The present thesis entitled “On Derivations and Commutativity of Rings” includes the research work carried out by the author during more than past four years at the Department of Mathematics, Aligarh Muslim University, Aligarh. The exposition may be said to be divided into two parts. The first part having only one chapter deals with the commutativity of certain rings and near rings under some polynomial constraints whereas second part comprises three chapters in which derivations in rings are studied. Besides research material arranged into the mentioned two parts, Chapter 1 contains preliminary notions, basic definitions and some important well known results relevant to our study needed for the development of the subject matter in the subsequent chapters. Each chapter is subdivided into various sections. The definitions, examples, results and remarks etc. have been specified with double decimal numbers. The first figure denotes the number of the chapter, second represents the section in the chapter and the third points out the number of definition, the example, the result or the remark as the case may be in a particular chapter. For example, Theorem 2.3.4 refers to the fourth theorem appearing in the third section of the second chapter.

Although the famous classical theorem, namely a finite division ring must be commutative was established as earlier as 1905 by Wedderburn, but it was after the development of structure theory of rings in the course of forties and fifties of the last century that significant contributions were made by many mathematicians in this direction. Since then the subject has been attracting a wide circle of algebraists like Amitsur, Braur, Kaplansky, Jacobson, Faith, Baer, McCoy, Herstein, Rowen, Ligh, Yaqub, Tominga, Luh, Richoux, Chacron, Bell, Quadri and Ashraf - ; to mention a few. There has been a great deal of work in the mathematical literature concerning the investigations of the classes of rings which turn commutative under some constraints, mostly satisfying certain polynomial conditions. Chapter 2 is devoted to study same type of work. In 1951, Herstein proved that if in a ring $R$, 

there exists an integer \( n > 1 \) such that \( x^n - x \) is central, for all ring elements \( x \), then such a ring must be commutative. In Section 2.2, we generalize this result by proving that if \( R \) is a semiprime ring satisfying \((xy)^n - xy \in Z(R)\), the centre of \( R \), then \( R \) must be commutative. Observing Example 2.2.1, we can think that the mentioned non-commutative ring does not contain unity. One may therefore hope that the above property could yield commutativity in rings with unity. Working in this direction we obtain in Section 2.3, rather a stronger result with weaker hypothesis as follows: If \( R \) is a ring with unity \( 1 \) satisfying \([xy]^n - xy, x] = 0\), for all \( x, y \in R \), then \( R \) is commutative. In Section 2.4 commutativity of periodic rings is investigated.

It is not always easy to obtain near ring theoretic analogues of ring theoretic results. Many of them do not hold in general either. As an example, we know that a Boolean ring is necessarily commutative but there exist enough Boolean near rings which are non-commutative (cf. Example 2.5.1). Nevertheless, a part of the recent work has been concerned with the generalizations of some well known commutativity theorems in rings to near rings. In Section 2.5 we too investigate commutativity of distributively generated (d-g) near rings under certain conditions.

Let \( R \) be an associative ring. An additive mapping \( d : R \rightarrow R \) is said to be a derivation on \( R \) if \( d(xy) = d(x)y + xd(y) \) holds, for all \( x, y \in R \). Although the notion of derivation has been existing in literature since the advent of twentieth century, yet it was during the past five decades that the study of derivation in rings started attracting a wide circle of mathematicians after E.C.Posner [119] established two very striking results on derivations which state as follows: (a) In a 2-torsion free prime ring if iterate of two derivations is again a derivation, then at least one of them must be zero and (b) if a prime ring \( R \) admits a nonzero centralizing derivation, then \( R \) must be commutative.
In recent years many well known algebraists such as Beidar, Bell, Bergen, Bresar, Herstein, Kaya, Martindale, Mason, Posner, Vukman and Ashraf etcetera, have made remarkable contributions to this area of study. The interest in this area was partially motivated by its many useful applications to various branches of Mathematics (see for examples [11],[49],[126],[134]).

The notion of derivation has been generalized in various directions such as left derivation \((\theta,\phi)\)-derivation, semiderivation, generalized derivation, Jordan derivation and Lie derivation etcetera. During the last two decades there has been some work concerning generalized deviation in the context of algebras on certain normed spaces. Bresar [48] defined generalized derivation in rings as follows. An additive mapping \(F : R \rightarrow R\) is called a generalized derivation on \(R\) if there exists a derivation \(d : R \rightarrow R\) such that \(F(xy) = F(x)y + xd(y)\) holds for all \(x, y \in R\). Recently Hvala [93] initiated the algebraic study of generalized derivation and extended some results concerning derivation to generalized derivation. There has been also ongoing interest between the commutativity of rings and the existence of certain specific types of derivations of rings (for reference see [38],[41],[43],[72],[74]). Chapter 3 deals with the investigation of commutativity of rings satisfying some functional identities. In section 3.2, we obtain commutativity of a prime ring \(R\) admitting a generalized derivation \(F\) satisfying any one of the conditions \(F(xy) - d(x)d(y) = 0\) and \(F(xy) + d(x)d(y) = 0\), for all elements \(x, y\) in some distinguished subset of \(R\). Very recently Daif and Bell [72] proved that if a semiprime ring \(R\) admits a derivation \(d\) such that either \(d([x,y]) + [x,y] = 0\) or \(d([x,y]) - [x,y] = 0\), for all \(x,y \in I\), a nonzero ideal of \(R\), then \(R\) must be commutative. Further, Hongan [92] generalized the result by considering \(R\) satisfying either of the conditions \(d([x,y]) + [x,y] \in Z(R)\) and \(d([x,y]) - [x,y] \in Z(R)\), for all \(x,y \in I\). In Section 3.3, we explore commutativity of a ring \(R\) admitting a generalized derivation \(F\) satisfying any one of the following : (i) \(F([x,y]) - [x,y] \in Z(R)\), (ii) \(F([x,y]) + [x,y] \in Z(R)\),
(iii) $F(x \circ y) - (x \circ y) \in Z(R)$ and (iv) $F(x \circ y) + (x \circ y) \in Z(R)$, for all $x, y$ in some appropriate subset of $R$.

Inspired by the definition of $(\theta, \phi)$-derivation the notion of generalized $(\theta, \phi)$-derivation has been defined by Asharaf et al. [18] as follows: An additive mapping $F : R \to R$ is said to be a generalized $(\theta, \phi)$-derivation on $R$ if there exists a $(\theta, \phi)$-derivation $d : R \to R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ holds for all $x, y \in R$. We shall call a generalized $(\theta, I)$-derivation as a generalized $\theta$-derivation where $I$ is the identity automorphism of $R$. Similarly a generalized $(I, \phi)$-derivation will be called as a generalized $\phi$-derivation. Finally in section 3.4 we obtain commutativity of a prime ring $R$ admitting a generalized $\phi$-derivation satisfying any one of the following conditions: (i) $F(xy) - xy \in Z(R)$, (ii) $F(xy) + xy \in Z(R)$, (iii) $F(xy) - yx \in Z(R)$, (iv) $F(xy) + yx \in Z(R)$, (v) $F(x)F(y) - xy \in Z(R)$ and (vi) $F(x)F(y) + xy \in Z(R)$, for all $x, y$, in a nonzero ideal of $R$.

Chapter 4 deals with the study of biderivations on prime and semiprime rings. Let $R$ be an associative ring. A symmetric biadditive mapping $D(., .) : R \times R \to R$ is said to be a symmetric biderivation if for any fixed $y \in R$, the mapping $x \mapsto D(x, y)$ is a derivation. In 1980, Gy. Maksa [109] introduced the concept of symmetric biderivation. It was shown in [110] that symmetric biderivations are related to general solution of some functional equations. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping $f : R \to R$ gives rise to a biderivation on $R$. In Section 4.2 we extend the two famous theorems of E.C.Posner ([119], Theorem 1 and Theorem 2) mentioned above for symmetric $(\sigma, \sigma)$-biderivation. Section 4.3 starts with a result due to Vukman which states that if $R$ is a prime ring with characteristic different from 2 and 3 admitting a symmetric biderivation $D(., .) : R \times R \to R$ such that the mapping $x \mapsto [f(x), x]$, where $f$ stands for the trace of $D$, is centralizing on $R$, then $R$ must be commutative. We obtain the result
for \((\sigma, \sigma)\)-biderivation on \(R\).

In [74], Deng and Bell generalized the concept of centralizing and commuting mappings by defining \(n\)-centralizing and \(n\)-commuting mappings. Let \(n\) be an arbitrary positive integer and \(S\) be a nonempty subset of a ring \(R\). A mapping \(f : R \to R\) is said to be \(n\)-centralizing (resp. \(n\)-commuting) on \(S\) if 
\[ [x^n, f(x)] \in Z(R) \] (resp. \([x^n, f(x)] = 0\)) holds, for all \(x \in S\). Finally in Section 4.4, we establish the following result: Suppose \(n > 1\) is a positive integer. Let \(R\) be a \(2, 3\) and \((2^n - 1)\)-torsion free semiprime ring and \(I\) be a nonzero ideal of \(R\). If \(D(.,.) : R \times R \to R\) is a symmetric biderivation such that the trace \(f : R \to R\) is \(n\)-centralizing on \(I\), then \(f\) is \(n\)-commuting on \(I\).

Chapter 5 is devoted to the study of derivations on a ring which act as homomorphisms or as anti-homomorphisms. Recently Bell and Kappe [40] proved that if \(R\) is a prime ring and \(d\) is a derivation which acts as a homomorphism or an anti-homomorphism on \(I\), a nonzero ideal of \(R\), then \(d = 0\). Very recently Ashraf et al. [19] obtained the result for \((\theta, \phi)\)-derivation on \(R\) which acts as a homomorphism or an anti-homomorphism on \(I\). In section 5.2, we generalize the mentioned results by proving that if \(R\) is a prime ring and \(d\) is a left \((\theta, \phi)\)-derivation on \(R\), which acts as a homomorphism or as an anti-homomorphism on a Jordan ideal \(J\) of \(R\), then \(d = 0\) on \(R\). Finally we extend the above result for generalized \((\theta, \phi)\)-derivation in the setting of Lie ideals of a prime ring.

At places, examples are provided to justify the conditions imposed on the hypothesis of various results. The extensions of some of the results presented in the exposition may not be outrightly ruled out but choice of our examples shows that they cannot be generalized arbitrarily. Also suitable remarks are given sometime to explain the theory and sometime to conjecture the possible extensions of the results.
In the end, an exhaustive bibliography of the existing material related to the subject matter of our thesis is included which may serve as source material for those, interested in the domain of our research.

One paper of the author related to some portion of Chapter 5 has already been published in Int. Math. J. vol. 2 (2007), 1105-1110, where as one paper based on the material of Chapter 4 has been accepted for publication in the Aligarh Bull. Math. Several papers related to the material of other chapters are in the process of acceptance.