CHAPTER-III

AN ALGEBRAIC STUDY OF A CLASS OF INTEGRAL FUNCTIONS

3.1 INTRODUCTION. In 1969, Sen [69] studied the topological and algebraic structure of the set of all complex valued functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), where \( |a_n| \) is bounded.

Let \( R \) denote the set of all complex valued functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), where \( \lbrack n \rbrack! |a_n| \) is bounded. Here \( \lbrack n \rbrack! \) denotes the q-factorial function given by

\[
\lbrack n \rbrack! = \prod_{\ell=1}^{n} [\ell],
\]

where for the real number \( \ell < q < 1 \) the q-number \( \lbrack n \rbrack \) is defined as

\[
\lbrack n \rbrack = \frac{1-q^n}{1-q}.
\]

The number \( \lbrack n \rbrack \rightarrow n \) as \( q \rightarrow 1 \).

It can easily be verified that the elements of \( R \) are all integral functions. In \( R \) we define addition and multiplication as follows:

\[
f(z) + g(z) = \sum_{n=0}^{\infty} \left( a_n + b_n \right) z^n
\]

and

\[
f(z) \cdot g(z) = \sum_{n=0}^{\infty} \lbrack n \rbrack! a_n b_n z^n
\]
where \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) are two elements of \( R \).

In this chapter a study has been made of the topological and algebraic structure of \( R \).

\( \S 3.2 \). In this section, we are going to prove that \( R \) is a commutative ring with identity element.

**Lemma 1.** \( R \) is closed with respect to the two operation '+' and 'o' given by (3.1.3) and (3.1.4).

**Proof.** Since \( \binom{n}{\lfloor n \rfloor} \frac{|a_n b_n|}{\lfloor n \rfloor} \leq \frac{\binom{n}{\lfloor n \rfloor} |a_n| + \binom{n}{\lfloor n \rfloor} |b_n|}{\ln n} \) therefore \( f(z) + g(z) \) is an element of \( R \) when \( f(z) \in R \) and \( g(z) \in R \).

Again \( \binom{n}{\lfloor n \rfloor} \frac{|a_n b_n|}{\lfloor n \rfloor} \leq \frac{\binom{n}{\lfloor n \rfloor} |a_n|}{\lfloor n \rfloor} \frac{\binom{n}{\lfloor n \rfloor} |b_n|}{\ln n} \). This implies that when \( f(z) \) and \( g(z) \) are two elements of \( R \), then \( f(z) \cdot g(z) \in R \).

Hence the lemma.

**Lemma 2.** \( e_q (1-q)z = \sum_{n=0}^{\infty} \frac{z^n}{\binom{n}{\lfloor n \rfloor}} \) is the identity element of \( R \).

**Proof.** It is obvious that \( e_q (1-q)z \) is an element of \( R \).

Now for \( f(z) \in R \), we have
\[ e_q ((1-z))of(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \cdot \sum_{n=0}^{\infty} a_n z^n \]

\[ = \sum_{n=0}^{\infty} \frac{[n]!}{[n]!} a_n z^n \]

\[ = \sum_{n=0}^{\infty} a_n z^n \]

\[ = f(z). \]

Similarly,

\[ f(z) \cdot e_q ((1-q)z) = \sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \]

\[ = \sum_{n=0}^{\infty} \frac{[n]!}{[n]!} a_n \frac{1}{[n]!} z^n \]

\[ = \sum_{n=0}^{\infty} a_n z^n \]

\[ = f(z). \]

Hence \( e_q ((1-q)z) \) is the identity \( * \) element of \( R \).

We remark that the remaining ring axioms are an complex fields—Hence we have the following theorem.

**Theorem.** \( R \) is a commutative ring with identity.

\( \S \) 3.3. In this section we are going to point out some of the
interesting properties of certain elements of \( R \). From what follows notations of the type \( E_q,^n((1-q)z) \) will mean

\[
E_q,^n((1-q)z) \equiv E_q((1-q)z) \circ E_q((1-q)z) \cdots \circ E_q((1-q)z)
\]

Since \( \sin_q((1-q)z), \sin_q((1-q)z), \cos_q((1-q)z), \cos_q((1-q)z), \sin_h_q((1-q)z), \sin_h_q((1-q)z), \cos_h_q((1-q)z), \cos_h_q((1-q)z) \) \( E_q((1-q)z) \) and \( E_q((1-q)z) \) are all elements of \( R \), for these elements, we have

(i) \( \cos_q,^n((1-q)z) + \sin_q,^n((1-q)z) = e_q((1-q)z) \)

(ii) \( \cos_q,^n((1-q)z) - \sin_q,^n((1-q)z) = e_q(-(1-q)z) \).

Now by \((1.3.17)\)

\[
\cos_q,^n((1-q)z) = \cos_q((1-q)z) \circ \cos_q((1-q)z)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{[2n]!} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{[2n]!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{[2n]!} \frac{(-1)^n z^{2n}}{[2n]!}
\]

\[
= \sum_{n=0}^{\infty} \frac{z^{2n}}{[2n]!} \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (3.3.1)
\]

Again, by \((1.2.16)\)
\[\sin_q, _2((1-q)z) = \sin_q((1-q)z) \cdot \sin_q((1-q)z)\]

\[= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{[?n+1]!} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{[2n+1]!}\]

\[= \sum_{n=0}^{\infty} \frac{2^{2n+1}}{[?n+1]!} \]

\[\text{(3.3.2)}\]

From (3.3.1) and (3.3.2) the results follows.

(iii) \[\cos_q((1-q)z) \cdot \cos_q((1-q)z) + \sin_q((1-q)z) \cdot \sin_q((1-q)z)\]

\[= E_q(-(1-q)z)\]

(iv) \[\cos_q((1-q)z) \cdot \cos_q(-(1-q)z) - \sin_q((1-q)z) \cdot \sin_q((1-q)z)\]

\[= E_q((1-q)z)\]

(v) \[\cos_q, _2((1-q)z) + \sin_q, _2((1-q)z) = \Phi_{\text{ee}} \left[ \begin{array}{c} - ; (1-q)z/q^2 \\ \end{array} \right] \]

(vi) \[\cos_q, _2((1-q)z) - \sin_q, _2((1-q)z) = \Phi_{\text{eo}} \left[ \begin{array}{c} - ; -(1-q)z/q^2 \\ \end{array} \right] \]

(vii) \[\cos_q, _n((1-q)z) + \sin_q, _n((1-q)z) = e_q((1-q)z) ; \]

\[\cos_q, _n((1-q)z) - \sin_q, _n((1-q)z) = e_q(-(1-q)z),\]
when \( n \) is an even positive integer.

\[(viii) \ \cos_{q,n}(1-qz) + \sin_{q,n}(1-qz) = \phi_{q,n}\left[\begin{array}{c} -(1-qz)/q^n \\ q^n \end{array}\right]\]

\[\cos_{q,n}(1-qz) - \sin_{q,n}(1-qz) = \phi_{q,n}\left[\begin{array}{c} -(1-qz)/q^n \\ q^n \end{array}\right]\]

\[(ix) \ \sin_{q,m}(1-qz) \cdot \sin_{q,m}(1-qz) + \cos_{q,m}(1-qz) = \phi_{q,m}\left[\begin{array}{c} -(1-qz)/q^m \\ q^m \end{array}\right]\]

\[\cos_{q,m}(1-qz) \cdot \cos_{q,m}(1-qz) - \sin_{q,m}(1-qz) \cdot \sin_{q,m}(1-qz) = \phi_{q,m}\left[\begin{array}{c} -(1-qz)/q^m \\ q^m \end{array}\right]\]

where \( m \) is any positive integer.

\[(x) \ \sin_{q}(1-qz) \cdot \cos_{q}(1-qz) = c \]

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\[\sin_{q}(1-qz) \cdot \cos_{q}(1-qz) = c \]
(xi) \( \sin h_{q,n}((1-q)z) + \cos h_{q,n}((1-q)z) = e_q((1-q)z) \),
\( \cos h_{q,n}((1-q)z) - \sin h_{q,n}((1-q)z) = e_q(-(1-q)z) \),
where \( n \) is any positive integer.

(xii) \( \sin h_{q}((1-q)z) \) \( \sin h_{q}((1-q)z) + \cos h_{q}((1-q)z) \)
\( \cos h_{q}((1-q)z) = E_q((1-q)z) \),
\( \cos h_{q}((1-q)z) - \sin h_{q}((1-q)z) \)
\( \sin h_{q}((1-q)z) = E_q(-(1-q)z) \)

(xiii) \( \cos h_{q,m}((1-q)z) \) \( \cos h_{q,m}((1-q)z) + \sin h_{q,m}((1-q)z) \)
\( \cos h_{q,m}((1-q)z) = E_q(-(1-q)z) \)
\( \sin h_{q,m}((1-q)z) = \Phi \left[ \begin{array}{c} - \n \end{array} \right] \left( \frac{(1-q)z}{q^m} \right) \)
\( \cos h_{q,m}((1-q)z) \) \( \cos h_{q,m}((1-q)z) - \sin h_{q,m}((1-q)z) \)
\( \sin h_{q,m}((1-q)z) \)
\( \Phi \left[ \begin{array}{c} - \n \end{array} \right] \left( \frac{(1-q)z}{q^m} \right) \)
\( \Phi \left[ \begin{array}{c} - \n \end{array} \right] \left( \frac{(1-q)z}{q^m} \right) \)
where \( m \) is any positive integer.

(xiv) \( \cos h_{q,m}((1-q)z) + \sin h_{q,m}((1-q)z) = E_q((1-q)z) \)
\( \Phi \left[ \begin{array}{c} - \n \end{array} \right] \left( \frac{(1-q)z}{q^m} \right) \)}
\[ \cosh q^m (1-q)z - \sinh q^m (1-q)z = \Phi_0 \begin{bmatrix} -; \frac{(-1)^m (1-q)z}{q^m} \\ \end{bmatrix} \]

\( m \) is any positive integer.

\((\pi)\) \( \sin q (1-q)z \) \( \cos q (1-q)z = 0 \)

\( \sin q (1-q)z \) \( \cos q (1-q)z = 0 \)

\( \sin q (1-q)z \) \( \cos q (1-q)z = 0 \)

\( \sin q (1-q)z \) \( \cos q (1-q)z = 0 \)

\((xvi)\) \( e_{q,m} (1-q)z = e_q (1-q)z \);

\[ E_{q,m} (1-q)z = \Phi_0 \begin{bmatrix} -; \frac{(-1)^n (1-q)z}{q^n} \\ \end{bmatrix} \]

where \( n \) is any positive integer.

\((xvii)\)

\[ \Phi_{1c} \begin{bmatrix} q^z ; (1-q)z \\ -; \end{bmatrix} \cdot \Phi_{1c} \begin{bmatrix} -; \end{bmatrix} \]

\[ = \sum_{n=0}^{\infty} \frac{(q^z)^n z^n}{[n]!} \cdot \sum_{n=0}^{\infty} \frac{z^n}{(1-q)^n [n]!} \]

\[ = \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \cdot \frac{1}{(1-q)^n} \frac{z^n}{[n]!} \]

\[ = e_q (1-q)z \]
Similarly,

\[
\Phi \left[ \begin{array}{c} q^{a}, (1-q)z \\ -, q^{1/2} \end{array} \right] \circ \Phi \left[ \begin{array}{c} -(1-q)z \\ q^{a} \end{array} \right] = \Phi \left[ \begin{array}{c} -, (1-q)z \\ q^{a} \end{array} \right]
\]

\[
\Phi \left[ \begin{array}{c} (1/4)n(n+1) \\ \Omega \end{array} \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{(q^{a})^{n} q^{(1/2)n(n+1)}}{n!} \frac{(1/4)n(n+1)}{(q^{a})^{n} [n]!} z^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(1/2)n(n+1)}{[n]!} z^n
\]

\[
= \Phi \left[ \begin{array}{c} q^{a}, (1-q)z \\ (b_{v}), q^{l_{1}} \end{array} \right] \circ \Phi \left[ \begin{array}{c} (c_{v}), (1-q)z \\ (a_{v}), q^{l_{2}} \end{array} \right]
\]

\[
= \Phi \left[ \begin{array}{c} -(1-q)z \\ q^{a} \end{array} \right] \circ \Phi \left[ \begin{array}{c} (c_{v}), (1-q)z \\ (a_{v}), q^{l_{1}+l_{2}} \end{array} \right]
\]

\[
= \sum_{n=0}^{\infty} \alpha \alpha_{n} z^{n}
\]

**3.4** In this section, we are going to show that \( R \) is a Banach algebra.

Let \( c \) denote that field of complex numbers. For \( a \in c \) and \( f(z) \in R \), scalar multiplication is defined as

\[
a f(z) = \sum_{n=0}^{\infty} \alpha a_{n} z^{n}
\]
now it can easily be verified that the following theorem holds:

**Theorem 1.** The set $R$ is a linear space over the field of complex numbers.

The next theorem is stated below.

**Theorem 2.** The set $R$ is a commutative algebra with the identity element.

**Proof.** We have

$$
\lambda(f(z)) g(z) = \lambda \left( \sum_{n=0}^{\infty} a_n z^n \right) \cdot \left( \sum_{n=0}^{\infty} b_n z^n \right)
$$

$$
= \lambda \left( \sum_{n=0}^{\infty} [n]! a_n b_n z^n \right)
$$

$$
= \sum_{n=0}^{\infty} [n]! (\lambda a_n) b_n z^n
$$

$$
= \sum_{n=0}^{\infty} (\lambda a_n) z^n \cdot \left( \sum_{n=0}^{\infty} b_n z^n \right)
$$

$$
= \lambda \sum_{n=0}^{\infty} a_n z^n \cdot \left( \sum_{n=0}^{\infty} b_n z^n \right)
$$

$$
= (\lambda f(z)) g(z).
$$

Hence the theorem follows from the results of § 3.2.
Theorem and § 3.4 theorem

Now define the norm of \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{R} \) by

\[
\| f(z) \| = \sup_n \left| \frac{a_n}{n!} \right|
\]

since \( \frac{1}{n!} a_n \) is bounded so \( \sup_n \left| \frac{a_n}{n!} \right| \) exists.

(i) Now \( \| f(z) \| = \sup_n \left| \frac{a_n}{n!} \right| \geq 0 \)

and \( \| f(z) \| = 0 \) iff \( \sup_n \left| \frac{a_n}{n!} \right| = 0 \)

i.e. \( \| f(z) \| = 0 \) for all \( n \).

i.e. iff \( f(z) = 0 \).

i.e. iff \( f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} 0 z^n = 0 \)

i.e. iff \( f(z) = 0 \).

(ii) \( \| f(z) + g(z) \| = \sup_n \left| \frac{a_n + b_n}{n!} \right| \)

\[
= \sup_n \left| \frac{a_n}{n!} \right| + \sup_n \left| \frac{b_n}{n!} \right|
\]

\( \leq \sup_n \left| \frac{a_n}{n!} \right| + \sup_n \left| \frac{b_n}{n!} \right| \)

i.e. \( \| f(z) + g(z) \| \leq \| f(z) \| + \| g(z) \| \)

i.e. \( \| f(z) + g(z) \| \leq \| f(z) \| + \| g(z) \| \)
(iii) \[ |a f(z)| = |\sum_{n=0}^{\infty} a_n z^n| \]
\[ = |\sum_{n=0}^{\infty} a_n z^n| \]
\[ = \sup_n (n!) |a_n| \]
\[ = \sup_n (n!) |a_n| \]
\[ = |a| \sup (n!) |a_n| \]
\[ = |a| |f(z)| \]

Hence we have the following result.

**Theorem 3.** \( R \) is a normed linear space.

Our next aim is to prove the following result.

**Theorem 4.** \( R \) is a Banach space.

**Proof.** Consider the sequence \( f_\lambda(z) \), where \( f_\lambda(z) = \sum_{n=0}^{\infty} a_{p_n} z^n \) is an element of \( R \).

Let the sequence be a Cauchy sequence. Hence there exists a positive integer \( p_\alpha \) for every \( \epsilon > 0 \) such that
\[ || f_p(z) - f_q(z) || < \varepsilon \text{ for } p, q \geq p_0. \]

Hence

\[ \sup_{n} [n]! |a_{pn} - a_{qn}| < \varepsilon \text{ for } p, q \geq p_0. \]

This implies that

\[ [n]! |a_{pn} - a_{qn}| < \varepsilon, \quad (3.4.1) \]

for \( p, q \geq p_0 \text{ and for every } n. \)

We regard \( n \) fixed and consider the sequence

\[ a_{1n}, a_{2n}, a_{3n}, \ldots, a_{mn}, \ldots. \]

On account of (3.4.1) this sequence would converge to a limit \( a_n \) (say) according to Cauchy test from (3.4.1)

\[ [n]! |a_n - a_{pn}| < \varepsilon, \quad (3.4.2) \]

for \( p \geq p_0 \text{ and for all } n. \)

Hence \( [n]! |a_n| \) is bounded for

\[ [n]! |a_n| = [n]! |a_n - a_{pn} + a_{pn}| \]

\[ \leq [n]! |a_n - a_{pn}| + [n]! |a_{pn}| \]
Then \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is an element of \( R \).

Now we find from (5.4.2) that

\[
\sup_n [n!] |a_n - a_{pn}| < \varepsilon, \text{ for } p > p_0
\]

that is

\[
|f_p(z) - f(z)| < \varepsilon, \text{ for } p > p_0
\]

Therefore \( f_p(z) \to f(z) \in R \) when \( p \to \infty \).

This implies that \( R \) is complete considering this result, the proof of the theorem follows from theorem 3.

**Theorem 3.** \( R \) is a commutative Banach algebra with identity element.

**Proof.** Let \( f(z) \) and \( g(z) \) be two elements of \( R \).

\[
|f(z) \circ g(z)| = \left| \sum_{n=0}^{\infty} [n!] a_n b_n z^n \right|
\]

\[
= \sup ( [n!]^2 |a_n b_n|)
\]

\[
\leq \sup_n [n!] |a_n| \sup |b_n|
\]

\[
|f(z) \circ g(z)| \leq \|f(z)\| \|g(z)\|.
\]
Again \( e_q(1-q)z \) is the identity element of \( R \). For this element \( e_q((1-q)z) \), we have

\[
||e_q((1-q)z)|| = \sup_n \left[ \frac{n!}{n^n} \right] = 1
\]

Hence the theorem.