CHAPTER V

PSEUDO-METRIC SEMI-SYMMETRIC CONNECTIONS

5.1. INTRODUCTION.

In 1975 Prvanovic [17] introduced pseudo-metric semi-symmetric connections on a Riemannian manifold $N$ as follows:

Let $\nabla$ be the Riemannian connection on $N$ with respect to the Riemannian metric $g$. Consider the connections

\[ \nabla^1_X Y = D_X Y + \pi(X)Y - g(X,Y)P, \]
\[ \nabla^2_X Y = D_X Y + \pi(Y)X - g(X,Y)P, \]

where $\pi$ is a 1-form and $P$ a vector field defined by

\[ g(X,P) = \pi(X), \]

as well as the connections

\[ \nabla^3(\pi(Y)) = \omega(\nabla^1 X) + (\nabla^2 \omega)(Y), \]
\[ \nabla^4(\pi(Y)) = \omega(\nabla^2 X) + (\nabla^1 \omega)(Y), \]

where $\omega$ is a 1-form. Then the connection $\nabla^2$ is the semi-symmetric metric connexion ([24], [30]), and $\nabla^3$ and $\nabla^4$ are called pseudo-metric semi-symmetric connections [17].
Following Prvanovic' [17] definitions and adopting them in the setting of almost contact metric manifold by identifying \( \pi \) with the contact form \( A \), and \( P \) with the characteristic vector field \( t \), we construct two connexions

\[
(5.1.6) \quad \nabla (AX) = A(\nabla X) + (\nabla A)(Y),
\]

and

\[
(5.1.7) \quad \nabla (AY) = A(\nabla Y) + (\nabla A)(Y),
\]

in terms of the connexions

\[
(5.1.8) \quad \nabla^1 X Y = D_X Y + A(X)Y - g(X,Y)t,
\]

and

\[
(5.1.9) \quad \nabla^2 X Y = D_X Y + A(Y)X - g(X,Y)t,
\]

where \( \nabla \) is the semi-symmetric metric connexion studied in Chapter III ([24], [30]), and \( \nabla^3 \) and \( \nabla^4 \) are the pseudo-metric semi-symmetric connexions, i.e.,

\[
\nabla^3_k g_{ij} = 0, \quad \nabla^4_k g = 0,
\]

but

\[
\nabla^4_k g_{ij} = \nabla^1_k g_{ij} = -2A_k g_{ij} + A^j_k g_{ix} + A^j_k g_{ik},
\]

and

\[
\nabla^3_k g = \nabla^1_k g = 2A_k g - t \delta_k - t \delta_k,
\]

where \( A_k \) and \( t^i \) are the components of \( A \) and \( t \) with respect to local co-ordinates.

In what follows, we study, in particular, the curvature...
tensor of those connections.

5.2. CURVATURE TENSOR OF THE CONNECTIONS $\triangledown^3$ AND $\triangledown^4$.

As has been shown by Prvanovic [17], the curvature tensor $R(X,Y,Z)$ of the connection $\triangledown^3$ can be written as

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z - \nabla_{Y\cdot X} Z.$$

Then, after some calculation, we find that in an almost contact metric manifold $\mathcal{E}$ and $\mathcal{F}$ are related by (1.27)

$$R(X,Y,Z) = R(X,Y,Z) + \alpha(X,Y)Z - \alpha(Y,Z)X$$

where $\alpha$ is a tensor field of type $(0,2)$ defined by

$$\alpha(X,Y) = (\nabla_X \alpha)(Y) - \nabla_X (\alpha(Y)) + g(Y, Z),$$

and $\beta$ is a tensor field of type $(1,1)$ defined by

$$\beta(X, Y) = \alpha(X, Y).$$

Let us put

$$\rho(X,Y) df^2 (0^3 R)(X,Y),$$

where $0^3$ stands for contraction in the third slot. Then (5.2.2) gives
\[(5.2.6) \quad \alpha(X,Y) = \frac{1}{n-1} \rho(X,Y), \]

and

\[(5.2.7) \quad \beta(X) = \frac{1}{n-1} \rho(X), \]

where

\[(5.2.8) \quad g(\rho(X),Y) = \rho(X,Y). \]

Substituting (5.2.6) and (5.2.7) in (5.2.2), we get

\[
R(X,Y,Z) = \frac{1}{n-1} \left[ \rho(X,Z) - \rho(Y,Z)X + g(Y,Z) \rho(X) - g(Y,Z) \rho(X) \right]
\]

\[(5.2.9) \quad = k(X,Y,Z) + g(Y,Z)X - g(X,Y). \]

It is easily seen that the vanishing of \( R(X,Y,Z) \)

implies

\[(5.2.10) \quad K(X,Y,Z) = - \{ g(Y,Z)X - g(X,Y) \}, \]

which in turn shows that

**Theorem 5.2.1** - If an almost contact metric manifold \( M \) admits a pseudo-metric semi-symmetric connection \( \nabla \) whose curvature tensor \( R(X,Y,Z,U) \) vanishes, then the Riemannian curvature of \( M \) is constant, and is in fact equal to \(-1\).

As has been shown by Prvanovic [17], the curvature tensor \( R(X,Y,Z) \) of the connection \( \nabla \) can be written as
\[(5.2.11) \quad R(x, y, z) = \begin{vmatrix} 2 & 1 \\ 4 & x \\ \end{vmatrix} \begin{vmatrix} 2 \\ y \\ \end{vmatrix} - \begin{vmatrix} 1 \\ y \\ \end{vmatrix} \begin{vmatrix} 2 \\ x \\ \end{vmatrix} + \begin{vmatrix} 2 \\ y \\ \end{vmatrix} \begin{vmatrix} 1 \\ x \\ \end{vmatrix} \begin{vmatrix} 1 \\ y \\ \end{vmatrix} \begin{vmatrix} 1 \\ x \\ \end{vmatrix} \begin{vmatrix} y \\ \end{vmatrix} \begin{vmatrix} x \\ \end{vmatrix}.
\]

Then we see that in an almost contact metric manifold, \( R \) and \( K \) are related by \( [27] \)
\[
R(x, y, z) = K(x, y, z) + T(x, y)z - T(y, z)x \tag{5.2.12}
\]
where
\[
\begin{align*}
R(x, y, z) & = g(x, z)g(y) - g(x, z)g(y) - \delta(x, y)z \\
+ & \delta(x, y)y - g(x, z)E(y) + g(y, z)E(x),
\end{align*}
\]
and \( G \) and \( E \) are the tensor fields of type \((1, 1)\) defined by
\[
(5.2.13) \quad (a) \quad T(x, y) = (D_x A)(y), \quad (b) \quad \delta(x, y) = A(x)A(y) - g(x, y),
\]
and \( G \) and \( E \) are the tensor fields of type \((1, 1)\) defined by
\[
(5.2.14) \quad (a) \quad g(G(x), y) = T(x, y), \quad (b) \quad g(E(x), y) = \delta(x, y).
\]

Let us put
\[
\begin{align*}
\rho(x, z) & \overset{\text{def}}{=} (c_1 R)(x, z) ; \quad \kappa(x, z) \overset{\text{def}}{=} (c_1 K)(x, z), \\
\tilde{\rho}(x, z) & \overset{\text{def}}{=} (c_2 R)(x, z) ; \quad \tilde{\kappa}(x, z) \overset{\text{def}}{=} (c_2 K)(x, z), \\
\tilde{\tilde{\rho}}(x, y) & \overset{\text{def}}{=} (c_3 R)(x, y) ; \quad \tilde{\tilde{\kappa}}(x, y) \overset{\text{def}}{=} (c_3 K)(x, y) = 0,
\end{align*}
\]
\[
\begin{align*}
\mathbf{t} & = \rho(x_1, x_1) ; \quad \mathbf{t} = \tilde{\rho}(x_1, x_1),
\end{align*}
\]
\[
\begin{align*}
\mathbf{r} = \frac{1}{4} \rho(x_1, x_1) + k = \frac{1}{4} \rho(x_1, x_1) = \frac{1}{4} \rho(x_1, x_1),
\end{align*}
\]

\(x_1\) being \(n\) orthonormal vectors. Then (5.2.15) gives

\[
\rho(y, z) = \nu(x, y) + \tau(x, y) = (n-1) \nu(x, y) - \delta(z, y) - (G-E) \gamma(x, y),
\]

(5.2.15)

and

\[
\rho(x, y) = (n-1) \left[ \nu(x, y) - \delta(x, y) \right].
\]

(5.2.16)

From (5.2.16) and (5.2.17), we get

\[
\begin{align*}
G &= \frac{1}{n-1} \left( \frac{1}{4} \mathbf{r} + k \right), \\
E &= \frac{1}{n-1} \left( \frac{1}{4} \mathbf{r} - \frac{1}{4} \mathbf{r} + k \right),
\end{align*}
\]

and hence

(5.2.19)

\[
G = E = \frac{1}{n-1} \frac{1}{4} \mathbf{r}.
\]

From (5.2.15) and (5.2.17), we find that

(5.2.19)

\[
\nu(x, y) = \frac{1}{n-1} \left[ \nu(x, y) + \frac{1}{n-1} \left( \frac{1}{4} \rho(x, y) - \frac{1}{4} \rho(x, y) \right) \right],
\]

and

\[
\nu(x, y) = \frac{1}{n-1} \left[ \nu(x, y) + \frac{1}{n-1} \left( \frac{1}{4} \rho(x, y) - \frac{1}{4} \rho(x, y) \right) \right].
\]
\[
5(x, y) = \frac{1}{n-1} \left[ x(x, y) + \frac{2-\alpha}{n-1} m(y, x) - m(y, x) \right]
\]

(5.2.20)

\[
- \frac{1}{n-1} e(x, y)
\]

Then, in consequence of (5.2.18), (5.2.19) and (5.2.20), equation (5.2.19) takes the form

\[
R(x, y, z) = \frac{1}{4} \left[ \left( x(x, y) + g(x, z) \right) + \left( y(y, z) + g(y, x) \right) \right]
\]

(5.2.21)

\[
- \frac{1}{n-1} \left[ \left( y(y, z) + g(y, z) \right) + \left( x(x, z) + g(x, z) \right) \right] X
\]

\[
+ \frac{1}{n-1} \left[ \left( y(y, z) + g(y, z) \right) + \left( x(x, z) + g(x, z) \right) \right] Y
\]

\[
= K(x, y, z) - \frac{1}{n-1} \left[ x(x, y)x - x(x, z) \right]
\]

We now assume that \( R(x, y, z, u) = 0 \). Then (5.2.21) becomes

\[
K(x, y, z, u) = \frac{1}{n-1} \left[ x(x, z)g(x, u) - x(x, z)g(y, u) \right]
\]

(5.2.22)

Contracting in \( Y, Z \), we get

\[
\psi(x, u) = \frac{1}{n} g(x, u)
\]

(5.2.23)
which shows that the manifold then is an EINSTEIN SPACE.
From (5.2.22) and (5.2.23) it follows that the Riemannian curvature is constant.

Hence we conclude:

**THEOREM 5.2.2**  - If an almost contact metric manifold $N$ admits a pseudo-metric semi-symmetric connexion $\nabla$ whose curvature tensor $R(X,Y,Z,U)$ vanishes, then the Riemannian curvature of $N$ is constant and is in fact equal to $r/n(n-1)$.

The following results are immediate consequences of (5.2.21):

**COROLLARY 5.2.3**  - If an almost contact metric manifold admits a pseudo-metric semi-symmetric connexion $\nabla$ whose Ricci tensors $^R\rho(X,Y)$ and $^R\rho(X,Y)$ vanish, then the curvature tensor $R(X,Y,Z,U)$ of $\nabla$ is equal to the projective curvature tensor of the Riemannian connexion (cf. [17]).

**COROLLARY 5.2.4**  - If the curvature tensor $R(X,Y,Z,U)$ of the pseudo-metric semi-symmetric connexion $\nabla$ vanishes, then the projective curvature tensor of the Riemannian connexion also vanishes (cf. [17]).