CHAPTER - 2
COMMUTATIVITY OF CERTAIN RINGS 
AND NEAR RINGS

2.1. INTRODUCTION

During the second half of the last century, a number of commutativity theorems were obtained by mathematicians like Bell, Jacobson, McCoy, Kaplansky, Faith and Herstein which sparked off a great interest among a lot of algebraists and numerous research papers have started pouring in the mathematical literature concerning the investigations of the classes of rings which turn out to be commutative under some constraints, mostly satisfying certain polynomial identities. The present chapter includes the same type of work.

In Section 2.1 we extend a theorem of Herstein [81] which in turn generalizes a theorem due to Jacobson [96]. The theorem under reference states that a ring must be commutative if there exists a positive integer \( n > 1 \) such that \( x^n = x \), for all ring elements \( x \). In fact, we prove our result for semiprime rings which is further extended in the next section for rings with unity. Section 2.3 deals with the commutativity of rings satisfying some rather more complicated identities defined in terms of the expansions what is called words by Putcha and Yaqub [120]. In the last section we discuss the commutativity of a special class of near rings what we call distributively generated abbreviated as d-g near rings.

2.2 COMMUTATIVITY OF SEMIPRIME RINGS

Long ago, Herstein [81] extended a well known theorem of Jacobson [96] as follows:

**Theorem H.** If in a ring, there exists an integer \( n > 1 \) such that \( x^n - x \) is central, for all ring elements \( x \), then such a ring must be commutative.
In the above quoted paper, Herstein conjectured certain possible extensions of Theorem H, some of which he himself proved later.

One of the natural questions arises as to what we can say if the constraint $x^n - x$, central, is replaced by $(xy)^n - xy$, central for every pair of ring elements $x$ and $y$. The existence of enough non-commutative rings with $(xy)^n - xy$, central rules out the possibility of generalizing Theorem H in this direction. The following example justifies our observation.

**Example 2.2.1.** Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ be the ring of strictly upper triangular matrices over $\mathbb{Z}$, the ring of integers. Then it is easy to note that $R$ is a non-commutative nil ring of index 3 satisfying the property $(xy)^2 - xy \in Z(R)$, the centre of $R$.

However, the result holds for some restricted classes of rings. We state the following theorem without supplying details of its proof. Just substitute $x^{-1}y$ for $y$ in the identity for any $0 \neq x \in R$ and use Theorem H given above.

**Theorem 2.2.1.** Let $D$ be a division ring in which there exist a positive integer $n > 1$ such that $(xy)^n - xy \in Z(D)$, the centre of $D$. Then $D$ must be a field.

In order to extend the mentioned result for some wider classes of rings, we begin with the following lemma:

**Lemma 2.2.1.** Let $R$ be a prime ring satisfying the identity

$$(P) : \quad (xy)^n - xy \in Z(R), \quad \text{for a fixed positive integer } n > 1.$$ 

Then $R$ has no nonzero nilpotent elements.
Proof. Suppose $a \in R$ such that $a^2 = 0$. Put $ax$ for $x$ and $ya$ for $y$ in (P) to get

$$(axya)^n - axya \in Z(R).$$

In particular,

$$(axya)^n - axya, ax] = 0, \text{ for all } x, y \in R$$

This on expanding and using $a^2 = 0$, yields

$$ax(axya)^n - axaxy = 0$$

i.e., $axaxya = 0$, for all $x, y \in R$ \hfill (2.2.1)

Let $r \in R$ be an arbitrary element of $R$. Replacing $y$ by $y + r$ in (2.2.1), we have

$$axax(y + r)a = 0$$

Using (2.2.1), we get

$$axaxra = 0$$

i.e., $(ax)^2Ra = (0)$. Since $R$ is prime, $(ax)^2 = 0$ or $a = 0$. If $a \neq 0$, then $aR$ is a nil right ideal of bounded index 2, which is not possible by Theorem 1.4.6 and so $R$ has no nonzero nilpotent elements.

Notice that our ring $R$ is zero-commutative. Indeed, if for any $a, b \in R$ with $ab = 0$, then $(ba)^2 = b(ab)a = 0$ and so in view of Lemma 2.2.1, we have $ba = 0$.

Now let $ab = 0$ for some $a, b \in R$. So that $ba = 0$ and $bax = 0 = b(ax) = axb$, for an arbitrary $x \in R$, $bax = b(ax) = axb = 0$. Thus $aRb = (0)$ for some $a, b \in R$. But primeness of $R$ forces either $a = 0$ or $b = 0$. This proves the following:

Lemma 2.2.2. Let $R$ be a prime ring satisfying condition (P). Then $R$ has no nonzero divisors of zero.
Theorem 2.2.1. Let $R$ be a prime ring satisfying the property $(P)$. Then $R$ is necessarily commutative.

Proof. Consider the subring $S = Ry$ for a fixed $y \in R$. Then in view of Lemma 2.2.2, $S$ is also prime such that there exists an integer $n > 1$ satisfying $s^n - s \in Z(S)$, the centre of $S$, for all $s \in S$.

Thus by Theorem H given above, $S$ is commutative and $y^2 \in S$ commute with every element of $S = Ry$ and obviously for any $x \in R$ i.e., $xy \in S$,

$$xyy^2 = y^2xy$$ (2.2.2)

For $y \neq 0$, cancelation of $y$ on the right is permissible in view of Lemma 2.2.2 and the fact that $Ry$ is a subring of $R$, yielding that $xy^2 = y^2x$, for all $x \in R$. This gives that $y^2 \in Z(R)$, centre of $R$. But it is well known that a prime ring with square of every element central is commutative. This proves our theorem.

Further suppose that $R$ is a semiprime ring satisfying $(P)$. By structure theory, a semiprime ring is isomorphic to a subdirect sum of prime rings $R_i$, each of which as a homomorphic image of $R$ satisfies the hypothesis placed on $R$. Thus by the above theorem each of $R_i$ is commutative forcing that $R$ is commutative. Thus we have established the following:

Theorem 2.2.2. Let $R$ be a semiprime ring satisfying $(P)$. Then $R$ must be commutative.

2.3 Commutativity of rings with unity 1

If we revisit to Example 2.2.1, we note that the mentioned non-commutative ring $R$ does not contain unity. One may therefore hope that property $(P)$ could yield commutativity in rings with unity. We have succeeded in establishing rather
a stronger result with weaker hypothesis.

**Theorem 2.3.1.** Let $R$ be a ring with unity 1. If there exists an integer $n > 1$ such that

$$(P^*) : \quad [(xy)^n - xy, x] = 0, \quad \text{for all } x, y \in R,$$

then $R$ must be commutative.

For development of the proof of our theorem, we require the following lemmas proved by Bell:

**Lemma 2.3.1.** ([31]). Let $R$ be a ring satisfying an identity $q[x] = 0$, where $q[x]$ is a polynomial in a finite number of non-commuting indeterminates with relatively prime integers as its coefficients. If there exists no prime $p$ for which the ring of $2 \times 2$ matrices over $GF(p)$ satisfies $q[x] = 0$, then $R$ has nil commutator ideal $C^*(R)$ and set $N(R)$ of nilpotent elements of $R$ forms an ideal.

Since the choice of $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ fails to satisfy the polynomial identity $(P^*)$, we get the following in view of the above lemma.

**Lemma 2.3.2.** ([31]). In a ring $R$ with unity 1 satisfying the property $(P^*)$ the commutator ideal $C(R)$ is nil and totality $N(R)$ of all nilpotent elements of $R$ is an ideal of $R$.

**Lemma 2.3.3.** In the ring $R$ of Theorem 2.3.1, $N(R) \subseteq Z(R)$.

**Proof.** Let $u \in N(R)$. So $(1 + u)$ is invertible. Substituting $x = 1 + u$ and $y = (1 + u)^{-1} y$ in $(P^*)$, we get $[y^n - y, 1 + u] = 0 = [y^2 - y, u]$, for all $y \in R$. It follows above that $N(R)$ is commutative and hence $N^2(R) \subseteq Z(R)$. Thus for any $x \in R$,
\[ [xu, x] = 0 = x[u, x]. \]  
(2.3.1)

Put \( x + 1 \) for \( x \) in (2.3.1) and use (2.3.1) to get

\[ [x, u] = 0, \quad \text{for all } x \in R. \]

Hence \( u \in Z(R) \) so that \( N(R) \subseteq Z(R) \).

**Corollary 2.3.1.** In view of Lemma 2.3.2, we have \( C(R) \subseteq Z(R) \).

The following result is essentially proved in [95].

**Lemma 2.3.4.** Let \( x, y \in R \) and \( [x, y] \) commute with \( x \). Then for all positive integers \( m \geq 1 \), \( [x^m, y] = mx^{m-1}[x, y] \).

**Proof of Theorem 2.3.1** Our identity \( (P^*) \) can also be written as

\[ [(xy)^n, x] = [xy, x], \quad \text{for all } x, y \in R. \]  
(2.3.2)

Substituting \( ky \) for \( y \) in (2.3.2), for an arbitrary integer \( k > 1 \), we get

\[ k^n[(xy)^n, x] = k[xy, x] \quad \text{i.e.,} \]

\[ (k^n - k)[xy, x] = 0, \quad \text{for all } x, y \in R. \]  
(2.3.3)

Thus substituting \( 1 + x \) for \( x \) in (2.3.3), we get \( (k^n - k)((x + 1)y, x + 1) = 0 \) i.e.,

\( (k^n - k)[xy + y, x] = 0 \) i.e., \( (k^n - k)[xy, x] + (k^n - k)[y, x] = 0 \), for all \( x, y \in R \). This together with (2.3.3), yields

\[ (k^n - k)[x, y] = 0, \quad \text{for all } x, y \in R. \]  
(2.3.4)

Case I. If the additive group \( < R, + > \) is torsion free, then we are through.

Case II. Let \( < R, + > \) be not torsion free.
Assume that $R$ is subdirectly irreducible and there is a unique prime $p$ for which $R$ is $p$-torsion. It follows from (2.3.4) that $p[x,y] = 0$ also by the Corollary 2.3.1, $C(R) \subseteq Z(R)$ and invoking Lemma 2.3.4, we have for all $x, y \in R$, $[x^p, y] = px^{p-1}[x, y] = 0$. Thus,

$$x^p \in Z(R), \text{ for all } x \in R. \quad (2.3.5)$$

At this point we note that

(a) In any ring satisfying (2.3.5), there is no distinction between left zero divisors and right zero divisors and the set $D$ of zero divisors has the property $RD \subseteq D$ and $DR \subseteq D$.

(b) If $R$ is a subdirectly irreducible with heart $H$, then any central zero divisor annihilates $H$.

(c) If $R$ is any ring with $z \in Z(R)$, then the set $I(z) = \{x \in R \mid xz = x\}$ is an ideal.

Now by our property $(P^*)$, we get

$$n(xy)^{n-1}[xy, x] = [xy, x], \text{ for all } x, y \in R. \quad (2.3.6)$$

Multiplying (2.3.6), $p$-times by $n(xy)^{n-1}$, we get

$$n^p(xy)^{p(n-1)}[xy, x] = [xy, x], \text{ for all } x, y \in R. \quad (2.3.7)$$

Let $y \in D$ and $x \in R$. Note that $n^p(xy)^{p(n-1)} \in Z(R)$, so that using (2.3.7), we get $[xy, x] \in I(n^p(xy)^{p(n-1)})$. Set $T = I(n^p(xy)^{p(n-1)}) \neq 0$. Then $H \subseteq T$. So if $w$ is a nonzero element of $H$, we have $n^p((xy)^{p(n-1)}w) = w$. However $n^p(xy)^{p(n-1)}$ is a central zero divisor by (2.3.5) and (a). Hence by (b) $n^p(xy)^{p(n-1)}$ must annihilate $H$. Then $T = (0)$ and hence for a fixed $y \in D$, we have $[xy, x] = 0 = x[x, y]$, for all $x, y \in R$. Again putting $x = x + 1$, we have $D \subseteq Z(R)$. Now let $z \in Z(R)$ be an arbitrary central element. Substituting $zy$ for $y$ in our identity $(P^*)$ and proceeding as above in getting (2.3.4), we find

$$(z^n - z)[x, y] = 0, \text{ for all } x, y \in R.$$
If \([x, y] \neq 0\), then \((z^n - z)\) is a zero divisor for all \(z \in Z(R)\). Thus for \(x \not\in D\), we get \(x^{pn} - x^p \in D\). Consequently \(x^{m-(p-1)} - x \in D \subseteq Z(R)\), for all \(x \in R\). Thus in any case \(x^{m-(p-1)} - x^p \in Z(R)\), for all \(x \in R\). Hence by Theorem H, \(R\) is commutative and this completes the proof of our theorem.

### 2.4 Commutativity of Periodic Rings

**Definition 2.4.1 (Periodic Ring).** A ring \(R\) is said to be periodic if for every \(x \in R\), there exist two distinct positive integers \(m\) and \(n\) depending on the element \(x \in R\) such that \(x^m = x^n\).

A sufficient condition for \(R\) to be periodic is Chacron’s Criterion [59]: For each \(x \in R\), there exists an integer \(m = m(x) \geq 1\) and a polynomial \(f(x) \in \mathbb{Z}[x]\), the ring of polynomials in \(x\) with integer coefficients such that \(x^m = x^{m+1} f(x)\).

Boolean rings (satisfying \(x = x^2\)) and \(J\)-rings (satisfying \(x = x^{n(x)}\)) are particular cases of periodic rings and it is well known that these both types of periodic rings are commutative. But there exist enough periodic rings (even with unity) which are not commutative.

**Example 2.4.1.** Consider the ring \(R\) of \(2 \times 2\) matrices over \(GF(2)\). Then it can be easily verified that every element of \(R\) satisfies the identity \(x^2 = x^8\). However, \(R\) is not commutative.

During the past few decades many researchers have attempted to obtain the conditions which turn periodic rings commutative or near commutative in the sense that commutator ideal is nil (see [5], [6], [32], [34], [36], [123] where one can find more references). In this section we too shall investigate the commutativity of such rings. In order to be able to state and establish main result of this
section, we pause to discuss some notions borrowed from Putcha and Yaqub [120] and Bell [33]. Suppose \( X_1, X_2, \ldots, X_n \) shall be \( n \) elements of \( R \). By word \( w(X_1, X_2, \ldots, X_n) \) we shall mean a product in which each factor is \( X_i \) for some \( i = 1, 2, 3, \ldots n \). A polynomial \( f(X_1, X_2, \ldots, X_n) \) is then an expression of the form 
\[ c_1 w_1(X_1, X_2, \ldots, X_n) + c_2 w_2(X_1, X_2, \ldots, X_n) + \ldots + c_n w_n(X_1, X_2, \ldots, X_n) \]
where \( c_i \)'s are integers. The degree of \( X_i \) in the word \( w(X_1, X_2, \ldots, X_n) \) is the number of times \( X_i \) appears as a factor in \( w(X_1, X_2, \ldots, X_n) \) and degree of \( X_i \) in the polynomial 
\[ f(X_1, X_2, \ldots, X_n) = c_1 w_1(X_1, X_2, \ldots, X_n) + c_2 w_2(X_1, X_2, \ldots, X_n) + \ldots + c_n w_n(X_1, X_2, \ldots, X_n) \]
is the smallest value among the degrees of \( X_i \) in \( w_1(X_1, X_2, \ldots, X_n), w_2(X_1, X_2, \ldots, X_n), \ldots w_n(X_1, X_2, \ldots, X_n) \).

Let \( \mathbb{Z} < X, Y > \) denote the ring of polynomials with integer coefficients in two non-commuting indeterminates \( X \) and \( Y \). Thus the symbol \( w(X, Y) \) will denote a word in \( X \) and \( Y \) i.e., an element of \( \mathbb{Z} < X, Y > \) of the form 
\[ Y^{j_1} X^{k_1} Y^{j_2} X^{k_2} \ldots Y^{j_s} X^{k_s} \]
where \( j_i \) and \( k_i \) are nonnegative integers such that \( \sum_{i=1}^{s} j_i + k_i > 0 \) and the symbols \( |w|_x \) and \( |w|_y \) will denote \( \sum_{i=1}^{s} k_i \) and \( \sum_{i=1}^{s} j_i \) respectively. We shall call \( P(X, Y) \in \mathbb{Z} < X, Y > \) an admissible polynomial if 
\[ P(X, Y) = \sum_{i=1}^{k} c_i w_i(X, Y), \]
where each \( c_i \) is an integer and each \( w_i(X, Y) \) is a word with \( |w_i|_x \geq 2 \) and \( |w_i|_y \geq 2 \).

In the mentioned papers commutativity and structures of rings satisfying identities involving polynomials in two or more indeterminates are studied. We continue the study by considering the rings satisfying 

\((*) : \quad xy = P(x, y)\)

where \( P(x, y) \) is an admissible polynomial in \( \mathbb{Z} < X, Y > \).
The following lemma is due to Bell.

**Lemma 2.4.1 ([33]).** If $R$ is a periodic ring with all nilpotent elements central, then $R$ is commutative.

**Theorem 2.4.1.** If $R$ is a ring satisfying condition $(\ast)$, then $R$ must be commutative.

**Proof.** Notice that $R$ satisfying $(\ast)$ is zero-commutative. Indeed if $xy = 0$, then by condition $(\ast)$, $yx = P(y, x) = 0$, for all $x, y \in R$, where $P(X, Y) = \sum_{i=1}^{\ell} n_i w_i(X, Y)$ as mentioned above. By taking $y = x$ in condition $(\ast)$, we see that $x^2 = P(x, x)$, where $P(X, Y) = \sum_{i=1}^{\ell} n_i w_i(X, Y)$. Thus $R$ is periodic satisfying Chacron’s condition for periodicity and moreover $u^2 = 0$, for all $u \in N(R)$. Let $x \in R$, then $(ux)u = P(ux, u)$, where $P(X, Y) = \sum_{i=1}^{\ell} n_i w_i(X, Y)$. If $Y$ precedes an $X$ in $w_i(X, Y)$, then clearly $w_i(ux, u) = 0$; Otherwise $w_i(X, Y) = X^j Y^k$ with $j, k \geq 2$ and again $w_i(X, Y) = 0$. Hence

\[(ux)u = 0, \text{ for all } u \in N(R) \quad x \in R.\]  

(2.4.1)

Now using (2.4.1) we find that $xu = P(x, u) = 0$, where $P(X, Y) = \sum_{i=1}^{\ell} n_i w_i(X, Y)$. Since $R$ is zero-commutative, $ux = 0$, for all $x \in R$. Hence $RN(R) = N(R)R = (0)$ and $N(R) \subseteq Z(R)$. Since $R$ is periodic and $N(R)$ is central, by Lemma 2.4.1, $R$ is commutative.

**2.5 Commutativity of Distributively Generated 
(d - g) Near Rings**

Although the classical structure theory of near rings runs word by word parallel to that of general rings, one should not jump off to the conclusion that all the near rings analogies of ring theoretic results can be easily obtained. Many of them do not quite hold either. As an example, it is well known that Boolean rings are
necessarily commutative, but there are numerous examples of near rings satisfying Boolean condition $x^2 = x$ which are badly non-commutative.

**Example 2.5.1.** Let $R = \{0, a\}$ with addition “$+$” and multiplication “$*$” defined as follows:

\[
\begin{array}{c|cc}
+ & 0 & a \\
0 & 0 & a \\
a & a & 0 \\
\end{array}
\quad
\begin{array}{c|cc}
* & 0 & a \\
0 & 0 & a \\
a & a & 0 \\
\end{array}
\]

It is easily checked that $(R, +, *)$ is a near ring with $x^2 = x$ but $R$ is not commutative.

Also one can notice that in a left near ring $R$, $x0 = 0$, for all $x \in R$ but not necessarily $0x = 0$ (cf. above example).

Despite such bad behaviour of near rings, researchers have not altogether given up attempts to examine ring theoretic analogues of many ring theoretic results. Instead, a part of the recent work has been concerned with the generalizations of some well known commutativity theorems in rings to near rings (see [5], [7], [4], [30], [123], [124]). In the present section we also investigate the commutativity of near rings under condition (*) mentioned in the previous section.

**Lemma 2.5.1** ([44]). Let $R$ be a d-g near ring such that for each $x \in R$, there exist a positive integer $n = n(x)$ and an element $s$ in the subnear ring generated by $x$ for which $x^n = x^n s$. If $N(R) \subseteq Z(R)$, then $R$ is periodic and commutative.

**Lemma 2.5.2.** Let $R$ be a near ring satisfying condition (*). Then nilpotent elements annihilate $R$ on both sides.

**Proof.** Notice that $R$ satisfying condition (*) is zero-symmetric. Take $x = 0$ and $y = x$ in condition (*), to get $0x = P(0, x)$, where $P(X, Y) = \sum_{i=1}^{f} n_i w_i(X, Y)$,
implies that $0x = 0$ (because in a left near ring $x0 = 0$). Hence $R$ is also zero-commutative satisfying condition $(\ast)$ as we have in the proof of Theorem 2.4.1. Again arguing in the similar manner as we have done in the proof of Theorem 2.4.1, we obtain

$$N(R)R = RN(R) = (0). \quad (2.5.1)$$

**Theorem 2.5.1.** If $R$ is a d-g near ring satisfying condition $(\ast)$, then $R$ is commutative.

**Proof.** In view of (2.5.1) we can find that $N(R) \subseteq Z(R)$. Now replacing $y$ by $x$ in condition $(\ast)$, we get an element $r$ in the subnear ring generated by $x$ such that $x^2 = xr$. Hence by Lemma 2.5.1, $R$ is periodic and commutative.

The following example justifies that restriction on $R$ to be a d-g near ring can be dropped in the above theorem.

**Example 2.5.2.** Let $R = \{0, a, b, c\}$ with addition “$+$” and multiplication “$*$” defined below:

$$
\begin{array}{ccc|ccc}
+ & 0 & a & b & c \\
\hline
0 & 0 & a & b & c \\
0 & a & 0 & c & b \\
0 & b & c & 0 & a \\
0 & c & b & a & 0 \\
\end{array} 
\quad \quad 
\begin{array}{ccc|ccc}
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & a \\
b & 0 & 0 & 0 & 0 \\
c & 0 & c & 0 & c \\
\end{array}
$$

It is easy to check that $R$ is a near ring satisfying $xy = x^2y^3x^2$, for all $x, y \in R$. However $R$ is not commutative.