CHAPTER III

FIXED POINT THEOREMS FOR
A CLASS OF MAPPINGS
CHAPTER-III

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3.1. INTRODUCTION:

In an attempt to generalize some of the results of Iséki [58], Husain and Sehgal [53] obtained some common fixed point theorems employing a functional inequality. In the same spirit, motivated by Fisher [37] and Sessa [112], we present yet another extension of the results of Husain and Sehgal [53] which in turn generalizes several known results due to Fisher [37,38], Das-Naik [26], Iséki [58], Kasahara [67], Ranganathan [99], Singh [118] and Yeh [132,134]. Illustrative examples, and convergence theorems for sequences of mappings and their fixed points are also discussed. In the end of the chapter we extend our main result to uniformly convex Banach spaces.

3.2. SELF-MAPPINGS ON A COMPLETE METRIC SPACE:

Throughout this section \((x, d)\) stands for a complete metric space, and \(\mathbb{R}^+\) the set of non-negative reals. Let \(\Psi\) denotes a family of mappings \(\varphi : (\mathbb{R}^+)^5 \to \mathbb{R}^+\) such that each \(\varphi \in \Psi\) is upper semi-continuous and nondecreasing in each co-ordinate variable. Also for a mapping \(\gamma : \mathbb{R}^+ \to \mathbb{R}^+\), we put \(\gamma(t) = \varphi(t, t, a_1t, a_2t, t)\) where \(a_1 + a_2 = 3\).
The following lemma due to Singh and Meade [122] (also see Chang [17]) is the key in proving our main result.

**Lemma 3.4.1.** For any \( t > 0 \), \( \tau(t) < t \) if and only if
\[
\lim_{n \to \infty} \tau^n(t) = 0 \quad \text{where} \quad \tau^n \quad \text{denotes the composition of} \quad \tau
\]
with itself \( n \) times.

The following result is essentially inspired by Fisher [37] whereas its proof is patterned after Husain and Sehgal [55].

**Theorem 3.2.3.** Let \( A \) be any self-mapping of \( X \), \( S \) and \( T \) be continuous self-mappings on \( X \) satisfying the following conditions:

(i) \( \{A,S\} \) and \( \{A,T\} \) are weakly commuting pairs such that
\[
A(X) \subseteq S(X) \cap T(X),
\]

(ii) there exists a \( \psi \in \Phi \) such that for all \( x, y \in X \),
\[
d(Ax, Ay) \leq \psi(d(Sx, Ty), d(Sx, Ax), d(Sx, Ay),
\]
\[
d(Ty, Ax), d(Ty, Ay))
\]
where \( \psi \) satisfies the condition:

(iii) for any \( t > 0 \), \( \psi(t, t, a_1 t, a_2 t, t) < t \) for all
\[
(a_1, a_2) \in \{(2,1), (1,2)\}.
\]

Then \( A, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** Let \( x_0 \in X \) be an arbitrary point in \( X \). Then as
argued in Lemma 2.2.1 one can always choose points $x_{2n+1}$ and $x_{2n+2}$ in $\Lambda$ such that $Sx_{2n+1} = Ax_{2n}$ and $Tx_{2n+2} = Ax_{2n+1}$ for $n = 0, 1, 2, \ldots$.

Let $d_n = d(Ax_n, Ax_{n+1})$, $n = 0, 1, 2, \ldots$. We prove that $d_{2n} \leq d_{2n-1}$ for all $n$. Suppose that $d_{2n} > d_{2n-1}$ for some $n$. Then

$$
 d_{2n} = d(Ax_{2n}, Ax_{2n+1}) \\
 = d(Ax_{2n+1}, Ax_{2n}) \\
 \leq \Phi(d(x_{2n+1}, Tx_{2n}), d(Sx_{2n+1}, Ax_{2n+1}), d(Sx_{2n}, Ax_{2n})) \\
 = \Phi(d(Ax_{2n}, Ax_{2n+1}), d(Ax_{2n}, Ax_{2n+1}), d(Ax_{2n}, Ax_{2n})) \\
 \leq \Phi(d_{2n}, d_{2n}, 0, d_{2n-1} + d_{2n}, d_{2n-1}) \\
 \leq \Phi(d_{2n}, d_{2n}, d_{2n}, 2d_{2n}, d_{2n}) \\
 < d_{2n},
$$
a contradiction. Hence $d_{2n} \leq d_{2n-1}$ for all $n$. Similarly, one can prove that $d_{2n+1} \leq d_{2n}$ for $n = 0, 1, 2, \ldots$.

Consequently, $\{d_n\}$ is a non-increasing sequence of non-negative reals. Then
\[
d_1 = d(Ax_1, Ax_2) \\
\leq \theta(d(Ax_1, Ax_2), d(Sx_1, Ax_1), d(Sx_1, Ax_2), d(Tx_2, Ax_1), \\
d(Tx_2, Ax_2)) \\
= \theta(d(Ax_0, Ax_1), d(Ax_0, Ax_1), d(Ax_0, Ax_2), d(Ax_1, Ax_1), \\
d(Ax_1, Ax_2)) \\
\leq \theta(d_0, d_0, d_0 + d_1, 0, d_1) \\
\leq \theta(d_0, d_0, 2d_0, d_0, d_0) \\
= \gamma(d_0).
\]

In general, we have \( d_n \leq \gamma^n(d_0) \). So if \( d_0 > 0 \), then the Lemma 3.4.1 gives

\[
\lim_{n \to \infty} d_n = 0.
\]

For \( d_0 = 0 \), we clearly have \( \lim_{n \to \infty} d_n = 0 \), since then \( d_n = 0 \) for each \( n \).

Now we wish to prove that \( \{Ax_n\} \) is a Cauchy sequence. Since \( \lim_{n \to \infty} d_n = 0 \), it is sufficient to show that the sequence \( \{Ax_{2n}\} \) is a Cauchy sequence. Suppose that \( \{Ax_{2n}\} \) is not a Cauchy sequence. Then there is an \( \varepsilon > 0 \) such that for each even integer \( 2k, k = 0, 1, 2, \ldots \), there exist even integers \( 2n(k) \) and \( 2m(k) \) with \( 2k \leq 2n(k) < 2m(k) \) such that
Let, for each even integer $2k$, $2m(k)$ be the least integer exceeding $2n(k)$ and satisfying (3.2.2.1). Therefore

(3.2.2.2) \[ d(Ax_{2n}(k), Ax_{2m}(k)-2) \leq \varepsilon \text{ and } d(Ax_{2n}(k), Ax_{2m}(k)) > \varepsilon. \]

Then, for each even integer $2k$ we have

\[ \varepsilon < d(Ax_{2n}(k), Ax_{2m}(k)) < d(Ax_{2n}(k), Ax_{2m}(k)-2) + d(Ax_{2m}(k)-2, Ax_{2m}(k)-1) + d(Ax_{2m}(k)-1, Ax_{2m}(k)). \]

So by (3.2.2.2) and $d_n \to 0$, we obtain

\[ \lim_{k \to \infty} d(Ax_{2n}(k), Ax_{2m}(k)) = \varepsilon. \]

It follows immediately from the triangular inequality that

\[ \left| d(Ax_{2n}(k), Ax_{2m}(k)-1) - d(Ax_{2n}(k), Ax_{2m}(k)) \right| \leq d_{2m}(k)-1 \]

and

\[ \left| d(Ax_{2n}(k)+1, Ax_{2m}(k)-1) - d(Ax_{2n}(k), Ax_{2m}(k)) \right| \leq d_{2m}(k)-1 + d_{2n}(k). \]

Hence by (3.2.2.2), as $k \to \infty$

(3.2.2.3) \[ d(Ax_{2n}(k), Ax_{2m}(k)-1) \to \varepsilon \text{ and } d(Ax_{2n}(k)+1, Ax_{2m}(k)-1) \to \varepsilon. \]

Now

\[ d(Ax_{2n}(k), Ax_{2m}(k)) \leq d(Ax_{2n}(k), Ax_{2n}(k)+1) + d(Ax_{2n}(k)+1, Ax_{2m}(k)) \]

\[ \leq d_{2n}(k) + \varepsilon d(Ax_{2n}(k), Ax_{2m}(k)-2) + d_{2n}(k), \]

\[ d(Ax_{2n}(k), Ax_{2m}(k)), d(Ax_{2m}(k)-1, Ax_{2n}(k)+1), \]

\[ d_{2m}(k)-1. \]
Using (3.2.2.3) \( \lim_{n \to \infty} d_n = 0 \), and upper semicontinuity and nondecreasing property of \( \phi \) in each co-ordinate variable, we have

\[
\varepsilon \leq \phi (\varepsilon, 0, \varepsilon, 0, \varepsilon) \leq \gamma(\varepsilon) < \varepsilon,
\]
as \( k \to \infty \), which is a contradiction. Thus \( \{A_n\} \) is a Cauchy sequence and hence by completeness of \( X \), there is a \( z \in X \) such that \( A_n \to z \). Since the sequences \( \{a_{2n+1}\} \) and \( \{T_{2n}\} \) are subsequences of \( \{A_n\} \) they have the same limits \( z \).

As \( \sigma \) and \( \tau \) are continuous, we have \( S_{2n} \to Sz \) and \( T_{2n+1} \to Tz \). Now consider

\[
d(S_{2n}, T_{2n}) = d(S_{2n-1}, T_{2n}) \\
\leq d(a_{2n-1}, a_{2n}) + d(T_{2n}, T_{2n}).
\]

Using (ii) and the weak commutativity of \( \{A, S\} \) and \( \{A, T\} \), we get

\[
d(S_{2n}, T_{2n}) \leq d(a_{2n-1}, a_{2n}) + d(T_{2n}, T_{2n}) + d(A_{2n}, T_{2n}) \\
\leq d(a_{2n-1}, a_{2n}) + \phi(S_{2n-1}, T_{2n}) \\
d(a_{2n-1}, a_{2n}) + d(T_{2n}, T_{2n}) + d(A_{2n}, T_{2n}) \\
d(A_{2n}, T_{2n}) + d(a_{2n-1}, a_{2n}) + \phi(S_{2n-1}, T_{2n}) \\
d(S_{2n-1}, S_{2n}) + d(S_{2n-1}, S_{2n}) + \phi(S_{2n-1}, T_{2n}).
\]
\textsuperscript{1} Suppose that \( d(S_3, T_3) \geq 0 \). Then (iii) implies that as \( n \to \infty \), we have
\[
d(S_n, T_n) \leq k(d(S_3, T_3), 0, d(S_3, T_3), d(T_3, S_3), 0),
\]
\[
\leq r(d(S_3, T_3)),
\]
\[
< d(S_n, T_n),
\]
a contradiction. Therefore, \( S_3 = T_3 \).

Now we shall prove that \( A_3 = S_3 \). To do this, consider the inequality
\[
d(SA_{2n+1}, A_3) \leq d(SA_{2n+1}, SA_{2n+1}) + d(A_{2n+1}, A_3)
\]
Again using (ii) and the weak commutativity of \( \{A_n, S\} \), we have
\[
d(SA_{2n+1}, A_3) \leq d(SA_{2n+1}, A_3) + d(A_{2n+1}, T_3),
\]
\[
d(SA_{2n+1}, A_3) + d(SA_{2n+1}, A_3),
\]
\[
d(T_3, A_3) + d(SA_{2n+1}, A_3),
\]
Taking \( n \to \infty \), we are left with
\[
d(S_3, A_3) \leq \frac{1}{4}(d(S_3, T_3), d(S_3, S_3), d(S_3, A_3), d(T_3, S_3), d(T_3, A_3)).
\]
\[
d(z, z) = d(0, 0, d(z, z), 0, d(z, z)) \\
\leq 2d(z, z),
\]
giving thereby \( z = Az \). Thus \( Az = z = Tz \). It now follows that
\[
d(Az, Ax_{2n}) = d(d(Sz, Tx_{2n}), d(Sz, z), d(Sz, Ax_{2n}), d(Tx_{2n}, Ax_{2n}))).
\]
Then as \( n \to \infty \), we get
\[
d(Az, z) \leq d(d(z, z), 0, d(z, z), d(z, z), 0),
\]
\[
\leq d(z, z),
\]
a contradiction, and therefore \( Az = z = Sz = Tz \). Thus \( z \) is a common fixed point of \( A, S \) and \( T \).

The unicity of the common fixed point is not hard to verify. This completes the proof.

**Remark.** As noted in Fisher [38], the proof of Theorem 3.2.2 reflects that the continuity of the mapping \( \varphi \) is unnecessarily stringent in Fisher [37].

The following is an immediate consequence of Theorem 3.2.2.

**Corollary 3.2.3.** Let \( F \) be a family of self-mappings on \( X \). Suppose there are two continuous mappings \( S \) and \( T \) in \( F \) such that to each mapping \( A \) in \( F \), the conditions (1) and
(ii) hold for all \( x \) and \( y \) in \( A \). Then the family \( F \) has a unique common fixed point.

As a partial generalization of Theorem 3.2.2 we have the following result.

**Theorem 3.4.4.** Let \( A, B, T \) and \( I \) be three self-mappings on \( A \) satisfying the following:

(i) \( A = B, \quad A = T, \quad A(x) \subseteq S^p(x) \cap T^q(x) \), where \( p \) and \( q \) are some positive integers,

(ii) there exists a positive integer \( n = n(A) \) and a \( \delta = \delta(A) \) such that for all \( x, y \in A \)

\[
\delta(A^nx, A^ny) \leq \delta(S^p x, T^q y), \delta(S^p x, A^nx), \delta(S^p x, A^ny), \delta(T^q y, A^nx), \\
\delta(T^q y, A^ny)
\]

(iii) \( S^p \) and \( T^q \) are continuous.

Then \( A, B, T \) and \( I \) have a unique common fixed point.

**Proof.** The same as that of Theorem 3.2.3.

3.3. **ILLUSTRATIVE EXAMPLES:**

In this section we furnish examples to discuss the validity of the hypotheses and degree of generality of Theorem 3.4.4. *Example 3.3.1* below exhibits the mappings \( A, B, T \) satisfying the hypotheses of Theorem 3.4.4.
Example 3.3.1. Consider $\mathbb{X} = [c, 1]$ with the usual metric.

Define $A x = \frac{x}{2 + x}$, $S x = \frac{x}{2}$ and $I x = \frac{3x}{4}$ for every $x \in \mathbb{X}$.

Let $f(t_1, t_2, t_3, t_4, t_5) = \frac{1}{5}(t_1 + t_2 + t_3 + t_4 + t_5)$.

Then $A(x) = [c, \frac{1}{2}] \subset [c, \frac{1}{2}] \cap [c, \frac{3}{4}] = S(x) \cap T(x)$.

Also for every $x \in \mathbb{X}$, we have

$$d(Ax, Sx) = \frac{3x}{6 + 3x} = \frac{3x}{8 + 4x} \leq d(Ax, Tx).$$

The weak commutativity of $(A, S)$ is verified in Example 1.5.2.

Also for all $x, y \in [c, 1]$, it is easy to verify the condition (ii) of Theorem 3.2.2.

Thus all the conditions of Theorem 3.2.2 are satisfied and $c$ is the unique common fixed point of $A$, $S$ and $T$.

Clearly the involved maps are not commutative.

Example 3.3.2. This example shows that Theorem 3.2.2 is stronger than the main theorem of Fisher [37] for commutative maps.

Let $\mathbb{X} = \mathbb{R}$, the set of reals endowed with the Euclidean metric. Define $A$, $S$, $T : \mathbb{X} \to \mathbb{X}$ as follows:
Let $\theta : (\mathbb{R}^+)^5 \to \mathbb{R}$ be given by

$$
\theta(t_1, t_2, t_3, t_4, t_5) = \begin{cases} 
\frac{t_1}{1+t_1} & \text{if } 0 \leq t_1 \leq 1, \\
\frac{1}{2} t_1 & \text{if } t_1 > 1.
\end{cases}
$$

Then $S$, $T$, and $\theta$ are continuous with $\theta$ non-decreasing. Also, $A(x) = [0, \frac{1}{2}] \subset [0, 1] \cap [0, +\infty] = S(X) \cap T(x)$. Further, by a routine calculation we can verify that $\{A, T\}$ and $\{A, S\}$ are commuting pairs. Note that here

$$
\gamma(t) = \theta(t, t, t, t, t) = t, t_1 + 2 = 3.
$$

Now we shall discuss different possibilities in the following manner.

**Case 1.** If $x \leq 0$, $y \leq 0$, then $d(Ax, Ay) = 0 = \gamma(d(Sx, Ty))$.

**Case 2.** If $x \leq 0$, $0 < y \leq 1$, then $d(Ax, Ay) = \frac{y}{1+\gamma} = \gamma(y) = \gamma(d(Sx, Ty))$. 

Case 3. For $x < 0, y > 1$, we get $d(Ax, Ay) = \frac{1}{2} < \frac{1}{2} y = \gamma(d(Sx, Ty))$.

Case 4. For $0 < x \leq 1, 0 < y \leq 1$, we get $d(Ax, Ay) =$

\[
\frac{x}{x+1} \cdot \frac{y}{y+1} = \frac{\lvert x-y \rvert}{(x+1)(y+1)} \leq \frac{\lvert x-y \rvert}{1+\lvert x-y \rvert} = \gamma(\lvert x-y \rvert) \gamma(d(Sx, Ty)), \text{because} \lvert x-y \rvert < 1.
\]

Case 5. For $0 < x \leq 1$, we have $Ax = \frac{x}{x+1}, Ay = \frac{1}{2}, Sx = x, Ty = y$

with the following chain of implications:

\[
\begin{align*}
x+1 < 2 & \implies \frac{x}{2} < \frac{x}{x+1} \implies \frac{x}{x+1} < 1 \\
2 & \implies \frac{x}{x+1} < \frac{1}{2} < \frac{1}{2} \frac{1}{y-x} \implies \frac{1}{2} < \frac{1}{2} \frac{1}{y-x} \text{ with } y > 1.
\end{align*}
\]

Then for $y-x > 1$, we deduce that $d(Ax, Ay) = \frac{1}{2} - \frac{1}{x+1} < \frac{1}{2}(y-x)$

\[
= \gamma(d(Sx, Ty)).
\]

If $y-x \leq 1$, we find

\[
\begin{align*}
1 + y - x & \leq 2 \implies \frac{1}{1+y-x} \geq \frac{1}{2} \\
\frac{y-x}{1+y-x} & \leq \frac{(y-x)}{2} \\
\frac{1+y-x}{2} & \geq \frac{1}{2} \frac{y-x}{1+y-x}
\end{align*}
\]

and therefore

\[
d(Ax, Ay) = \frac{1}{2} \frac{1}{1+y-x} \leq \frac{y-x}{1+y-x} \leq \gamma(d(Sx, Ty)).
\]

Case 6. Lastly, if $x > 1, y > 1$, we get
Thus all the assumptions of Theorem 3.2.2 are satisfied but the theorem of Fisher [37] can not be applied because for 
$x = 0$ and $0 < y < 1$, Fisher's theorem would yield

$$d(Ax, Ay) = \frac{y}{1+y} \leq c d(Sx, Ty) = ay.$$ 

which, for $y \to 0$, gives $c > 1$, a contradiction to the assumption $c < 1$.

The ideas of these examples appear in Sessa [112].

The contents of this section have appeared in Khan-Imdad [77].

3.4. SELF-MAPPINGS ON SEPARATED L-SPACES:

After a careful examination, it is found that the Banach Contraction Principle and several other fixed point theorems do not actually require all the metric properties, particularly the axiom of triangular inequality in their proofs by the method of iteration. With this in mind, Kasahara [68] has published a very useful treatment of the fixed point theory by introducing the notion of L-spaces. We observe that our results, with some restriction on $\Psi$, can be extended to the case when $x$ is an L-space. As a sample, we state the following result in which $\Psi$ denotes a family of mappings $\Psi: (R^+)^3 \rightarrow R^+$. 
with \( \theta \) upper semi-continuous and non-decreasing in each coordinate variable. Let \((x, \to)\) denote a separated L-space.

**Theorem 3.4.1.** Let \( A, S \) and \( T \) be three continuous self-mappings on \((X, \to)\) satisfying (1). Suppose that \((X, \to)\) is \( d \)-complete for some semi-metric \( d \) on \( X \), and there exists a \( \Phi \) such that for all \( x, y \in X \)

\[
d(ax, ay) < \Phi(d(ax, Ty), d(ax, ax), d(Ty, ay))
\]

where \( \Phi \) satisfies the condition: for \( t > 0 \)

\[
\Phi(t, t, t) < 1.
\]

Then \( A, S \) and \( T \) have a unique common fixed point.

**Remarks.** (1) As in Theorem 3.2.2 we can avoid the continuity of the mapping \( A \) in Theorem 3.4.1 provided \( d \) is continuous.

(2) Results due to Kasahara [57], and Tewari and Singh [128] can be regarded as particular cases of Theorem 3.4.1.

3.5. **Convergence of Sequences and Their Fixed Points:**

The theorems of the type presented in this section concerning the convergence of sequences and their common fixed points for mappings satisfying condition (ii) of Theorem 3.2.2 have not been seen before by the author. Our results are indeed generalizations of those due to Husain-Sehgal [55] and Iseki [58]. Throughout this section we follow the notations of Section 3.2.
Theorem 3.3.1. Let \( \{A_n\} \), \( \{S_n\} \) and \( \{T_n\} \) be sequences of self-mappings on \( X \) converging uniformly to self-mappings \( A \), \( S \) and \( T \) on \( X \) where \( S \) and \( T \) are continuous.

Suppose that for each \( n \geq 1 \), \( x_n \) is a common fixed point of \( A_n \) and \( S_n \), and \( y_n \) is a common fixed point of \( A_n \) and \( T_n \). Further, let \( A \), \( S \) and \( T \) satisfy conditions (i) and (ii) of Theorem 3.2.2. If \( x_0 \) is the common fixed point of \( A \), \( S \) and \( T \), and \( \sup d(x_n, x_0) < \infty \) and \( \sup d(y_n, x_0) < \infty \) then \( x_n \to x_0 \) and \( y_n \to x_0 \).

Proof. \( x_0 \) is indeed the unique common fixed point of \( A \), \( S \) and \( T \). Since \( A_n \to A \) and \( S_n \to S \) uniformly, we have

\[
d(A_n x_n, Ax_n) = d(x_n, Ax_n) \to 0
\]

and

\[
d(S_n x_n, Sx_n) = d(x_n, Sx_n) \to 0.
\]

Now using inequalities

\[
d(S_n x_n, x_0) \leq d(S_n x_n, Sx_n) + d(x_n, x_0)
\]

\[
d(S_n x_n, Ax_n) \leq d(S_n x_n, Sx_n) + d(A_n x_n, Ax_n)
\]

and

\[
d(x_0, Ax_n) \leq d(x_0, x_n) + d(A_n x_n, Ax_n)
\]

it follows from (ii) that

\[
d(x_n, x_0) \leq d(x_n, Ax_n) + d(A_n x_n, Ax_0)
\]

\[
\leq d(x_n, Ax_n) + \frac{1}{2} (d(S_n x_n, Tx_0) + d(S_n x_n, Ax_n), d(S_n x_n, Ax_0))
\]

\[
= d(Tx_0, Ax_n) + d(Tx_0, Ax_0)
\]
or \( d(x_n, x_0) \leq d(x_n, Ax_n) + \theta(d(x_n, x_0), d(Sx_n, Ax_n), d(Sx_n, x_0), d(x_0, Ax_n)) \).

Let \( \varepsilon = \lim \sup d(x_n, x_0) \). Then the above inequality implies that

\[
\varepsilon \leq \theta(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) = \gamma(\varepsilon) < \varepsilon,
\]

and hence \( \varepsilon = 0 \), and consequently, \( x_n \to x_0 \). Similarly, we can show that \( y_n \to y_0 \). This completes the proof.

Remark. If \( S_n = T_n = I_x \) for each \( n \) in Theorem 3.3.1 then we get Theorem 3 of Husain and Sehgal [55] which is in turn an improvement of Theorem 2 of Iseki [56].

**Theorem 3.3.2.** Let \( \{A_n\}, \{S_n\} \) and \( \{T_n\} \) be sequences of self-mappings on \( X \) such that for each \( n \), \( S_n \) and \( T_n \) are continuous, \( A_n \), \( S_n \) and \( T_n \) satisfy conditions (i) and (ii) of Theorem 3.2.2. Let \( x_n \) be the unique common fixed point of \( A_n \), \( S_n \) and \( T_n \) for each \( n \). Let \( A \), \( S \) and \( T \) be self-mappings on \( X \) such that \( A_n \to A \), \( S_n \to S \) and \( T_n \to T \). If \( x_0 \) is any cluster point of the sequence \( \{x_n\} \) then \( Ax_0 = Sx_0 = Tx_0 = x_0 \).

**Proof.** There is a subsequence \( \{x_{n_1}\} \) of \( \{x_n\} \) such that \( x_{n_1} \to x_0 \). Since \( S_{n_1} \) and \( T_{n_1} \) are continuous for each \( i \geq 1 \), \( x_{n_1} \to S_{n_1} x_0 \) and \( x_{n_1} \to T_{n_1} x_0 \). This implies that for each \( i \geq 1 \), \( x_0 = S_{n_1} x_0 = T_{n_1} x_0 \). Also the convergence of \( T_{n_1} \) to \( T \) and \( S_{n_1} \) to \( S \) imply that as \( i \to \infty \).
Now for each $i \geq 1$, we have

$$d(x_o, Ax_o) \leq d(x_o, x_1) + d(Ax_1, Ax_o) + d(Ax_o, Ax_1).$$

By (ii) we have

$$d(x_1, Ax_1) \leq d(x_1, x_1) + d(Ax_1, Ax_1) + d(Ax_1, Ax_1).$$

Furthermore, for each $i \geq 1$,

$$d(x_1, Ax_1) \leq d(x_1, x_1) + d(x_1, Ax_1) + d(Ax_1, Ax_1).$$

and

$$d(x_1, Ax_1) \leq d(x_1, Ax_1) + d(Ax_1, Ax_1).$$

Therefore, as $i \to \infty$ we have

$$d(x_o, Ax_o) \leq d(x_o, Ax_o) + d(Ax_1, Ax_1).$$

which gives $x_o = Ax_o$. Thus $x_o$ is a common fixed point of $A$, $S$ and $T$, as required.
Remark. Taking \( s_n = t_n = t_x \) for each \( n \) in Theorem 3.5.2, we get Theorem 4 of Husain and Sehgal [55] which generalizes Theorem 3 of Iséki [58].

The contents of the last two sections form a part of Khan-Imdad [79].

3.6. COMMON FIXED POINTS IN UNIFORMLY CONVEX BANACH SPACES:

This section offers a Banach space version of Theorem 3.2.2. For this kind of work one can see Bose [8] which improves the results of Husain and Sehgal [55], Iséki [58] and Srivastava and Gupta [126] by taking as domain of two mappings a closed convex subset of a uniformly convex Banach space. Our result further refines Theorem 3.2.2. by relaxing the hypothesis of convexity on the domain.

Throughout this section we follow the notations of Section 3.2 except that here \( X \) stands for a Banach space.

The following lemma due to Goebel, Kirk and Shimi [50] is crucial in proving our result.

Lemma 3.6.1. Let \( X \) be uniformly convex and \( B_r \) the closed ball in \( X \) centred at the origin with radius \( r > 0 \). If \( x_1, x_2, x_3 \in B_r \) such that

\[
\| x_1 - x_2 \| \geq \| x_2 - x_3 \| \geq d > 0
\]

and

\[
\frac{1}{2} \| x \| \geq (1 - \frac{1}{2} \delta(\frac{d}{r})) \cdot r
\]
then \[ \|x_1 - x_3\| \leq \eta(1 - \frac{1}{2}(\frac{d}{\delta})), \|x_1 - x_2\|, \]

where \(\delta\) denotes the modulus of convexity of \(x\) and \(\eta\) the inverse of \(\delta\).

**Theorem 3.5.2.** Let \(x\) be a uniformly convex and \(K\) a nonempty closed subset of \(x\). Let \(A, S\) and \(T\) be three self-mappings of \(K\) satisfying the following conditions:

(a) \(S\) and \(T\) are continuous, \(A(K) \subseteq S(K) \cap T(K)\),

(b) \([A, S]\) and \([A, T]\) are weakly commuting pairs on \(K\),

(c) there exists a function \(\theta \in \Phi\) such that for every \(x, y \in x\)

\[ \|Ax - Ay\| \leq \theta(\|Ax - Ty\|, \|Ax - Ax\|, \|Ax -Ay\|, \|Ty - Ax\|, \|Ty - Ay\|) \]

where \(\theta\) satisfies the additional conditions:

(d) for \(t > 0\), \(\theta(t, t, 0, t, t) \leq ft\) and \(\theta(t, t, 0, t, t) \leq ft\),

here \(f < 1\) for \(a < 2\), and \(f = 1\) for \(a = 2\), with \(f, \alpha \in \mathbb{R}\),

(e) \(\theta(t, 0, t, t, t) < t\), for \(t > 0\).

Then there exists a \(z \in K\) such that \(z\) is the unique common fixed point of \(A, S\) and \(T\).

**Proof.** Let \(x_0\) be arbitrary in \(K\). Since \(A(K) \subseteq S(K) \cap T(K)\), we can always define a sequence \([Ax_n]\) such that \(Tx_{2n+2} = Ax_{2n+1}\) and \(Sx_{2n+2} = Ax_{2n}\) for \(n = 0, 1, 2 \ldots \).
Using the notations of Theorem 3.2.2 let us put

\[ d_n = \|Ax_n - Ax_{n+1}\|, \quad n = 0,1,2, \ldots \]

Now for an even \( n \) we have

\[ d_n = \|Ax_n - Ax_{n+1}\| = \|Ax_{n+1} - Ax_n\| \leq \|(Ax_n - Ax_{n+1})(Ax_{n+1} - Ax_n)\|, \quad 0 \]

which implies

\[ d_n \leq \|d_{n-1}, d_n, 0, d_{n-1} + d_n, d_{n-1}\|.

Similarly, for an odd \( n \), we obtain

\[ d_n \leq \|d_{n-1}, d_n, 0, d_{n-1} + d_n, 0, d_n\|.

Let us assume (on contrary) \( d_n > d_{n-1} \) for some \( n \geq 1 \).

Then \( d_{n-1} + d_n = ad_n \) with \( a < 2, \quad a \in \mathbb{R}^+ \).

Since \( \| \) is nondecreasing in each co-ordinate variable, we have

\[ d_n \leq \begin{cases} \|d_n, d_n, 0, d_n, d_n\| & \text{if } n \text{ is even,} \\ \|d_n, d_n, d_n, 0, d_n\| & \text{if } n \text{ is odd.} \end{cases} \]

In both the cases by (d) we get \( d_n \leq fd_n \) which gives a contradiction if \( f < 1 \). So \( d_n \leq d_{n-1} \) for \( n = 1,2, \ldots \).

Without loss of generality, we can postulate that the origin of \( x \) belongs to \( K \) and \( 0 < \gamma = \limsup \|Ax_n\| \).
Let us choose \( \gamma \in \mathbb{R}^+ \) in such a way that \( \gamma' < \gamma \) and
\[
\gamma \left( 1 - \frac{1}{2} \left( \frac{d}{\gamma} \right) \right) < \gamma'.
\]
Certainly there exists a sequence \( \{n(k)\}, \ k = 0, 1, 2, \ldots, \ n(0) \geq 1 \) of positive integers such that
\[
\|A_n(k)\| \geq \gamma \left( 1 - \frac{1}{2} \left( \frac{d}{\gamma} \right) \right), \quad \text{whereas}
\]
\[
\|A_n(k)\| < \gamma \quad \text{for any} \ n \geq n(0).
\]

Since \( d_n(k)-1 \geq d > 0 \) for \( k = 0, 1, 2, \ldots, \) from Lemma 3.6.1 it follows that
\[
\|A_n(k)-1\| = \|A_n(k+1)\| \leq \gamma' \|A_n(k)\|.d_n(k)-1 \quad \text{where}
\]
\[
\gamma' \gamma < 2 \quad \text{being} \ \gamma' \gamma < 1.
\]

Then using (a), (b) and (c), we have
\[
\frac{\gamma \left( \frac{d}{\gamma} \right)}{\gamma' \gamma} \cdot d_n(k)-1 \cdot \gamma' \gamma \cdot d_n(k)-1 \cdot d_n(k)-1 \quad \text{if} \ n(k) \text{is even}
\]
\[
\frac{\gamma \left( \frac{d}{\gamma} \right)}{\gamma' \gamma} \cdot d_n(k)-1 \cdot \gamma' \gamma \cdot d_n(k)-1 \cdot d_n(k)-1 \quad \text{if} \ n(k) \text{is odd}.
\]

In both the cases (d) implies \( d_n(k) \leq \gamma' \gamma \cdot d_n(k)-1 \).

Observing that \( \gamma' \gamma \) does not depend on \( k \), the foregoing inequality, as \( k \to \infty \), yields \( d \leq \gamma' \gamma \), and thus \( d = 0 \).

The rest of the proof works on the lines of the proof of Theorem 3.4.2. This ends the proof.

Remark. It is trivial to note that condition (iii) of Theorem 3.4.2 can be restricted by the conditions (d) and (e).

Example 3.6.3. We again reframe Example 3.3.1 to demonstrate Theorem 3.6.2.
Let \( A \) be the set of reals with Euclidean norm and \( K = [0,1] \), and \( A, S : K \to K \) are defined by \( Ax = \frac{x}{4+x} \) and \( Sx = \frac{x}{2+x} \) for any \( x \in K \). Then for any \( x \in K \)

\[
\|SAx - ASx\| \leq \left| \frac{x}{8+3x} - \frac{x}{8+5x} \right| = \frac{2x^2}{(8+3x)(8+5x)}.
\]

So the pair \( \{A,S\} \) is weakly commuting but not commuting because \( SAx \neq ASx \) for every \( x \in K = [0] \).

Clearly \( S \) is continuous and \( A(K) = [0,1/3] \subset C[0,1/3] = 3(K) \). Assuming \( \Phi \in \mathcal{F} \) as \( \Phi(t_1,t_2,t_3,t_4,t_5) = \frac{t_1}{1+t_1} \) for every \( (t_1,t_2,t_3,t_4,t_5) \in \mathbb{R}^5 \), we have, for any \( x,y \in K \),

\[
\|Ax - Ay\| = \frac{4\|x-y\|}{(4+x)(4+y)} \leq \frac{2\|x-y\|}{2\|x-y\| + (2+x)(2+y)}.
\]

Since \( \Phi \) obviously satisfies conditions (d) and (e), all the assumptions of Theorem 3.6.2 are satisfied and \( \Phi \) is the unique common fixed point of \( A, S \) and \( T \).