2.1 INTRODUCTION. The Bessel polynomials were introduced by Krall and Frink [53] in connection with the solution of the wave equation in spherical coordinates. They are the polynomial solution of the differential equation

\[ x^2 y''(x) + (ax + b x^2) y'(x) = n (n-a-1) y(x) \]  \hspace{1cm} (2.1.1)

where \( n \) is a positive integer and \( a \) and \( b \) are arbitrary parameters. These polynomials are orthogonal on the unit circle with respect to the weight function.

\[ \int (x, x) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-1)} (-2/x)^n . \]  \hspace{1cm} (2.1.2)

Seven other authors, including ['3], ['4], ['5], ['6], ['7], ['8], ['9], ['10], [18], [19], [23], [24], [33], [34], [64], and [65] have contributed to the study of the Bessel polynomials.

In 1962, Aoki [2] defined \( c \)-Bessel polynomials and discussed some of the important properties. He defined this polynomial \( J(q; c, n; x) \) and defined it as

\[ J(q; c, n; x) = \frac{(q^n)_{-n}}{(q)_n} \Phi_1 \left(q^{-n}, q^{c \cdot n}; q^{-c}; x \right) \]  \hspace{1cm} (2.1.3)
Motivated by the above interesting works on Bessel and q-Bessel polynomials, Ismail [41], in a recent paper, made a detailed study of a basic analogue of the Bessel polynomials

\[ Y_n(x) = {}_2F_0 \left( -n, n+1; -; -\frac{x}{2} \right) \]  \hspace{1cm} (2.1.4)

and the generalized Bessel polynomials

\[ Y_n(x; a) = {}_2F_0 \left( -n, n+a-1; -; -\frac{x}{2} \right) \]  \hspace{1cm} (2.1.5)

His basic analogue is different from the one obtained by Abdi. In fact his polynomials are orthogonal, contrary to Abdi's polynomials, on the unit circle and arise from the basic Bessel functions in the same way the Bessel polynomials and their generalization arise from the ordinary Bessel functions. The simple basic Bessel polynomials turn out to be

\[ Y_n(x; q^2) = \frac{(l/2)^n(n-1)}{2\Phi_1} \left( q^{-n}, q^{n+1}; -q, q, -2xq \right) \]  \hspace{1cm} (2.1.6)

and are generalized to

\[ Y_n(x; a; q^2) = q^{n(n-1)/2} \frac{\Phi_1}{2\Phi_1} \left( q^{-n}, q^{n+a-1}; -q, q, -2xq \right) \]  \hspace{1cm} (2.1.7)

bearing the analogy to (2.1.4) and (2.1.5) respectively.
In 1975, Karande and Thakare [45] gave the unification of Bernoulli and Euler polynomials. In fact, they defined the polynomials \( D_n(x; a, k) \), where \( a \) is a non-zero real number and \( k \) is an integer, by the following generating relation

\[
2(\frac{t}{e^t - 1})^k e^{xt} = \sum_{n=0}^{\infty} D_n(x; a, k) \frac{t^n}{n!}
\] (2.1.8)

They also established relationships between their polynomials \( D_n(x; a, k) \) and Bernoulli and other polynomials.

The present chapter deals with the recurrence relations, characterizations and integral representations of q-Bessel polynomials of Abdi and the q-Bessel polynomials of Ismail.

The chapter also deals with the q-analogue of the polynomials given by Karande and Thakare to unify Bernoulli and Euler polynomials. The resulting q-polynomial unifies q-Bernoulli, q-Euler, q-Eulerian and q-Genocchi polynomials as well as q-Bernoulli, q-Euler and q-Genocchi numbers.

\section*{2.2 Definitions of q-Bessel Polynomials}

We shall adopt in this chapter a somewhat different notation from that used by Abdi for his non-orthogonal q-Bessel polynomials and by Ismail for his orthogonal q-Bessel polynomials.

In the notation of q-hypergeometric series, the q-Bessel
polynomials corresponding to Aodi's q-Bessel polynomials are given by

\[ J_{n,q}(x) = \sum_{k=0}^{n} \frac{(q^{-n})_k (q^{n+\alpha+1})_k}{(q)_k} x^k \]  \hspace{1cm} (2.2.1)

Thus

\[ J_{n,q}(x) = \frac{(q^n)_n}{(q^{1+\alpha})_n} J(q; l+\alpha, n; x) \]  \hspace{1cm} (2.2.2)

Similarly, the q-Bessel polynomials corresponding to Ismail's orthogonal q-Bessel polynomials, are given by

\[ Y_{n,-q}(x) = q^{(1/2)n(n-1)} \sum_{l=1}^{n} \frac{(q^{-n})_l (q^{n+\alpha-1})_l}{(q)_l (-q)_l} x^l \]  \hspace{1cm} (2.2.4)

Thus

\[ Y_{n,-q}(x) = Y_n(x; \alpha/q^2) \]  \hspace{1cm} (2.2.6)

§ 3.3 Recurrence Relations. From formulae (2.2.2) and (2.2.5) we see that
These suggest the difference formulae

\[ \Delta_{x} J_{n,q}^{(\alpha)}(x) = q^{n+\alpha+1}(1-q^{-n}) \times J_{n-1,q}^{(\alpha+2)}(x) \]  

(2.3.3)

and

\[ \Delta_{x} Y_{n,-q}^{(\alpha)}(x) = \frac{2x(1-q^{n})q^{n+\alpha}}{1+q} Y_{n-1,-q^{2}}^{(\alpha+2)}(x) \]  

(2.3.4)

where \( \Delta_{x} f(x) = f(x+1) - f(x) \).

The \( q \)-derivatives of the \( q \)-Bessel polynomials (2.2.1) and (2.2.4) are themselves \( q \)-Bessel polynomials of their respective kinds with the parameter increased by two. Indeed we find from formulae (2.2.2) and (2.2.3) the following

\[ (1-q) \nu_{q} J_{n,q}^{(\alpha)}(x) = (1-q^{-n})(1-q^{n+\alpha+1}) J_{n-1,q}^{(\alpha+2)}(x) \]  

(2.3.5)

and

\[ (1-q^{2}) \nu_{q} Y_{n,-q}^{(\alpha)}(x) = (-2)(1-q^{-n})(1-q^{n+\alpha})q^{n} Y_{n-1,-q^{2}}^{(\alpha+2)}(x) \]  

(2.3.6)
which can also be written as

$$\frac{1}{1-q} D_q J_{n,q}(x) = (-\alpha_j)_{n+\alpha_j+1} J_{n-1,q}(x) \quad (2.3.7)$$

and

$$\frac{1+q}{1-q} D_q, x \ y, n,-q \ x) = (-2\alpha_j)_{n+\alpha_j} c^n y \ (x+2) \ (x) \quad (2.3.8)$$

respectively.

From (2.3.3) and (2.3.5), we see that the q-Bessel polynomials (2.2.1) satisfy the mixed equation

$$\Delta_x J_{n,q}(x) = \frac{xq^{n+r}}{n+r+1} D_q J_{n,q}(x) \quad (2.3.9)$$

Similarly, from (2.3.4) and (2.3.6), we find that the q-Bessel polynomials (2.2.4), satisfy the mixed equation

$$\Delta_x J_{n, -q}(x) = \frac{xq^{-n+r}}{n-x} D_q J_{n, -q}(x) \quad (2.3.10)$$

The following recurrence relations can be verified directly:

$$J_{n+1,q}(x) - J_{n,q}(x) = (q^{n+1} - q^{-n+1}) x J_{n,q}(x) \quad (2.3.11)$$
and
\[ q^{-\alpha} \eta(\alpha+1) \frac{\eta(x)}{\eta(\alpha+1)} = q^{-\alpha} \eta(\alpha+1) \frac{\eta(x)}{\eta(\alpha+1)} \]
\[ = \frac{2x(q^{-\alpha} - q^{-\alpha+1})}{1 - q} \eta(x) \quad (2.3.12) \]

Now (2.3.11) and (2.3.12) can also be respectively written as
\[ \alpha, \beta \eta(\alpha+1) \frac{\eta(x)}{\eta(\alpha+1)} = (q^{\alpha} - q^{-\alpha+1}) x^{\eta+1} \eta(x) \eta(\alpha+1) \]
(2.3.13) and
\[ \eta(x) = \frac{2x(q^{-\alpha} - q^{-\alpha+1})}{1 - q} \eta(x) \quad (2.3.14) \]
where
\[ \eta(x) = \frac{2x(q^{-\alpha} - q^{-\alpha+1})}{1 - q} \]

§ 2.4 Characterizations. In this section we obtain some characterizations of the q-Jacobi polynomials (2.2.1) and (2.2.4), similar to those for (i) the Jacobi polynomials obtained by Carlitz [22] and (ii) the q-Jacobi polynomials obtained by Carlitz [9]. We prove here the following.

**Theorem.** Given a sequence \( \{ f_{\alpha}(x) \} \) of q-polynomials
in x where deg \( f_{n, q}^x(x) = n \), and \( x \) is a parameter, such that

\[
(1-q) \ D_q f_{n, q}^x(x) = (1-q^n)(1-q^{n+1}) f_{n-1, q}^x(x+2)
\]

and \( f_{n, q}^x(0) = 1 \).

Then \( f_{n, q}^x(x) = j_{n, q}^x(x) \).

**PROOF:** Assume \( f_{n, q}^x(x) = \sum_{k=0}^{n} a_{x, q}^x(i, n) x^i \).

Now by (2.4.1), we have

\[
a_{x, q}^x(i, n) = \frac{(1-q^n)(1-q^{n+1})}{(1-q^x)} a_{x-1, q}^x(i+2, n-1).
\]

Since \( f_{n, q}^x(0) = 1 \) so \( a_{0, q}^x(i, n) = 1 \). Consequently, we obtain

\[
a_{x, q}^x(i, n) = \frac{(q^n)(q^{n+1})}{(q)^i} a_{x-1, q}^x(i+2, n-1).
\]

which proves the theorem.

**Remark 1.3.** Given a sequence \( \{ f_{n, q}^x(x) \} \) of \( q \)-polynomials in \( x \) where deg \( f_{n, q}^x(x) = n \), and \( x \) is a parameter.
such that

\[(1-q^{-2}) D_q, x f_{n,-q}^{(n)}(x) = 2(1-q^n, 1-q^{n+1}) f_{n-1,-q,2}^{(n)}(x)\]

and \(f_{n,-q}^{(n)}(x) = q^{(1/2)n(n-1)}\) for all \(n\). Then

\[f_{n,-q}^{(n)}(x) = f_{n,-q}^{(n)}(x)\]

\[\text{Proof: Assume } f_{n,-q}^{(n)}(x) = q^{(1/2)n(n-1)} \sum_{k=0}^{n} a_{k,-q}^{(n)}(x) x^k\]

Now by (2.4.2), we have

\[A_{k,-q}^{(n)}(x,\alpha) = \frac{(-2\zeta, -q^{-n})(1-q^{n+1})}{(1-q^x)(1-q^x)} A_{k-1,-q}^{(n-1)}(x+2, n-1, 1-q^n)\]

since \(f_{n,-q}^{(n)}(x) = x^n\) for all positive integral values of \(n\), we have \(A_{0,-q}^{(n)}(x,\alpha) = 0\). Thus

\[A_{k,-q}^{(n)}(x,\alpha) = \frac{(-2\zeta, -q^{-n})(1-q^{n+1})}{(q^x)(-q^x)}\]

which proves the theorem.

Other characterizations are suggested by (2.3.3) and (2.3.4). Indeed, we have
**Theorem 2(a).** Given a sequence of $q$-functions $\{f_n(x)\}$ such that

$$\Delta_x f_n(x) = q^{n+1} (1-q^{-n}) x f_{n-1}(x)$$

(2.4.3)

and

$$f_n(0) = 1, f_0(x) = 1$$

(2.4.4)

Then $f_n(x) = f_n(x)$.

**Proof.** From (2.4.3) it is evident that $f_n(x)$ is a $q$-polynomial in $x$ of degree $n$. Hence we can write

$$f_n(x) = \sum_{r=0}^{n} \Lambda_r(n,x) \frac{(q^{n+1})^r}{(q)_r}$$

Hence (2.4.3) implies

$$\Lambda_r(n,x) = (q^{-n}) x^r.$$ 

This proves the theorem.

**Theorem 2(b).** Given a sequence of $q$-functions $\{f_n(x)\}$ such that
\[ \Delta_{\alpha} f_{n,-q}(x) = \frac{(-2xq)(1-q^{-n})q^{2n+\alpha-1}}{l+q} f_{n,-q^2}(x) \quad (2.4.5) \]

and

\[ f_{n,-q}(0) = q^{(l/2)n(n-1)} \quad f_{0,-q}(x) = 1 \quad (2.4.6) \]

Then \( f_{n,-q}(x) = y_{n,-q}(x) \).

**Proof:** From (2.4.5) it is evident that \( f_{n,-q}(x) \) is a \( q \)-polynomial in \( q^x \) of degree \( n \). Hence we can write

\[ f_{n,-q}(x) = q^{(l/2)n(n-1)} \sum_{r=0}^{n} A_r(n,x) \frac{(q^{n+\alpha})_r}{(q)_r(-q)_r} \]

Thus (2.4.5) implies

\[ A_r(n,x) = (-2xq)(1-q^{-n})A_{r-1}(n-1,x) \]

i.e. \( A_r(n,x) = (q^{-n})_r(-2xq)_r \)

This proves the theorem.

Equations (2.3.9) and (2.3.10) imply the following.

**Theorem 3(A).** If the sequence \( \{f_{n,q}(x)\} \), where \( f_{n,q}(x) \) is a polynomial of degree \( n \) in \( x \), and \( \alpha \) is a parameter, satisfies
such that \( f_{n,q}^{(0)}(x) = j_{n,q}^{(0)}(x) \), then

\[
f_{n,q}^{(x)}(x) = j_{n,q}^{(x)}(x).
\]

The proof of this theorem is similar to that of Theorems 1(A) and 2(A).

**Theorem 3(B):** If the sequence \( \{f_{n,-q}^{(x)}(x)\} \), where \( f_{n,-q}^{(x)}(x) \) is a polynomial of degree \( n \) in \( x \) and \( \alpha \) is a parameter, satisfies

\[
\Delta_\alpha f_{n,-q}^{(x)}(x) = \frac{x q^{n+\alpha}}{[n+\alpha]} D_q f_{n,-q}^{(x)}(x) \quad (2.4.3)
\]

such that \( f_{n,-q}^{(0)}(x) = j_{n,-q}^{(0)}(x) \), then \( f_{n,-q}^{(x)}(x) = j_{n,-q}^{(x)}(x) \).

The proof of this theorem is similar to that of Theorems 1(B) and 2(B).

Finally, we give the theorems suggested by the formulae (2.3.13) and (2.3.14) respectively:

**Theorem 4(A):** Given a sequence of \( q \)-functions \( \{f_{n,q}^{(x)}(x)\} \) such that
\[ \Delta_n f_n^{(\alpha)} (x) = (q^{n+1} - q^{-n-1}) x f_{n,q}^{(\alpha+1)} (x) \]

and \( f_0^{(\alpha)} (x) = 1 \) for all \( x \) and \( \alpha \)

Then \( f_n^{(\alpha)} (x) = f_{n,q}^{(\alpha)} (x) \).

**Theorem 4(b).** Given a sequence of \( q \)-functions \( \{ f_{n,-q}^{(\alpha)} (x) \} \) such that

\[ \Delta_n f_n^{(\alpha)} (x) = \frac{2x(q^{-n} - q^{n+1})}{1+q} f_{n,-q}^{(\alpha+1)} (x), \]

where \( \Delta_n f_n^{(\alpha)} = (q^{-1/2})^n f_n^{(\alpha+1)} f_n^{(\alpha)} - q^{-1/2} f_n^{(\alpha-1)} f_n^{(\alpha)} \)

and \( f_0^{(\alpha)} (x) = 1 \) for all \( x \) and \( \alpha \)

Then \( f_n^{(\alpha)} (x) = f_{n,-q}^{(\alpha)} (x) \).

The proofs of the above theorems follow by induction on \( n \).

§ 2.5 **Integral Representations.** The following \( q \)-integral representations of the \( q \)-Bessel polynomials (2.2.4) due to Ismail[41] hold:

\[ \frac{1}{1-\text{q}} \int_0^1 t^{a-1} (1-\text{qt}) b^{-1} Y_{n,-q}^{(\alpha)} (xt) \text{d}(t;\text{q}) = \]
\[
\begin{align*}
&\frac{(q)_\infty}{(q^a)_\infty (q^b)_\infty} \frac{(1/2)n(n-1)}{q^{a+b}} \Phi_2^1 \left[ \begin{array}{c} q^{-n}, q^{n+a-1}, q^b, -2xq \\ -q, q^{a+b} \end{array} \right] \\
\text{If } a+b = n+a' = 1 \text{ then } (2.5.1) \text{ reduces to} \\
&\frac{1}{1-q} \int_0^1 \frac{t^{a-1} (1-qt)^{b-1} y_n(x)}{q,n,-q} (xt) d(t; q) \\
&\frac{(q)_\infty}{(q^{a+b})_\infty (q^{1/2})_\infty} \frac{(q^{n+a-1})_\infty}{(q^{n+a-b-1})_\infty (q^b)_\infty} y_n(a-b) (x) \tag{2.5.2}
\end{align*}
\]

Similarly,
\[
\begin{align*}
&\frac{1}{1-q} \int_0^1 \frac{t^{a-1} (1-qt)^{b-1} y_n(x) (x [1-q^b t]) d(t; q)}{q,n,-q} \Phi_2^1 \left[ \begin{array}{c} q^{-n}, q^{n+a-1}, q^b, -2xq \\ -q, q^{a+b} \end{array} \right] \tag{2.5.3}
\end{align*}
\]

For \( a+b = n+a'-1 \), (2.5.3) becomes
\[
\begin{align*}
&\frac{1}{1-q} \int_0^1 \frac{t^{a-1} (1-qt)^{b-1} y_n(x) (x [1-q^b t]) d(t; q)}{q,n,-q} \\
&\frac{(q)_\infty}{(q^{a+b})_\infty (q^{1/2})_\infty} \frac{(q^{n+a-1})_\infty}{(q^{n+a-b-1})_\infty (q^a)_\infty} y_n(a-a') (x) \tag{2.5.4}
\end{align*}
\]
Further,

$$\frac{1}{1-q} \int_0^1 t^{n-2} (1-qt) \, \gamma_{n-1}^{(a)}(xt) \, d(t; q)$$

$$= \frac{(q)^n}{(q^2-1)^n} \gamma_n^{(a-n)}(x) \quad (2.5.7)$$

and

$$\frac{1}{1-q} \int_0^1 \frac{t_q(xt)}{2 \Phi_2 \left[ \begin{array}{cc} -n, & q^{n+1} \ -2xq \\ q^{1+c}, & -q; \end{array} \right] t^c \, d(t; q)$$

$$= q^{(a)(n-1)} \gamma_n^{(a)}(x) \quad (2.5.6)$$

Similar results can be obtained for $j_n^{(a)}(x)$

### 2.6 Unification of $q$-Bernoulli and $q$-Euler Polynomials

The study of Bernoulli, Euler and Eulerian polynomials has contributed much to our knowledge of the theory of numbers. These polynomials are of basic importance in several parts of Analysis and Calculus of finite differences, have applications in various field such as Statistics, Numerical Analysis etc.

In recent years, the Eulerian numbers and certain generalizations have been encountered in a number of Combinatorial problems [24,26,27].

In 1975, Karande and Thakare [45] defined the polynomials
\( D_n (x; a, k) \), where \( a \) is a non-zero real number and \( k \) is an integer, by the following generating relation:

\[
2 \left( \frac{t}{e^t - 1} \right)^k e^x = e^t \sum_{n=0}^{\infty} D_n (x; a, k) \frac{t^n}{n!} \quad (2.6.1)
\]

He stated the following relationship between his polynomials \( D_n (x; a, k) \) and Bernoulli and other polynomials:

(i) Bernoulli polynomials: when \( a = k = 1 \), we have

\[
D_n (x; a, k) = B_n (x) \quad (2.6.2)
\]

where the Bernoulli polynomials \( \{B_n (x)\} \), \( n = 0, 1, 2 \) are defined by

\[
\frac{t e^x}{e^t - 1} = \sum_{n=0}^{\infty} B_n (x) \frac{t^n}{n!} \quad (2.6.3)
\]

(ii) Bernoulli numbers: when \( a = k = 1 \) and \( x = 0 \), we have

\[
D_n (0, 1, 1) = B_n \quad (2.6.4)
\]

where the Bernoulli numbers \( \{B_n\} \), \( n = 0, 1, 2 \ldots \) are given by

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (2.6.5)
\]
(iii) Euler polynomials: when \( \alpha = -1, \beta = 0 \), we have

\[
\varphi_n(x; \alpha, \beta) = \phi_n(x)
\]  

(2.6.6)

where the Euler polynomials \( \{\phi_n(x)\}, n = 0, 1, 2 \ldots \), are defined by

\[
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!}
\]  

(2.6.7)

(iv) Genocchi numbers: when \( -\alpha = \beta = 1, \gamma = 0 \), we get

\[
2 \psi_n(0; -1, 1) = G_n
\]  

(2.6.8)

where these numbers are defined by

\[
\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}
\]  

(2.6.9)

(v) Eulerian polynomials: when we put \( \beta = 0, \alpha = 1/t \), with \( t \neq 1 \), we get

\[
\psi_n(x; 1/t, 0) = \frac{2^t}{t^{n+1}} \Phi_n(x, t/2)
\]  

(2.6.10)

where the Eulerian polynomials \( \{\Phi_n(x, t)\} \) as defined by Carlitz [2,5] are given by

\[
\frac{1 - \xi}{1 - \xi e^t} = \sum_{n=0}^{\infty} \Phi_n(x, \xi) \frac{t^n}{n!}
\]  

(2.6.11)
With $\zeta \neq 1$, but otherwise arbitrary. They are connected with Euler polynomials by the relation.

$$\hat{\phi}_n (x, -1) = \zeta_n (x)$$  \hspace{1cm} (2.6.12)

In this section a $q$-analogue of (2.6.1) has been defined and studied. In fact we define $H_{n,q} (x; a, k)$, where $a$ is a non-zero real number and $k$ is an integer, by the following generating relation.

$$t^k e_q ((1-q) xt) \bigg|_{e_q ((1-q)t)-a} = \sum_{n=0}^{\infty} H_{n,q} (x; a, k) \frac{t^n}{[n]!}$$  \hspace{1cm} (2.6.13)

where $[n]! = (q)_{n}/(1-q)^n$ and $e_q (x) = \sum_{n=0}^{\infty} x^n (q)_n^n$

We state the following relationship between our polynomials $H_{n,q} (x; a,k)$ and the $q$-Bernoulli and other polynomials:

(i) $q$-Bernoulli polynomials: when $a = k = 1$, we get

$$H_{n,q} (x; 1,1) = \phi_n (x)$$  \hspace{1cm} (2.6.14)

where the $q$-Bernoulli polynomials \{\( \beta_n (x) \), $n = 0, 1, 2, ...$ \} (cf. ni-Salam [11]) are given by

$$t e_q ((1-q) xt) \bigg|_{e_q ((1-q)t)-1} = \sum_{n=0}^{\infty} \beta_n (x) \frac{t^n}{[n]!}$$  \hspace{1cm} (2.6.15)

(ii) $q$-Bernoulli numbers: when $a = k = 1$ and $x = 0$, we get
and for \( a = k = x = 1 \) and \( t \) replaced by \((-t)\), we obtain

\[
H_n, q (1; 1, 1) = (-1)^n q^{(1/2)n(n-1)} b_n
\]

(2.6.17)

where the q-Bernoulli numbers \( B_n \) and \( b_n \) (cf. Al-Salam [11]) are given by

\[
\frac{t}{e_q(-1-q) - 1} = \sum_{n=0}^{\infty} \frac{t^n}{[n]!} B_n
\]

(2.6.13)

and

\[
\frac{t}{e_q(-(1-q)t) - 1} = \sum_{n=0}^{\infty} \frac{t^n}{[n]!} q^{(1/2)n(n-1)} b_n
\]

(2.6.19)

(iii) q-Euler polynomials: When \( a = -1, k = 0 \), we get

\[
\frac{e_q((-1-q)x) - 1}{e_q((-1-q)t) + 1} = \sum_{n=0}^{\infty} H_n, q (x; -1, 0) \frac{t^n}{[n]!}
\]

(2.6.20)

Thus \( E_n, q (x) = 2 H_n, q (x; -1, 0) \)

(2.6.21)

where the q-analogue of Euler's polynomials i.e. \( E_n, q (x) \) is given by
\[
\frac{2 e_q((1-q)xt)}{e_q((1-q)t)+1} = \sum_{n=0}^{\infty} \frac{E_{n,q}(x) t^n}{[n]!}
\] (2.6.22)

(iv) **q-Euler numbers** : when \( a = -1, k = 0 \) and \( x = 0 \), we have

\[
\frac{1}{e_q((1-q)t)+1} = \sum_{n=0}^{\infty} \frac{H_{n,q}(0; -1, 0) t^n}{[n]!}
\] (2.6.23)

Thus

\[
E_{n,q} = 2 H_{n,q}(0; -1, 0),
\] (2.6.24)

where the q-Euler numbers \( E_{n,q} \) are given by

\[
\frac{2}{e_q((1-q)t)+1} = \sum_{n=0}^{\infty} \frac{E_{n,q} t^n}{[n]!}
\] (2.6.25)

(v) **q-Genocchi polynomials** : when \( a = -1, k = 1 \), we get

\[
\sum_{n=0}^{\infty} H_{n,q}(x; -1, 1) \frac{t^n}{[n]!} = \frac{t e_q((1-q)xt)}{e_q((1-q)t)+1}
\] (2.6.26)

\[
= \frac{1}{2} \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{[n]!}
\] (2.6.27)

Thus,

\[
G_{n,q}(x) = 2 H_{n,q}(x; -1, 1),
\] (2.6.28)

where \( G_{n,q}(x) \) denote q-Genocchi polynomials.
(vi) **q-Genocchi numbers**: When $a = -1$, $k = 1$, $x = 0$, the q-Genocchi numbers $G_{n,q}$ are given by

$$G_{n,q} = 2 H_{n,q}(0, -1, 1), \quad (2.6.29)$$

where the q-Genocchi numbers $G_{n,q}$ are defined as

$$e_{q((1-q) + 1)} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{[n]!} \quad (2.6.30)$$

(vii) **q-Eulerian polynomials**: When $k = 0$, we have

$$e_{q((1-q)xt)} = \sum_{n=0}^{\infty} H_{n,q}(x; a, 0) \frac{t^n}{[n]!} \quad (2.6.31)$$

But

$$e_{q((1-q)t-a) - (1-a) n=0} = \frac{1}{1-a} \sum_{n=0}^{\infty} R_{n,q}(a; x) \frac{t^n}{[n]!} \quad (2.6.32)$$

Thus the q-Eulerian polynomials $R_{n,q}(a; x)$ are connected by the relation

$$R_{n,q}(a; x) = (1-q) H_{n,q}(x; a, 0) \quad (2.6.33)$$

(viii) **q-Eulerian numbers**: When $k = 0$, $x = 0$, and $a = x$, we get,
\[
\frac{1}{e_q((1-q)t-x)} = \sum_{n=0}^{\infty} \frac{n! q (0; x, 0)^n}{[n]!} \tag{2.6.34}
\]

But the q-Dedekind numbers are defined by (cf. Al-Salam [11])

\[
\frac{1}{e_q((1-q)t-x)} = \sum_{n=0}^{\infty} \frac{n! q (x)^n}{[n]!} \tag{2.6.35}
\]

Hence, we obtain

\[
\frac{1}{n!} (1-x) \frac{1}{e_q((1-q)\tau)} = \sum_{n=0}^{\infty} \frac{(n)! (x)^n}{[n]!} \tag{2.0.56}
\]

\[\text{If we write the q-derivative with respect to } x \text{ of both sides of (2.0.13), we get}
\]

\[
\sum_{n=0}^{\infty} \frac{n! q (x, a, k)^n}{[n]!} = \sum_{n=0}^{\infty} \frac{n! x^n}{[n]!} \tag{2.0.13}
\]

\[
\sum_{n=0}^{\infty} \frac{n! x^n}{[n]!} \tag{2.0.13}
\]

\[
\sum_{n=0}^{\infty} \frac{n! x^n}{[n]!} \tag{2.0.13}
\]

\[
\sum_{n=0}^{\infty} \frac{n! x^n}{[n]!} \tag{2.0.13}
\]
Thus, we obtain

\[ H_{n, q} (x; j, k) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \]

It is also evident from (2.6.13) that

\[
\sum_{n=0}^{\infty} \left\{ \frac{(1/a)^n H_{n, q}(x, a, k)}{H_{n, q}([x-1], a, k)} \right\} \frac{t^n}{n!}
\]

\[
= \frac{t^k}{e_q((1-q)t) - a} - \frac{t^k}{e_q((1-q)(x-1)t) - a}
\]

\[
= \frac{t^k}{e_q((1-q)t) - a} \left[ (1/a) e_q((1-q)xt) - e_q((1-q)(x-1)t) \right]
\]

\[
= \frac{t^k}{e_q((1-q)t) - a} \left[ (1/a) e_q((1-q)xt) - \frac{e_q((1-q)xt)}{e_q((1-q)t)} \right]
\]

\[
= (1/a)^k \frac{t^k}{e_q((1-q)(x-1)t)}
\]

\[
= (1/a)^k \sum_{n=0}^{\infty} \frac{(x-1)^n q^n t^{n+k}}{n!}
\]
\[= \left(\frac{1}{a}\right) \sum_{n=k}^{\infty} \frac{(x-1)^{n-k}q}{[n-k]!} t^n\]

Equating the coefficients of \(t^n\) on both sides we are led to

\[H_{n,q} (x; a,k) - aH_{n,q} ([x-1], a,k) = \binom{n}{k}_q \left(\begin{array}{l} \frac{[k]}{(x-1)^{n-k},q} \end{array}\right), \text{ with } n \geq k \quad (2.7.2)\]

The result (2.6.13) can also be transcribed as

\[\Sigma_{n=0}^{\infty} \frac{H_{n,q} ([x-1], a,k)}{[n]!} t^n\]

\[= \frac{t^k e_q ((1-q) [xt-t])}{e_q ((1-q)t) - a}\]

\[= \frac{t^k e_q ((1-q)xt)}{e_q ((1-q)t) - a} \Sigma_{n=0}^{\infty} \frac{q^{n(n-1)/2} (-t)^n}{[n]!}\]

\[= \Sigma_{r=0}^{\infty} H_{r,q} (x; a,k) \frac{t^r}{[r]!} \Sigma_{n=0}^{\infty} \frac{(-t)^n q^{n(n-1)/2}}{[n]!}\]

\[= \Sigma_{n=0}^{\infty} \Sigma_{q=0}^{\infty} H_{r,q} (x, a,k) \frac{t^n (-1)^{n-r}}{[r]![n-r]!} q^{(n-r)(n-r-1)/2}\]
This relation explicitly implies that

\[
H_{n, q} ([x-1], a, k)
\]

\[
= (-1)^n q^{n(n-1)/2} \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) q H_{r, q} (x, a, k) (-1)^r q^{-nr+{r(r-1)}/2}
\]

(2.7.3)

Elimination of \( H_{n, q} ([x-1], a, k) \) from (2.7.2) and (2.7.3)
leads to

\[
H_{n, q} (x, a, k) - \left( \begin{array}{c} n \\ k \end{array} \right) q [k] \cdot (x-1)_{n-k, q}
\]

\[
= a(-1)^n q^{n(n-1)/2} \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) q (-1)^r q^{-nr+{r(r-1)}/2}
\]

\[
H_{r, q} (x; a, k)
\]

(2.7.4)

\section{2.3 PARTICULAR CASES.} Here we enlist the results for
the \( q \)-Bernoulli and \( q \)-Euler polynomials as particular cases
of our results obtained in \( \xi \) 2.6.

\textbf{CASE I : q-Bernoulli polynomials.} On account of the relation
(2.6.14) we get from the results (2.7.1), (2.7.2) and (2.7.4)
after affecting the substitutions \( a = k = 1 \) in each, the
following results for the \( q \)-Bernoulli polynomials \( B_n(x) \)
respectively:
\[ B_n(x) = [n] B_{n-1}(x) \quad (2.8.1) \]

\[ B_n(x) = a B_n([x-1]) = [n] (x-1)_{n-1,q} \quad (2.8.2) \]

\[ B_n(x) = [n] (x-1)_{n-1,q} \]

\[ = a(-1)^n q^{n(n-1)/2} \sum_{r=0}^{n} \binom{n}{r} q (-1)^r \cdot q^{-nr+(r(r-1))/2} B_n(x). \quad (2.8.3) \]

**CASE II: q-Euler polynomials:** In view of the relationship (2.6.21) between (2.6.13) and q-Euler polynomials we get from (2.7.1), (2.7.2) and (2.7.4) after making the substitutions \( a = -1, k = 0 \) in each the following results involving the q-Euler polynomials respectively:

\[ E_{n,q}(x) = [n] E_{n-1,q}(x) \quad (2.8.4) \]

\[ E_{n,q}(x) + 2E_{n,q}([x-1]) = 2 (x-1)_{n,q} \quad (2.8.5) \]

\[ E_{n,q}(x) - 2(x-1)_{n,q} \]

\[ = (-1)^{n+1} q^{n(n-1)/2} \sum_{r=0}^{n} \binom{n}{r} q (-1)^r q^{-nr+(r(r-1))/2} E_{r,q}(x) \quad (2.8.6) \]