CHAPTER VI

CERTAIN FORMULAS INVOLVING GENERALIZED BISERIAL
HYPERGEOMETRIC FUNCTIONS

§ 6.1. INTRODUCTION. In 1968, Manscha (56) have a formula involving hypergeometric functions. In establishing his formula he used the following unpublished result due to Rainville (54).

\[
P^{q}_{q} \left[ \left( \begin{array}{c} \alpha_{p} \\ \beta_{q} \end{array} \right) ; x^{q} \right] = e^{t} \sum_{k=0}^{\infty} \frac{(t,\alpha_{q})_{t}}{(t,\beta_{q})_{t}} \left( \begin{array}{c} -k(\alpha_{q}) \\ k \end{array} \right) \frac{(z)^{k}}{k!}
\]

(6.1.1)

Srivastava (74) gave the following formulae

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_{n}}{(\beta_{q})_{n}} \left[ \begin{array}{c} \alpha_{p}, b_{q}, \ldots, b_{q} \\ \lambda, a_{q} + n, \ldots, a_{q} + n \end{array} \right] z^{n} = e^{z} \sum_{k=0}^{\infty} \frac{(\lambda)^{k}}{(\beta_{q})_{k}} \left[ \begin{array}{c} \alpha_{p}, \lambda, b_{q}, \ldots, b_{q} \\ \beta_{q}, a_{q} + n, \ldots, a_{q} + n \end{array} \right] z^{k}
\]

(6.1.2)

\[
\sum_{n=0}^{\infty} \frac{(\alpha_{j})_{n}}{(\beta_{q})_{n}} \left[ \begin{array}{c} c_{p}, (c_{q})_{n} \\ d_{q}, (d_{q})_{n} \end{array} \right] z^{n} = e^{z} \sum_{k=0}^{\infty} \frac{(\alpha_{j})_{k}}{(\beta_{q})_{k}} \left[ \begin{array}{c} c_{p}, (c_{q})_{k} \\ d_{q}, (d_{q})_{k} \end{array} \right] z^{k}
\]

(6.1.3)
In the present chapter a 'Bibasic' analogue of (6.1.1) has been obtained which has been used to establish a 'Bibasic' analogue of the formula due to Manocha.

The formula so established gives as a particular case, a $q$-analogue of a result due to Chaudhry [30] and also gives a Saalschützian summation theorem for $\Psi_2$.

'Bibasic' analogues of (6.1.2) and (6.1.3) have also been derived in this chapter and their applications discussed while studying the particular cases of the formulae derived in this chapter.

§ 6.2. In this section the following two Lemmas have been established using these Lemmas the 'Bibasic' analogue of a formula due to Manocha has been obtained in the form of a theorem.

**Lemma 1.** For $\lambda \geq 0$, $\lambda_1 \geq 0$, $|xt| < 1$, $|x| < 1$ and $|t| < 1$, we have

$$
\phi^{(s+1)} \left[ \begin{array}{c}
q (a_r) ; -; x^t \\
\vdots \\
q (b_s) ; q_1 ; q^\lambda q_1
\end{array} \right] = q_1 (t q_1) \sum_{n=0}^{\infty} q_1 (t q_1) \phi^{(s+1)} \left[ \begin{array}{c}
q (a_r) ; -n \\
\vdots \\
q (b_s) ; q_1 ; q^\lambda q_1
\end{array} \right] \frac{(1/2)n(n+1)(-t)^n}{[q_1]_n}.
$$

(6.2.1)
Proof. The right hand side of (6.1.4) is equal to

\[ e_{q_1}(tq_1) \prod_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\left[ q^{(s)} \right]_k \left[ q^{-n} \right]_k}{\left[ q^{(s)} \right]_k \left[ q_1 \right]_k \left[ q \right]_n} \times \frac{(1/2)\lambda_1(k+1)}{q_1} \frac{(1/2)\lambda_1(k+1)}{q_1} \frac{(1/2)\mu(n+1)}{t_n} (-t)^n \]

\[ = e_{q_1}(tq_1) \prod_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\left[ q^{(s)} \right]_k \left[ q_1 \right]_k \left[ q \right]_n (-t)^k}{\left[ q^{(s)} \right]_k \left[ q_1 \right]_k \left[ q \right]_n} \times \frac{(1/2)\lambda_1(k+1)}{q_1} \frac{(1/2)\mu(n+1)}{t_n} (-t)^{n+k} \]

\[ = e_{q_1}(tq_1) e_{q_1}(tq_1) \prod_{r=s+1}^{t} \left[ \frac{(s_r)}{q_1} ; - ; \frac{q_1}{q_1} \right] \left[ \frac{(s)}{q} ; q_1 ; q^\lambda q_1 \right] \]

This completes the proof of Lemma 1.

Lemma 2. For \( k > 0 \), \( \lambda_1 > 0 \) and \( |x| < 1 \), we have

\[ x^n = \frac{\left[ q^{(s)} \right]_n q_1^n}{\left[ q^{(s)} \right]_n q_1^{n/2} \lambda_1(n+1) \mu(n+1)} \times \prod_{k=0}^{n} \frac{\left[ q^{-n} \right]_k q_1^k}{\left[ q_1 \right]_k} \prod_{r=s+1}^{t} \left[ \frac{(s_r)}{q_1} ; q_1^{-k} ; q_1 \right] \]

\[ \left[ \frac{(s)}{q} ; q_1 ; q^\lambda q_1 \right] \]

..(6.2.2)
Proof. From (6.2.1), we have

\[ \lim_{\lambda \to 0} \sum_{n=0}^{\infty} \frac{[q(a_r)]_n x^n q^{(1/2) \lambda (n+1)} (1/2) \lambda (n+1)}{[q_s]_n [q_1]_n} \]

Equating coefficients of \( t^n \) on both sides of (6.2.3) we get (6.2.2). Now by means of (6.2.2), we shall prove the following:

**Theorem 1.** For \( \lambda \geq 0, \lambda_1 \geq 0, |x_t| < 1, |x| < 1 \) and \( |t| < 1 \), we have

\[ \Phi^{(n)}_{r+m-s+1} \left[ \begin{array}{c}
(a_r) ; q_1 ; x \\
(b_s) ; q_1 ; q_1 \lambda
\end{array} \right] = \sum_{k=0}^{\infty} \frac{[q_1]_k q^n_1}{[q_1]_n} \]

\[ \sum_{k=0}^{\infty} [q_1]_k \left[ \begin{array}{c}
(c_m) \\
(d_p)
\end{array} \right]_k (\lambda (k+1) (-t))^k \]
where the series \( \Phi_p \) is a basic hypergeometric series on a single base \( q_1 \).

**Proof.** The left-hand side of (6.2.4) is equal to

\[
\sum_{n=0}^{\infty} \left[ \begin{array}{c} (a_r) \n \n \end{array} \right]_n \left[ \begin{array}{c} (c_m) \n \n \end{array} \right]_n q^n \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n q^{n(n+1)} \left(\frac{1}{2}\right)^{n(n+1)} x^n t^n \]

\[
\sum_{n=0}^{\infty} \left[ \begin{array}{c} (b_s) \n \n \end{array} \right]_n \left[ \begin{array}{c} (d_p) \n \n \end{array} \right]_n q^n \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n q^{n(n+1)} \left(\frac{1}{2}\right)^{n(n+1)} x^n t^n \]

\[
= \sum_{n,k=0}^{\infty} \left[ \begin{array}{c} (c_m) \n \n \end{array} \right]_{n+k} q_1^{n+k} \left(\frac{1}{2}\right)^{n+k} \left(\frac{1}{2}\right)^{n+k} q^{(n+1)(\lambda+1)} (-1)^k x^n t^n \]

\[
\times \Phi_{r+1,s+1} \left[ \begin{array}{c} (a_r) \n \n \end{array} \right]_{r+1} q_1^{-k} x \]

\[
\Phi_{r+1,s+1} \left[ \begin{array}{c} (b_s) \n \n \end{array} \right]_{s+1} q_1^\lambda \lambda_{1} \]

\[ \times \Phi_{r+1,s+1} \left[ \begin{array}{c} (a_r) \n \n \end{array} \right]_{r+1} q_1^{-k} x \]

\[ \times \Phi_{r+1,s+1} \left[ \begin{array}{c} (b_s) \n \n \end{array} \right]_{s+1} q_1^\lambda \lambda_{1} \]

\[ \times \Phi_{r+1,s+1} \left[ \begin{array}{c} (a_r) \n \n \end{array} \right]_{r+1} q_1^{-k} x \]

\[ \times \Phi_{r+1,s+1} \left[ \begin{array}{c} (b_s) \n \n \end{array} \right]_{s+1} q_1^\lambda \lambda_{1} \]
This completes the proof of the theorem.

§6.3 In this section, the 'Bibasic' analogues of (6.1.2) and (6.1.3) have been obtained in the form of the following theorems.

**Theorem 1.** For \(|q| < 1\), \(|q_j| < 1\), \(|q_1| < 1\), and \(|z| < 1\), we have

\[
e_{q_1}(z) = \sum_{n=0}^{\infty} \frac{[q_1]_n [q_j]_n z^n}{[\bar{q}]_n [q_1]_n} q_1^{n+1} \left( \begin{array}{c} \frac{1}{2} \lambda (n+1) \\ \frac{1}{2} \lambda_1 (n+1) \end{array} \right) x
\]

\[
= \sum_{n=0}^{\infty} \frac{[q_1]_n [q_j]_n z^n}{[\bar{q}]_n [q_1]_n} q_1^{n+1} \left( \begin{array}{c} \frac{1}{2} \lambda (n+1) \\ \frac{1}{2} \lambda_1 (n+1) \end{array} \right) x
\]
Proof. The L.H.S. (6.3.1) is

\[ e_{q_1} (z) \sum_{n,k=0}^{\infty} \frac{[q_1]_{n+k} [q]_{n+k} x^{n+k} z^k}{[q_1]_n [q]_n \frac{q^k}{n+k}} = (q_1^{\alpha+\kappa-1}) q_1^{\frac{1}{2} \lambda_1 n(n+1)} \frac{1}{q_1^{\frac{1}{2} n(n+1)}} \]

\[ = e_{q_1} (z) \sum_{n=0}^{\infty} \frac{[q_1]_{n+k} [q]_{n+k} x^n q_1^{(1/2) \lambda_1 n(n+1)} (1/2) \lambda_1 n(n+1)}{[q_1]_n [q]_n \frac{q_1^{(1/2) \lambda_1 n(n+1)}}{q_1^{\frac{1}{2} n(n+1)}}} \]

\[ = \sum_{n=0}^{\infty} \frac{[q_1]_{n+k} [q]_{n+k} x^n q_1^{(1/2) \lambda_1 n(n+1)} (1/2) \lambda_1 n(n+1)}}{[q_1]_n [q]_n} \times \sum_{k=0}^{n} \left[ \frac{1}{k} \right] \left[ \frac{1}{k} \right] \]

Finally, a direct generalization of (6.3.1) has been established in the form of the following theorems:
Theorem 2. For \(|x| < 1, |z| < 1, |q| < 1\) and \(|q_1| < 1\),

we have

\[
\sum_{n=0}^{\infty} \left[ \left( \sum_{k=0}^{n} \binom{n}{k} \frac{a_{n-k}}{q_1^n \cdots q_1^1} \right) x^n \right] = \frac{1}{1 - (x z) q_1^n \cdots q_1^1}.
\]

The proof of (6.3.2) is straightforward.

§ 6.4. Particular Cases. In this section various particular cases of the results obtained in §6.2 and §6.3 have been given.

(i) We put \(q_1 = q, x = q, t = z/q, \lambda = s = \lambda_1, r = s = 3, m = 2\) and \(p = 1\) in (6.2.4) and then replace \(a_2, b_2\) and \(b_3\)
by $d_1$, $c_1$ and $c_2$ respectively, thus getting

$$
\Phi_1 \left[ \frac{c_1, c_2; z}{q} \right] = \sum_{k=0}^{\infty} \frac{(1/3)^k (z-1)^k}{q_1^k q_2^k} \left[ \frac{c_1}{q_1} \frac{c_2}{q_2} \right] (-z)^k
$$

$$(6.4.1)$$

(6.4.1) is a $q$-analogue of a result due to Chand [30].

(ii) Again, in (6.2.4) we put $r = s = \rho = 1$ and $m = 2,$

$$
\kappa = \omega = \lambda_1, \kappa = q, t = 1 \text{ and } q_1 = q \text{ and then replace } c_1 \text{ by } -c,
$$

using $q$-analogue of Vandermonde's theorem.

$$
\Phi_1 \left[ \frac{q^a, q^{-a}; q}{q^{b}, q^{-b}} \right] = \frac{[b-a]_q [q^a]_q}{[q^b]_q}
$$

We arrive at the known result.

$$
\Phi_2 \left[ \frac{-a, c_2; q}{c_1, d} \right] = \frac{[d-c]_q [q^a]_q}{[c_1]_q} \Phi_2 \left[ \frac{-a, b-a, c; q}{1+a-d} \right]
$$

Putting $d = 1+a+b-c$, (6.4.2) yields Saalschützian $\Phi_2$.

Some interesting particular cases of the result (6.3.1)
(a) Taking \( p = j = 1, \alpha = q, \xi_1 = q, \lambda = \sigma = \lambda_1 \) and applying q-analogue of Vandermonde's theorem, we get

\[
e_q(z_1) = \sum_{n=0}^{\infty} \frac{[\alpha]_n [\beta]_n q^n}{[\gamma]_n} \phi \left[ \begin{array}{c} \alpha+\nu, \beta+\nu; \frac{z}{q^{\lambda-1}} \\ \alpha, \beta+\nu; q^2 \end{array} \right]_{(1-q^{-1})_\infty (1-q^{-1+\alpha+\alpha-\beta})_{\infty}}
\]

\[
= \frac{(1-q^{1-b})_{\infty} (1-q^{1+\alpha+\alpha-\beta})_{\infty}}{(1-q^{1+b})_{\infty} (1-q^{1+\alpha+\alpha-\beta})_{\infty}} \phi \left[ \begin{array}{c} 1+\nu-b \\ \nu+b \\ \frac{z}{q^{\lambda-1}} \end{array} \right]_{(1-q^{-1})_\infty (1-q^{-1+\alpha+\alpha-\beta})_{\infty}}
\]

(6.4.2)

(b) Taking \( p = j, a_i = b_j, i = 1, 2, \ldots, p \) (or \( \sigma \)), \( \alpha = q, \lambda = \sigma = \lambda_1 \) in (6.4.1), we get

\[
e_q(z) = \sum_{n=0}^{\infty} \frac{[\gamma]_n c^n}{[\phi]_n} \phi \left[ \begin{array}{c} \alpha+\nu; z q^{-\lambda-1} \\ \alpha; q^2 \end{array} \right]_{(1-q^{-1})_\infty (1-q^{1+\alpha+\alpha-\beta})_{\infty}}
\]

\[
= \frac{q J_\alpha \left( \frac{\sigma \tau}{i \alpha} \right)}{(i \alpha \tau)^{\alpha}}
\]

(6.4.3)

where \( J_\alpha \) is a basic analogue of Bessel function [36].

(c) Putting \( \alpha = \sigma, \lambda = \sigma = \lambda_1 \) and \( q = q_1 \) in (6.4.1),
we have

\[
\phi_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[q^n]_n} \prod_{j=1}^{p+1} \left[ \frac{(a_p^j)^n q^2; x z/q^2}{(a_p^j)^n q^2; q^2} \right] \prod_{i=0}^{\infty} \left[ \frac{(b_i^j)^n q^2; x}{(b_i^j)^n q^2} \right]
\]

(6.4.4)