CHAPTER - 1
PRELIMINARIES

1.1 INTRODUCTION

This chapter is devoted to collect some basic notions and important terminology which we shall need for the development of the subject in the subsequent chapters of the present thesis. Of course, the knowledge of the elementary algebraic concepts as those of groups, rings, modules, fields and homomorphisms etcetera. has been preassumed and no attempt will be made to discuss them here. Most of the material included in this chapter occurs in standard literature namely: Herstein [84, 86], Jacobson [95, 96], Lambek [100], McCoy [116], Kurosh [99], Guenter Pilz [118] etcetera. Some key results and classical theorems related to our subject matter are also incorporated for ready reference. Suitable examples and necessary remarks are given at proper places to make the exposition self contained as much as possible.

1.2 SOME RING THEORETIC CONCEPTS

In the present section we give a brief exposition of some important terminology in ring theory. Throughout the thesis, unless otherwise mentioned, $R$ denotes an associative ring having at least two elements. For any pair of elements $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$ and anti-commutator $xy + yx$ by $x \circ y$. The symbols $N(R)$, $C(R)$ and $Z(R)$ denote the set of nilpotent elements, the commutator and the centre of the ring $R$.

**Definition 1.2.1 (Characteristic of a Ring).** If there exists a positive integer $n$ such that $na = 0$, for every element $a$ of the ring $R$, the smallest such positive integer is called the characteristic of $R$, which is generally expressed as $\text{char}R = n$. If no such positive integer exists, then $R$ is said to have characteristic zero.
Definition 1.2.2 (Torsion Free Element). An element $x \in R$ is said to be \( n \)-torsion free if \( nx = 0 \) implies \( x = 0 \). If \( nx = 0 \) implies \( x = 0 \), for every \( x \in R \), then we say that \( R \) is \( n \)-torsion free.

Remark 1.2.1 Obviously, if \( \text{char} R \neq m \), then \( ma = 0 \), for some \( a \in R \) implies that \( a = 0 \).

Definition 1.2.3 (Nilpotent Element). An element \( a \) of a ring \( R \) is said to be nilpotent if there exists a positive integer \( n \) such that \( a^n = 0 \), where \( a^n \) stands for \( a.a.a\ldots a \) \( (n-\text{factors}) \).

Definition 1.2.4 (Idempotent Element). An element \( e \) of a ring \( R \) is said to be idempotent if \( e^2 = e \).

Remark 1.2.2. In a ring \( R \) with unity and no nonzero divisors of zero the only idempotents are the zero and the unity.

Definition 1.2.5 (Direct Sum and Subdirect Sum of Rings). Let \( \{S_i\}, i \in U \) be a family of rings indexed by the set \( U \) and \( S \) denote the set of all functions defined on the set \( U \) such that for each \( i \in U \), the value of the function at \( i \) is an element of \( S_i \). If addition and multiplication in \( S \) are defined as \( (a + b)(i) = a(i) + b(i) \), \( ab(i) = a(i)b(i) \), for \( a, b \in S \), then \( S \) is a ring which is called the complete direct sum of the rings \( S_i, i \in U \). The set of all functions in \( S \) which take on the values zero at all but at most a finite number of elements \( i \) of \( U \) is a subring of \( S \) which is called the discrete direct sum of the rings \( S_i, i \in U \). However, if \( U \) is a finite set, the complete (discrete) direct sum of rings \( S_i, i \in U \), as defined above is called a direct sum of the rings \( \{S_i\}, i \in U \).
Let $T$ be a subring of the direct sum $S$ of rings $S_i$ and for each $i \in U$, let $\theta_i$ be a homomorphism of $S$ onto $S_i$ defined as $a\theta_i = a(i)$ for $a \in S$. If $T\theta_i = S_i$ for every $i \in U$, then $T$ is said to be a subdirect sum of the rings $S_i, i \in U$.

**Definition 1.2.6 (Centre of a Ring).** The centre $Z(R)$ of a ring $R$ is the set of all those elements of $R$ which commute with each element of $R$ and denoted as $Z(R)$ that is,

$$Z(R) = \{x \in R \mid xy = yx, \text{ for all } y \in R\}.$$

**Remark 1.2.3.** A ring $R$ is commutative if and only if $Z(R) = R$.

**Definition 1.2.7 (Centralizer).** Let $S$ be a nonempty subset of a ring $R$. Then the centralizer $C_R(S)$ of $S$ in $R$ is defined as

$$C_R(S) = \{a \in R \mid sa = as \text{ for all } s \in S\}.$$

**Definition 1.2.8 (Finitely Generated Ideal).** Let $S$ be a nonempty subset of a ring $R$. Then the ideal (right or left) $I$ of $R$ is said to be generated by $S$ if

(i) $S \subseteq I$.

(ii) For any (right or left) ideal $A$ of $R$, $S \subseteq A \Rightarrow I \subseteq A$.

**Definition 1.2.9 (Principal Ideal).** An ideal (right ideal) in $R$ generated by one element of $R$ is called a principal ideal (right ideal). An ideal (right ideal) generated by the element $a$ is denoted by $(a)((a),)$. 

**Definition 1.2.10 (Nilpotent Ideal).** A right (left, two sided) ideal $I$ of a ring $R$ is said to be a nilpotent ideal if there exists a positive integer $n > 1$ such that $I^n = (0)$.

**Definition 1.2.11 (Nil Ideal).** A right (left, two sided) ideal $I$ of a ring $R$ is said to be nil if each of its element is nilpotent.
Example 1.2.1. Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ and $I$ be an ideal of $R$ generated by $\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$. Then $I$ is nilpotent and also a nil ideal.

Remark 1.2.4.

(i) If every element of a ring $R$ is nilpotent, then $R$ itself is called a nil ring.

(ii) Every nilpotent ideal is nil but a nil ideal need not be necessarily nilpotent.

Example 1.2.2. Let $R = \prod_{n=1}^{\infty} \mathbb{Z}/2^n$. Let $A$ be the ideal of all nilpotent elements in $R$. Obviously $A$ is a nil ideal. But $A$ is not nilpotent, because if $A^n = (0)$ for some $n$, then $x^n = 0$, for all $x \in A$. Now take $x_n = (0,0,0,...,2,0,...)$, with $2$ at $(n + 1)^{th}$ place. We see that $x_n^{n+1} = 0$ but $x_n^n \neq 0$.

Definition 1.2.12 (Commutator ideal). The commutator ideal $C(R)$ of a ring $R$ is the ideal generated by all commutators $[x, y]$ with $x, y \in R$.

Definition 1.2.13 (Maximal Ideal). An ideal $M$ in a ring $R$ is said to be a maximal ideal in $R$. If

(i) $M \neq R$ and

(ii) there exists no ideal $I$ in $R$ such that $M \subset I \subset R$.

Remark 1.2.5. If $M \neq R$ is a maximal ideal in $R$, then for any ideal $I$ of $R$, $M \subset I \subset R$ holds only when either $I = M$ or $I = R$.

Definition 1.2.14 (Prime Ideal). An ideal $P$ in a ring $R$ is said to be a prime ideal if and only if it has the property that for any two ideals $A$ and $B$ in $R$ whenever $AB \subseteq P$ then $A \subseteq P$ or $B \subseteq P$.

Remark 1.2.6. Equivalently, an ideal $P$ in a ring $R$ is prime ideal if and only if any one of the following holds:
(i) If \( a, b \in R \) such that \( aRb \subseteq P \), then \( a \in P \) or \( b \in P \).

(ii) If \((a)\) and \((b)\) are principal ideals in \( R \) such that \((a)(b) \subseteq P\), then either \( a \in P \) or \( b \in P \).

(iii) If \( U \) and \( V \) are left (right) ideals in \( R \) such that \( UV \subseteq P \), then \( U \subseteq P \) or \( V \subseteq P \).

**Definition 1.2.15 (Semiprime Ideal).** An ideal \( I \) in a ring \( R \) is said to be a semiprime ideal if for any ideal \( A \) in \( R \), whenever \( A^2 \subseteq I \), then \( A \subseteq I \).

**Remark 1.2.7.**

(i) A prime ideal is necessarily semiprime but the converse need not be true in general.

(ii) Intersection of prime (semiprime) ideals is semiprime. Thus in the ring \( \mathbb{Z} \) of integers, ideal \( (2) \cap (3) = (6) \) is semiprime which is not prime.

**Definition 1.2.16 (Annihilator).** If \( M \) is a subset of a commutative ring \( R \), then annihilator of \( M \), denoted by \( \text{Ann}(M) \) is the set of all elements of \( r \) of \( R \) such that \( rm = 0 \) for all \( m \in M \).

\[
\text{Ann}(M) = \{ r \in R \mid rm = 0, \text{ for all } m \in M \}.
\]

**Definition 1.2.17 (Jacobson Radical).** The Jacobson radical \( J(R) \) of a ring \( R \) is the intersection of all maximal left (right) ideals of \( R \).

**Remark 1.2.8.**

(i) \( J(R) \) is a two sided ideal of \( R \).

(ii) \( J(R) \) is the set of all those elements of \( R \) which annihilates all the irreducible \( R \)-modules i.e.,

\[
J(R) = \{ r \in R \mid rM = 0, \text{ for every irreducible } R\text{-modules} \}.
\]
Definition 1.2.18 (Prime Ring). A ring $R$ is said to be prime if and only if the zero ideal $(0)$ is a prime ideal in $R$.

Remark 1.2.9. Equivalently, a ring $R$ is prime if and only if any one of the following holds:

(i) If $I_1$ and $I_2$ are ideals in $R$ such that $I_1I_2 = (0)$, then $I_1 = (0)$ or $I_2 = (0)$.

(ii) If $a, b \in R$ such that $aRb = 0$ then either $a = 0$ or $b = 0$.

Remark 1.2.10. Every division ring is a prime ring.

Definition 1.2.19 (Semiprime Ring). A ring $R$ is said to be semiprime if it has no nonzero nilpotent ideals.

Definition 1.2.20 (Simple Ring). A ring $R$ with more than one element is said to be a simple ring if its only ideals are the two trivial ideals namely $(0)$ and $R$.

Remark 1.2.11. A division ring is necessarily simple but not conversely. In fact, if $D$ is a division ring then the complete matrix ring $D_n$, for a positive integer $n$ is simple which of course, is not a division ring.

Definition 1.2.21 (Semisimple Ring). A ring $R$ with zero Jacobson radical is said to be semisimple.

Definition 1.2.22 (Boolean Ring). A ring $R$ is said to be Boolean ring if all of its elements are idempotent.

Definition 1.2.23 (Lie and Jordan Structures). Let $R$ be an associative ring. We can induce two new operations on $R$ as follows:

(i) For $x, y \in R$, the Lie product, $[x, y] = xy - yx$
(ii) For \( x, y \in R \), the Jordan product, \((x \circ y) = xy + yx\)

**Remark 1.2.12.** For any \( x, y, z \in R \), the following identities hold,

(i) \([xy, z] = x[y, z] + [x, z]y\).

(ii) \([x, yz] = y[x, z] + [x, y]z\).

(iii) \([[x, y], z] + [[y, z], x] + [[z, x], y] = 0\) (this identity usually called Jacobi identity).

(iv) \(x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z\).

(v) \((xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]\).

**Definition 1.2.24 (Lie (Jordan) Subring).** A nonvoid subset \( U \) of a ring \( R \) is said to be a Lie (resp. Jordan) subring of \( R \) if \( U \) is an additive subgroup of \( R \) and for \( x, y \in U \) implies that \([x, y]\) (resp. \((x \circ y)\)) is also in \( U \).

**Definition 1.2.25 (Lie (Jordan) Ideal).** An additive subgroup \( U \) of a ring \( R \) is said to be a Lie (resp. Jordan) ideal of \( R \) if whenever \( u \in U \) and \( x \in R \), then \([u, x]\) (resp. \((u \circ x)\)) is also in \( U \).

**Example 1.2.3.** Let \( R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} | a, b, c \in \mathbb{Z}_2 \right\} \). Then it can be easily seen that \( U = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} | a, b \in \mathbb{Z}_2 \right\} \) is a Lie ideal of \( R \) and \( J = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} | b \in \mathbb{Z}_2 \right\} \) is a Jordan ideal of \( R \).

**Definition 1.2.26 (Commuting Mappings).** Let \( S \) be a nonvoid subset of a ring \( R \). An additive mapping \( f : R \to R \) is said to be commuting on \( S \) if \([f(x), x] = 0\) holds, for all \( x \in S \).

**Definition 1.2.27 (Centralizing Mappings).** Let \( S \) be a nonvoid subset of a ring \( R \). An additive mapping \( f : R \to R \) is said to be centralizing on \( S \) if \([f(x), x] \in Z(R)\) holds, for all \( x \in S \).
**Definition 1.2.28 (Derivation).** A mapping $d : R \rightarrow R$ is said to be derivation on $R$ if it satisfies the following properties:

(i) $d(x + y) = d(x) + d(y)$ and

(ii) $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$.

**Example 1.2.4.** The most natural example of a nontrivial derivation is the usual differentiation on the ring $F[x]$ of polynomials defined over a field $F$.

For fixed $a \in R$, define $d : R \rightarrow R$ by $d(x) = [x, a]$, for all $x \in R$. The function $d$ so defined is additive and

$$
d(xy) = [xy, a] = x[y, a] + [x, a]y = xd(y) + d(x)y
$$

Thus, $d$ is a derivation which is called an inner derivation of $R$ associated with $a$ and generally denoted by $I_a$.

**Remark 1.2.13.** It is obvious to see that every inner derivation on a ring $R$ is a derivation. But converse need not be true in general.

**Example 1.2.5.** Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$ be a ring of $2 \times 2$ matrices over $\mathbb{Z}$, the ring of integers. Define a mapping $d : R \rightarrow R$ such that

$$
d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}
$$

Then, it can be verified that $d$ is a derivation but not an inner derivation on $R$.

### 1.3 SOME NEAR RING THEORETIC CONCEPTS

**Definition 1.3.1 (Near Ring).** A left near ring $R$ is a triple $(R, +, *)$ with two binary operations “$+$” and “$*$” such that
(i) \((R, +)\) is a group (not necessarily abelian).

(ii) \((R, \cdot)\) is a semigroup.

(iii) \(a \cdot (b + c) = a \cdot b + a \cdot c\), for all \(a, b, c \in R\).

Analogously, if instead of (iii), we have the right distributive law

\[(iii)' \ (a + b) \cdot c = a \cdot c + b \cdot c, \text{ for all } a, b, c \in R\]

holds, then \(R\) is said to be a right near ring.

As in both the cases, the theory of near rings runs completely parallel, we may consider left near rings throughout and for simplicity call them as near rings and the product \(a \cdot b\) will be denoted by \(ab\).

**Example 1.3.1.** The set of all identity preserving mappings acting on the right of an additive group \(G\) (not necessarily abelian) into itself with pointwise addition and composition of the mappings as multiplication is the most natural example of a right near ring.

**Definition 1.3.2 (Distributive Element).** An element \(x\) of a near ring \(R\) is said to be distributive if \((y + z)x = yx + zx\), for all \(y, z \in R\).

**Remark 1.3.1.** In any near ring \(R\), \(x0 = 0\), for all \(x \in R\), but not necessarily \(0x = 0\). However, if \(d\) is a distributive element in \(R\) then \(0d = 0\).

**Remark 1.3.2.** In any near ring \(R\), \(x(-y) = -xy\), for all \(x, y \in R\), but not necessarily \((-x)y = -xy\). However, if \(d\) is a distributive element in \(R\) then \((-x)d = -xd\).

**Definition 1.3.3 (Distributive Near Ring).** A near ring \(R\) is called distributive if each of its element is distributive.
Example 1.3.2. Let $R = \{0, a, b, c, x, y\}$ with addition “+” and multiplication “*” defined as follows:

$$
\begin{array}{cccc|cccc}
+ & 0 & a & b & c & x & y \\
\hline
0 & 0 & a & b & c & x & y \\
a & a & 0 & y & x & c & b \\
b & b & x & 0 & y & a & c \\
c & c & y & x & 0 & b & a \\
x & x & b & c & a & y & 0 \\
y & y & c & a & b & 0 & x \\
\end{array}
$$

$$
\begin{array}{cccc|cccc}
* & 0 & a & b & c & x & y \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & a & a & 0 & 0 \\
b & 0 & a & a & a & 0 & 0 \\
c & 0 & a & a & a & 0 & 0 \\
x & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

Then $(R, +, *)$ is a distributive near ring.

Definition 1.3.4 (Distributively Generated Near Ring). A near ring $R$ is said to be distributively generated (d-g), if it contains a multiplicative subsemigroup of distributive elements which generates the additive group $(R,+)$. 

Example 1.3.3. The near ring generated additively by all the endomorphisms of a group $(G, +)$ (not necessarily abelian) is a distributively generated near ring.

Definition 1.3.5 (Zero-symmetric Near Ring). A near ring $R$ is called zero-symmetric, if $0x = 0$, for all $x \in R$ (recall that left distributivity yields $x0 = 0$). 

Example 1.3.4. Let $R = \{0, a, b, c\}$ with addition “+” and multiplication “*” defined as follows:

$$
\begin{array}{ccc|ccc}
+ & 0 & a & b & c \\
\hline
0 & a & b & c & 0 \\
a & a & b & c & 0 \\
b & b & c & 0 & a \\
c & c & 0 & a & b \\
\end{array}
$$

$$
\begin{array}{ccc|ccc}
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & c & b & a \\
b & 0 & 0 & 0 & 0 \\
c & 0 & a & b & c \\
\end{array}
$$

It can be easily seen that $R$ is a zero-symmetric near ring.

Remark 1.3.3. A d-g near ring is always zero-symmetric.
Definition 1.3.6 (Zero-commutative Near Ring). A near ring $R$ is called zero-commutative, if $xy = 0$ implies $yx = 0$, for all $x, y \in R$.

Example 1.3.5. Let $R = \{0, a, b, c, u, v\}$ with addition “+” and multiplication “*” defined below:

$$
\begin{array}{c|cccccc}
+ & 0 & a & b & c & u & v \\
\hline
0 & 0 & a & b & c & u & v \\
a & a & 0 & v & u & c & b \\
b & b & u & 0 & v & a & c \\
c & c & v & u & 0 & b & a \\
u & u & b & c & a & v & 0 \\
v & v & c & a & b & 0 & u
\end{array}
\begin{array}{c|cccccc}
* & 0 & a & b & c & u & v \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a & a & 0 \\
b & b & 0 & b & b & b & 0 \\
c & c & 0 & c & c & c & 0 \\
u & u & 0 & 0 & 0 & 0 & 0 \\
v & v & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Then $(R, +, *)$ is a zero-commutative near ring.

Definition 1.3.7 (Additive Center). An additive center of a near ring $R$ is the set of all those elements of $R$ which commute with every element of $R$ under addition.

Multiplicative center of a near ring is defined in the same manner as we have defined in the case of rings (cf. Definition 1.2.6).

1.4 SOME IMPORTANT RESULTS ON RINGS AND NEAR RINGS

Theorem 1.4.1 (Wedderburn [133]). A finite division ring is a field.

Theorem 1.4.2 (Jacobson [94]). Let $R$ be a ring in which for every $x \in R$, there exist an integer $n = n(x) > 1$, depending on $x$ such that $x^{n(x)} = x$. Then $R$ is commutative.

Theorem 1.4.3 (Kaplansky [97]). Let $R$ be a prime ring in which for every $x \in R$, there exist an integer $n = n(x) > 1$, depending on $x$ such that $x^{n(x)} \in Z(R)$, for every $x \in R$. If in addition $R$ is semisimple, then it is also commutative.
Theorem 1.4.4 (Faith [77]). Let $D$ be a division ring and $A \neq D$, a subring of $D$. Suppose that for every $x \in D$, $x^{n(x)} \in A$, where $n(x) \geq 1$ depends on $x$. Then $D$ is commutative.

Theorem 1.4.5 (Bell and Martindale [43]). Let $R$ be a prime ring and $U$ a nonzero left ideal of $R$. If $R$ admits a nonzero derivation which is centralizing on $U$, then $R$ is commutative.

Theorem 1.4.6 (Herstein [84]). Let $R$ be a prime ring and $0 \neq \rho$ a right ideal of $R$. Suppose that, $a \in \rho$, $a^n = 0$ for a fixed integer $n$. Then $R$ has a nonzero nilpotent ideal.

Theorem 1.4.7 (Neumann [117]). The additive group of a division near ring is abelian.

Theorem 1.4.8 (Frohlic [80]). A d-g near ring $R$ is distributive if and only if $R^2$ is additively commutative.

Theorem 1.4.9 (Frohlic [80]). A d-g near ring $R$ with unity $1$ is a ring if $(R, +)$ is abelian or if $R$ is distributive.