(F(uv) - vu)v + uvd(v), v] = 0 \), for all \( u, v \in U \) and hence \( [uvd(v), v] = 0 \), for all \( u, v \in U \). We have

\[
(5.4.3) \quad uv[d(v), v] + [u, v]vd(v) = 0, \quad \text{for all } u, v \in U.
\]
Replace \( u \) by \( 2wu \) in (5.4.3) and use (5.4.3), to obtain \([w,v]ud(v) = 0\), for all \( u, v, w \in U \). Hence \([w,v]Ud(v) = \{0\}\), for all \( v, w \in U \). Thus by Lemma 5.4.3, either \([w,v] = 0\) or \( vd(v) = 0\). If \([w,v] = 0\), then by Proposition 5.2.4, we get \( U \subseteq Z(R)\), a contradiction. On the other hand, if \( vd(v) = 0\), then linearizing the above relation on \( v\), we obtain

\[
(5.4.4) \quad ud(v) + vd(u) = 0, \quad \text{for all } u, v \in U.
\]

Again replace \( v \) by \( 2vu \) in (5.4.4) and use the fact that \( \text{char}R \neq 2 \), to get \( ud(vu) + vu d(u) = 0 \), for all \( u, v \in U \). Thus (5.4.4) yields that \([u, vd(u)] = 0\), for all \( u, v \in U \). This gives that \( v[u,d(u)] + [u,v]d(u) = 0\), for all \( u, v \in U \).

Replacing \( v \) by \( 2vu \), we get \([u,w]vd(u) = 0\), for all \( u, v, w \in U \) i.e., \([u,w]Ud(u) = \{0\}\), for all \( u, w \in U \). Hence for each fixed \( u \in U \), by Lemma 5.4.3 either \([u,w] = 0\) or \( d(u) = 0\). Now, let \( U_1 = \{u \in U \mid d(u) = 0\}\) and \( U_2 = \{w \in U \mid [u,w] = 0\}\). Then \( U_1 \) and \( U_2 \) both are additive subgroups of \( U \) and \( U_1 \cup U_2 = U \). Thus either \( U_1 = U \) or \( U_2 = U \). If \( U_1 = U \), then \( d(u) = 0\), for all \( u \in U \) and by Lemma 5.4.4, we get \( d = 0\), a contradiction. On the other hand, if \( U_2 = U \), then \([u,w] = 0\), for all \( u, w \in U \). Thus by Proposition 5.2.4, \( U \subseteq Z(R)\), again a contradiction. Hence the result is proved. \( \Box \)

Using the same techniques with necessary variations we get the following:

**Theorem 5.4.13.** Let \( R \) be a 2-torsion free prime ring and \( U \) be a nonzero Lie ideal of \( R \) with \( u^2 \in U \), for all \( u \in U \). If \( R \) admits a generalized derivation \( F \) with associated derivation \( d \neq 0 \) such that \( F(uv) + vu \in Z(R)\), for all \( u, v \in U \), then \( U \subseteq Z(R) \).
Remark 5.4.6. The Example 5.3.6 demonstrates that \( R \) to be prime is essential in the hypotheses of the above theorems.