CHAPTER - I

HISTORICAL RESUME

1.1. Introduction:

The present thesis is based on certain investigations on linear positive operators in approximation theory. We study linear methods of approximation which are given by a sequence of linear positive operators.

The concept of 'Linear Positive Operators' in approximation theory was introduced by Korovkin ([1],[2]) in 1953, although the corresponding idea already occurred in 1952 in a paper by Bohman[1]. Since then many mathematicians have constructed a sequence of linear positive operators, problem of convergence and many other interesting properties being studied in connection with these operators. This theory has recently led to numerous results in this domain.

Before giving a brief résumé of the hitherto obtained results against the background of which the problem has been studied in the present thesis, the author proposes to state various definitions and notations which will be required in the sequel.
Definition (1.1.1) (Linear positive operators):

A linear positive operator is a function $L$ having the following properties:

(i) The domain $D$ of $L$ is a non-empty set of real functions, all having the same real domain $V$.

(ii) For every $f \in D$, $L(f)$ is again a real function with domain $V$.

(iii) If $f$ and $g$ belong to $D$, and if $a$ and $b$ are reals, then $af + bg \in D$, and

$$L(af + bg) = aL(f) + bL(g).$$

(iv) If $f \in D$, and $f(x) \geq 0$ for every $x \in V$, then $(Lf)(x) \geq 0$ for every $x \in V$.

Consequently, if $L$ is a linear positive operator and $f, g \in D$, then $f \leq g$ throughout $V$ implies $L f \leq L g$ there, and $|f| \leq g$ throughout $V$ implies $|Lf| \leq L g$ there.

1.2. Approximation Problem:

A general outline: In approximation theory we study the relation between a given function and its smooth version. The
problem of approximation of functions by polynomials can be described in the following way:

Let \( \Phi \) be a set of functions defined on \( A \). If \( f \) is a function on a space \( A \), can one find a linear combination

\[
P = a_1 \phi_1 + a_2 \phi_2 + \ldots + a_n \phi_n,
\]

where \( \phi_i \in \Phi \) and \( a_i \) are reals, which is closed to \( f \)?

Generally two problems arise:

(i) How to select \( \Phi \)?

(ii) How we measure the deviation of \( P \) from \( f \)?

We are mostly concerned with the second question i.e., for approximating \( f(x) \) to a polynomial \( P(x) \) (a linear combination); actually we have to show that how much \( P(x) \) is closed to \( f(x) \), i.e.,

\[
|f(x) - P(x)| \leq \varepsilon,
\]

for all \( x \), where \( \varepsilon \) characterises the closeness of \( P(x) \) with \( f(x) \)? Therefore, the measure of approximation is the quantity

\[
\Delta(P) = \max [f(x) - P(x)]
\]

which is called the distance between \( P(x) \) and \( f(x) \), or deviation of \( f(x) \) from \( P(x) \).

**Definition (1.2.1) (Approximation):** Let \( X \) be a Banach space of continuous functions on \([a,b]\) with the norm \( || \cdot || \) defined by

\[
||f|| = \sup_{x \in [a,b]} |f(x)|.
\]
Let $\Phi$ be a subset of $X$. An element $f$ of $X$ is called approximable by linear combination.

\begin{equation}
P = a_1 \phi_1 + a_2 \phi_2 + \cdots + a_n \phi_n ,
\end{equation}

where $\phi \in \Phi$, and $a_i$ are reals, if for each $\varepsilon > 0$, there is a polynomial $P$, such that

$$
|| f - P || < \varepsilon .
$$

Answering the fundamental question of approximation of function by polynomials in the affirmative, Weierstrass [1] has shown the possibility of representation of any continuous function subjected to the chosen algebraic polynomial $P(x)$. We can formulate his result as follows:

**Weierstrass First Theorem**: If $f(x) \in C[a,b]$, then for every $\varepsilon > 0$, there exists an algebraic polynomial $P(x)$ such that

$$
| f(x) - P(x) | < \varepsilon ,
$$

holds for every $x$ in the interval.

We are also concerned with the possibility of establishing approximation of continuous and periodic functions by means of trigonometric polynomials. For this we have the following theorem
**Weierstrass Second Theorem:** Let \( f(x) \in C_{2\pi} \), for each \( \varepsilon > 0 \), there exists a trigonometric polynomial \( T(x) \) that for all real \( x \):

\[
| T(x) - f(x) | < \varepsilon.
\]

Weierstrass' second theorem has shown that any function \( f(x) \in C_{2\pi} \) can be represented by a trigonometric polynomial at any prescribed accuracy. But the degree of approximating polynomial may come out to very high. Therefore, it is natural to ask what accuracy of approximation can be attained if the degree of approximating polynomial is limited beforehand.

**Definition (1.2.2) (Degree of approximation):**

Let \( \phi = \{ \phi_n \} \) be a sequence of functions. Then

\[
E_n^*(f) = E_n(f) = \inf_{a_1, a_2, \ldots, a_n} \| f - (a_1 \phi_1 + a_2 \phi_2 + \ldots + a_n \phi_n) \|
\]

\[
(1.2.2) = \inf_{a_1, a_2, \ldots, a_n} \| f - P \|,
\]

where \( P \) is defined in (1.2.1), is called the \( n^{th} \) degree of approximation of \( f \) by the sequence \( \{ \phi_n \} \).

**Definition (1.2.3) (Best approximation):** If the infimum in (1.2.2) is attained for some \( P \), then \( P \) is called a
polynomial (a linear combination) of best approximation.

**Remarks** (i): If the $P$ are algebraic polynomials of a given degree, then $n$ in (1.2.2) will refer to the degree of the polynomial rather than the number of functions $\phi_i$.

(ii) $E_n(f)$ is also called the error in approximating $f$ by the polynomial $P$.

1.3. **Classes of functions**:

In this section we list several classes of functions which will be constantly used later.

Let $f(x) \in C[a,b]$ (or $f(x) \in C_2$) and let $E_n$ be its best approximation by means of algebraic (or trigonometric) polynomials of order not higher than $n$. By Weierstrass's theorem it is found that

$$\lim_{n \to \infty} E_n = 0.$$

Naturally, the "simpler" the approximating function $f(x)$, the more accurately will it be represented by means of a polynomial (algebraic or trigonometric).
Now, we shall engage in the question of the influence exerted by an improvement in the structural properties of the approximated function on the order of the decrease of its best approximation $E_n$.

A convenient characteristic of the structural properties of a function is a quantity called the "modulus of continuity" of this function.

**Definition (1.3.1) (Modulus of continuity):** To measure the continuity of a function $f(x)$ defined on $<a,b>$ (this may be either a segment $[a,b]$ or the interval $(a,b)$, in particular, the entire axis $(-\infty, \infty)$ or the half-segment $[a,b)$, etc.), we define a function $\omega(\delta) = \omega(f; \delta)$ as follows:

\[ (1.3.1) \quad \omega(f; \delta) = \sup_{x,y \in <a,b>, \delta > 0} |f(x) - f(y)| : |x - y| \leq \delta \]

This function $\omega(\delta)$ is called the modulus of continuity of $f$.

The modulus of continuity $\omega(\delta)$ has the following fundamental properties:

(a) $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$,

(b) $\omega(\delta) \geq 0$ and non-decreasing,

(c) $\omega$ is subadditive, i.e.,

\[ \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2) \]
(d) \(\omega(\delta)\) is continuous,
(e) \(\omega(n\delta) \leq n\omega(\delta)\), if \(n\) is any natural number,
(f) \(\omega(\lambda\delta) \leq (\lambda+1) \omega(\delta)\), if \(\lambda\) is any positive number.

**Definition (1.3.2) (Lipschitz class Lip \(\alpha\)):** We suppose \(0 < \alpha \leq 1\). The Lipschitz class Lip \(\alpha\) is the space of all functions \(f\) satisfying the condition

\[
\|f\|_\alpha = \sup_{t > 0, x} \left( \frac{|f(x+t) - f(x)|}{t^\alpha} \right) < \infty.
\]

**Definition (1.3.3) (Hardy-Littlewood class Lip \((\alpha,p)\) [1]):**

The Hardy-Littlewood class Lip \((\alpha,p)\) \((p < \infty)\) is the space of all functions \(f\) satisfying the condition

\[
\|f\|_{\alpha,p} = \sup_{t > 0, x} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(x+t) - f(x)|}{t^\alpha} \right)^{1/p} dt < \infty.
\]

**Definition (1.3.4) (Lip \(K(t)\) class):** A function \(f(x)\), integrable \(L\) (Lebesgue integrable) is said to belong to Lip \(K(t)\) class, if

\[
\|f\|_K = \sup_{t > 0, x} \left( \frac{|f(x+t) - f(x)|}{K(t)} \right) < \infty,
\]
where \( K(t) \) is positive increasing function, such that \( \frac{K(t)}{t} \) is decreasing, \( K(t) \to 0 \) as \( t \to 0 \), and

\[ K(xy) \leq K(x) K(y). \]

**Remarks:** We notice that by taking

(i) \( K(t) = t^\alpha \) in \((1.3.4)\) for \( 0 < \alpha < 1 \), the above class reduces to \( \text{Lip } \alpha \),

(ii) \( K(t) = t^{\alpha - \frac{1}{p}} \) in \((1.3.4)\) for \( 0 < \alpha < 1 \) and \( p < \infty \), the above class reduces to \( \text{Lip } (\alpha, p) \).

1.4. Approximation of functions by linear positive operators:

The problem of approximating a real (or complex) valued function \( g(t) \) defined on the real line or on the subset of it by means of a suitable sequence \( \{L_n\} \) \((n = 1, 2, \ldots)\) of linear positive operators for points of continuity have been studied by several mathematicians. In general, such a procedure assumes the convergence \( L_n(f; x) \to f(x) \) as \( n \to \infty \), where \( x \) is a fixed point or it belongs to a set of points on which an approximation is desired, for some test functions \( f(t) \).
The second Weierstrass theorem establishes the possibility of an unlimited approach to periodic continuous functions with the aid of trigonometric polynomials.

In 1908 de la Vallee-Poussin (See Natanson [1] p. 7) introduced the operators $V_n(f; x)$ and gave a very simple proof of Weierstrass second approximation theorem.

Definition (1.4.1): Let $f(x) \in C_{2\pi}$. The integral

$$V_n(x) = V_n(f; x)$$

$$= \frac{(2n)!!}{2\pi(2n-1)!!} \int_{-\pi}^{\pi} f(t) \cos^{2n} \frac{t-x}{2} \, dt,$$

$$n = 1, 2, \ldots$$

is called the de la Vallée-Poussin singular integral.

The de la Vallée-Poussin Theorem: Let $f(x) \in C_{2\pi}$ and $V_n(x)$ be the de la Vallée-Poussin singular integral defined as in (1.4.1). Then

$$\lim_{n \to \infty} V_n(x) = f(x),$$

holds uniformly for all real $x$. 
Natanson [1] gave an alternate proof of Weierstrass second theorem by a trigonometric polynomial \( R^*_n(x) \) called Rogosinski singular integral.

**Definition (1.4.2):** Let \( f(x) \in C_{2\pi} \) and \( s_n(x) \) be a partial sum of the Fourier series of this function. We write

\[
R_n(x) = \frac{1}{2} \left[ s_n(x + \frac{\pi}{2n}) + s_n(x - \frac{\pi}{2n}) \right]
\]

\[
= \frac{1}{2} a_0 + \sum_{k=1}^{n} \left( a_k \cos kx + b_k \sin kx \right) \cos \left( \frac{\pi x}{2n} \right).
\]

Then the Rogosinski singular integral \( R^*_n(x) \) is defined as

\[
R^*_n(x) = \frac{1}{4\pi} \int_{-\pi}^{\pi} f(x+t) \cos nt \left[ \frac{1}{\sin(\frac{t}{2} + \frac{\pi}{2n})} - \frac{1}{\sin(\frac{t}{2} - \frac{\pi}{2n})} \right] dt.
\]

This trigonometric polynomial \( R^*_n(x) \) satisfies the property that

\[
\lim_{n \to \infty} R^*_n(x) = f(x),
\]

holds uniformly on the entire axis.

It has been observed by Korovkin ([1],[2]) that for a sequence \( \{L_n\} \) of positive linear operators convergence often
can be established quite simply by checking it for certain finite sets of functions $f$. This is one of the most important properties of linear positive operators. We state the following theorem due to Korovkin ([2] p. 14).

**Theorem 1.1**: If the three conditions

$$L_n(1; x) = 1 + \alpha_n(x),$$
$$L_n(t; x) = x + \beta_n(x),$$
$$L_n(t^2; x) = x^2 + \gamma_n(x)$$

are satisfied for the sequence of linear positive operators $L_n(f; x)$, where $\alpha_n(x), \beta_n(x), \gamma_n(x)$ converge uniformly to zero in the interval $a \leq x \leq b$, then the sequence $L_n(f; x)$ converges uniformly to the function $f(x)$ in this interval, if $f(t)$ is bounded, continuous in the interval $[a, b]$, continuous on the right at point $b$ and on the left at the point $a$. 
1.5. **Approximation of classes of functions**


**Theorem (1.3):** If \( \omega(\delta) \) is the modulus of continuity of \( f(x), 0 \leq x \leq 1 \), then for each \( n \) there is a polynomial \( P_n(x) \) of degree \( \leq n \) such that

\[
(1.3.1) \quad |f(x) - P_n(x)| \leq C \omega(\frac{1}{n})
\]

where \( C \) is an absolute constant, we may take for instance \( C = 3 \).

For example, \( |f(x) - P_n(x)| = O(n^{-\alpha}) \) if \( f(x) \) belongs to the class \( \text{Lip } \alpha \).

Of numerous mathematicians who have studied the degree of approximation of functions belonging to certain classes of functions by considering different types of operators we mention here Bernstein, Natanson, Butzer and Korevkin etc.

I. **The Regesinski singular integral**: In 1930 Bernstein [2] obtained the degree of approximation of functions \( f(x) \) belonging to the class \( \text{Lip } \alpha \) using the Regesinski singular integrals \( R_n(x) \) defined by (1.4.2).
Theorem (1.4): If \( f(x) \in \text{Lip } \alpha \), \( 0 < \alpha \leq 1 \), then

\[
\| h_n(x) - f(x) \|_C = O \left( n^{-\alpha} \right).
\]

II. The de la Vallée-Poussin singular integral: Natanson [1] determined the degree of approximation of functions belonging to \( C_{2n} \) by de la Vallée-Poussin singular integral \( V_n(x) \) defined by (1.4.1) in terms of modulus of continuity as follows:

Theorem (1.5): If \( f(x) \in C_{2n} \) has a modulus of continuity \( \omega (\delta) \), then for all \( x \)

\[
|V_n(x) - f(x)| \leq 3 \omega \left( \frac{1}{\gamma_n} \right).
\]

An immediate consequence of Theorem (1.5) was proved by Natanson [1] as follows:

Theorem (1.6): If \( f(x) \in \text{Lip } \alpha \), then

\[
|V_n(x) - f(x)| \leq \frac{3 M}{\gamma_n^{\alpha}}.
\]

Later on, for \( f(x) \in \text{Lip } (\alpha, p) \), Butzer [1] extended Theorem (1.6) as follows:

Theorem (1.7): If \( f(x) \in \text{Lip } (\alpha, p) \), \( p \geq 1 \), \( 0 < \alpha \leq 1 \), then

\[
\| V_n - f \|_p = O \left( n^{-\frac{\alpha}{2}} \right).
\]
III. The Jackson type operators: We assume that \( f \in C_{2n} \). Let

\[
U_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \frac{p(n)}{k} \cos kt \geq 0, \quad (-\pi \leq t \leq \pi)
\]

be a non-negative trigonometric polynomial of degree \( n \). Then the linear positive operator \( L_n \) associated with \( U_n(t) \) is defined by:

\[
L_n(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \, U_n(t) \, dt.
\]

For the operator defined by (1.5.7) Korovkin ([2], p. 72) proved the following theorem:

**Theorem (1.8):** If \( f \in C_{2n} \) and \( L_n \) is an operator of type (1.5.7), then

\[
|L_n(f; x) - f(x)| \leq \left(1 + \frac{m^2}{\sqrt{2}}\right) V\left(1 - \frac{p(n)}{n} \right) \omega(f; \frac{1}{n}),
\]

\((m > 0)\).

In 1965 Schurer [2] improved the estimate (1.5.8) provided by the Theorem (1.8) and obtained the degree of approximation of functions belonging to \( C_{2n} \) by means of the operators \( L_{n0-s} \) (\( n, s \) positive integers), defined by

\[
L_{n0-s}(f; x) = \frac{1}{\Lambda_{n0-s}} \int_{-\pi}^{\pi} f(x+t) \left( \frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t} \right)^{2s} \, dt,
\]
where
\[ A_{ns-s} = \int_{-\pi}^{\pi} \left( \frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t} \right)^{2s} \, dt. \]

**Theorem (1.9):** If \( f(x) \in C_{2k} \), then

\[ |L_{ns-s}(f(x)) - f(x)| \leq (1 + \frac{s}{2^n}) \omega(f; \frac{1}{n}), \quad (n = 1, 2, 3, \ldots; s = 3, 4, 5, 6). \]

Chapter II is devoted to study the extension of the foregoing results for wider classes of functions. It may be seen that Theorems (2.2) and (2.3) of Chapter II include Theorems (1.4) and (1.6) as special cases for \( K(t) = t^\alpha, \ 0 < \alpha < 1 \). While Theorem (2.1) generalizes Theorem (1.4) in the case when \( f(x) \in \text{Lip} (\alpha, p) \).

1.6. **Bernstein operators:**

In approximation theory of real functions, one of the most important theorems is the well-known theorem of Weierstrass. It states that every real function defined and continuous on a (finite) closed interval \([a, b]\) of the real axis can be approximated arbitrarily closely by polynomials. One of the most elegant proofs of the Weierstrass theorem was given by Bernstein [1] in 1912, it is based upon a consideration drawn from probability theory. In the
proof a sequence of polynomials is constructed, now bearing his name, which depend on the function to be approximated and if their order increases infinitely they tend uniformly to that function. This fact amplified and enriched the researches in this domain with new and important results.

The \( n \)th Bernstein polynomial is defined as follows:

**Definition (1.6.1) (Bernstein polynomial):** Let \( f \in C[0,1] \) (the set of real continuous functions defined on \([0,1]\)). The Bernstein polynomial of \( f \) degree \( n \) is defined by

\[
B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x),
\]

where

\[
p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad (x \in [0,1], \quad n=1,2,\ldots, \quad k = 0,1,2, \ldots).
\]

and \( \binom{n}{k} \) are binomial coefficients, \( f\left(\frac{k}{n}\right) \) is the value of the given function at the point \( x = \frac{k}{n} \).

Since the functions (1.6.2) are linear positive in \([0,1]\) therefore, for \( f \geq 0 \) implies \( B_n(f;x) \geq 0 \) on \([0,1]\), that is \( B_n(f;x) \) is a positive linear operator from \( C[0,1] \) to \( C[0,1] \).
Further, it possesses a property which plays a very important role in approximation theory: it is monotone, i.e., if $f(x)$ is non-negative on $[0,1]$ its image $B_n f$ is also non-negative on $[0,1]$. These properties of Bernstein operators are shared with other operators considered in the approximation theory.

In 1912 Bernstein [1] associated with the operators (1.6.1) and proved the following:

**Theorem (1.10):** For a function $f(x)$ bounded on $[0,1]$ the relation

(1.6.3) \[ \lim_{n \to \infty} B_n (f \cdot x) = f(x), \]

holds at each point $x$ of $f$, and the relation (1.6.3) holds uniformly on $[0,1]$ if $f(x)$ is continuous on this interval.

The Weierstrass first theorem follows directly from Bernstein theorem, if segment $[a,b]$ coincides with segment $[0,1]$. We observe however, that the Bernstein theorem is more productive than the Weierstrass theorem in the case, as it provides a sequence of well defined polynomials, while the Weierstrass theorem only establishes the existence of such a sequence of approximation without stating anything in regards to its construction.
1.7. Estimate of the order of approximation

In this section we are concerned with the estimates of the order of approximation of a function \( f \in C[0,1] \) by Bernstein operators. It may be simply described by means of the modulus of continuity. For instance, Popoviciu [1] proved in 1935 that Bernstein operators \( B_n(f; x) \) \((n = 1, 2, \ldots)\) possess the property that there exists a constant \( K > 0 \) such that

\[
(1.7.1) \quad |B_n(f; x) - f(x)| \leq K \omega_n^{-\frac{1}{2}},
\]

holds for all \( f \in C[0,1] \), all \( x \in [0,1] \) and \( n = 1, 2, \ldots \), if \( \omega(\delta) \), \( (\delta > 0) \) denotes the modulus of continuity of \( f \) on \([0,1]\). He showed that \( \frac{3}{2} \) may be taken as a value of \( K \).

In 1953 Lorentz [2] proved that \( K = \frac{3}{4} \) is also an admissible value. He considered also the class \( C^1[0,1] \) of all functions \( f \) which are defined on \([0,1]\) and continuously differentiable there and showed that the inequality

\[
(1.7.2) \quad |B_n(f; x) - f(x)| \leq K n^{-\frac{1}{2}} \omega_1(n^{-\frac{1}{2}}), \quad (n = 1, 2, \ldots)
\]

is valid with \( K = \frac{3}{4} \) for all \( f \in C^1[0,1] \), all \( x \in [0,1] \) and \( n = 1, 2, \ldots \), if \( \omega_1(\delta) = \omega(f', \delta) \), \( (\delta > 0) \) denotes the modulus of continuity of \( f' \) on \([0,1]\).
Later on the question arose as the speed with which \( B_n(f; x) \) tends to \( f(x) \). As an answer to this question has been given in different directions. One direction is that in which \( f(x) \) is supposed to be at least twice differentiable at a point \( x \) of \([0,1]\). As an illustration we state here a result due to Voronowskaja [1].

**Theorem 1.11**: Let \( f(x) \) be bounded in \([0,1]\) and suppose that the second derivative \( f''(x) \) exists at a certain point \( x \) of \([0,1]\), then

\[
B_n(f; x) - f(x) = \frac{x(1-x)}{2n} f''(x) + o\left(\frac{1}{n}\right).
\]

1.8. **Kantorovich type operators**:

Since ordinary Bernstein operators are not suitable for approximation to discontinuous functions of general type. However, replacing \( f\left(\frac{k}{n}\right) \) in the formula (1.6.1) by some
other expression, we may obtain better results. For instance, replacing $B_n$ by $B_{n+1}$, we obtain the operator \((\text{Kantorovitch}[1], \text{Lorentz}[1])\):

\[
(1.8.1) \quad K_n \left( f(t) \right) x = \frac{d}{dx} B_{n+1}^F (x)
\]

\[
= (n+1) \sum_{k=0}^{n} \left( \frac{k+1}{n+1} \int_0^x f(t) \ dt \right) p_{n,k}(x)
\]

where

\[
F(x) = \int_0^x f(t) \ dt,
\]

and $p_{n,k}(x)$ is as defined in (1.6.2), which approximate any continuous function $f(x)$ on $[0,1]$.

In Chapter III we have applied the Kantorovitch methodology for approximation of continuous functions for the finite
interval \([0,1]\) and defined the operator:

\[
A_n(f(t);x) = \frac{(-1)^s}{(n+1)} \sum_{m=0}^{n} \int_0^1 f(1-x)t \, dt \, a_{m,k}(x)
\]

where

\[
s = 0, \text{ for } f = 1 \\
s \text{ is even when } f \text{ is even} \\
s \text{ is odd when } f \text{ is odd}
\]

and

\[
a_{n,k}(x) = mC_k (-x)_k (1-x)^{-m},
\]

and studied the convergence and other properties of this operator.

It is well-known fact that Bernstein operator as well as most of its generalizations have been obtained from the identity

\[
\sum_{k=0}^{n} p_{n,k}(x) = 1,
\]

where \(p_{n,k}(x)\) is the binomial distribution.

In 1968 Stancu [1] has considered a Bernstein type operator \(P_n(a)(f;x)\) defined as

\[
P_n(a)(f;x) = \sum_{k=0}^{n} w_{n,k}(x,a) f \left( \frac{k}{n} \right),
\]

based on the Pólya distribution.
(1.8.5) \[ w_{n,k}(x;\alpha) = \binom{n}{k} \frac{\sum_{k=0}^{k-1} (x + \nu \alpha) \frac{n-k-1}{\mu \alpha}}{\lambda \sum_{k=0}^{n-1} (1 + \lambda \alpha)} \]

with

(1.8.6) \[ \sum_{k=0}^{n} w_{n,k}(x;\alpha) = 1, \]

\( \alpha \) being a non-negative parameter.

For \( \alpha = 0 \) the operator (1.8.4) reduces to the classic Bernstein operators (1.6.1).

Stancu [1] proved the following theorems for the operator \( p_n^{[\alpha]}(f; x) \).

**Theorem (1.12)**: If \( f \in C[0,1] \) and \( 0 \leq \alpha = \alpha(n) \rightarrow 0 \) as \( n \rightarrow \infty \), then the sequence \( \{p_n^{[\alpha]}(f; x)\} \) converges to \( f(x) \) uniformly in \( [0,1] \).

**Theorem (1.13)**: If \( f \in C[0,1] \) and \( \alpha \geq 0 \), then

\[ |f(x) - p_n^{[\alpha]}(f; x)| \leq \frac{3}{2} \omega(f; \left(\frac{1 + \alpha n}{n + \alpha n}\right)^{1/2}) \]

**Theorem (1.14)**: If \( f \in C^1[0,1] \), then the following inequality


\[ |f(x) - p_n^{[\alpha]}(f; x)| \leq \frac{3}{4} \left( \frac{1+\tan}{n+\tan} \right)^{\frac{1}{2}} \omega_1(f; \left( \frac{1+\tan}{n+\tan} \right)^{\frac{1}{2}}) \]

holds, where \( \omega_1(\delta) \) is the modulus of continuity of \( f' \).

**Theorem (1.15):** Let \( \alpha = \alpha(n) \to 0 \) as \( n \to \infty \). If \( f \) is bounded on \([0,1]\) and possesses a second derivative at a point \( x \) of \([0,1]\), then

\[ f(x) - p_n^{[\alpha]}(f; x) = -\frac{1+\tan}{1+\alpha} \frac{x(1-x)}{2n} f''(x) + \frac{\varepsilon_n^{[\alpha]}(x)}{n}, \]

where \( \varepsilon_n^{[\alpha]}(x) \) tends to zero when \( n \) tends to infinity.

In Chapter IV we have developed a corresponding method for approximation of continuous functions, in the same way as Kantorovich operators are constructed from Bernstein operators, for Stancu operators and defined the operators

\[ w_n^{[\alpha]} : C[0,1] \to C[0,1] \text{ as:} \]

\[ w_n^{[\alpha]}(f; x) = (n+1) \sum_{k=0}^{k+1} \frac{1}{n+1} \int_0^1 f(t) \, dt \, w_{n,k}(x; \alpha), \]

where \( w_{n,k}(x; \alpha) \) is the same as defined in (1.8.5).

One may observe that the results corresponding to Bernstein-Kantorovich operators can easily be obtained by our operator \( w_n^{[\alpha]}(f; x) \) as particular cases when \( \alpha = 0 \).
If $f(x)$ is defined on the interval $(0, b)$, $b > 0$, the Bernstein polynomial $B_n^b(x; b)$ for this interval is given by

$$B_n^b(x) = B_n(x; b) = \sum_{k=0}^{n} \binom{n}{k} \binom{b}{k} (\frac{x}{b})^k (1-\frac{x}{b})^{n-k}.$$

For the polynomial of this type Chlodovsky [1] proved the following theorem by taking $b = b_n$ as a function of $n$, which increases to $\infty$ with $n$, and $f(x)$ is defined in the infinite interval $0 \leq x < +\infty$.

**Theorem (1.16):** If $b_n = o(n)$ and the function $f(x)$ is bounded in $(0, +\infty)$, say $|f(x)| \leq M$, then $B_n(x) \longrightarrow f(x)$ holds at any point of continuity of the function $f$.

**Theorem (1.17):** If $b_n = o(n)$ and $M(b_n) = \frac{b_n}{n} \longrightarrow 0$ for each $a > 0$, then $B_n(x; b) \longrightarrow f(x)$ holds at each point of continuity of the function $f$.

In Chapter V we have modified the operator $W_n^a(f; x)$ in analogy with the Chlodovsky [1]. We set $y = x b^{-1}$ in the operators.
\( w_n^f(y) \) of the function \( g(y) = f(by), 0 \leq y \leq 1 \) to obtain the desired generalization of the Stancu operators as follows:

\[
(1.8.9) \quad w_n(x) = w_n^f(x; a, b) = (n+1) \sum_{k=0}^{n} \binom{n}{k} \int_{0}^{1} f(tb) \, dt \, w_{n,k} \left( \frac{x}{b} \right),
\]

where

\[
(1.8.10) \quad w_{n,k}(\frac{x}{b}) = \left( \begin{array}{c} n \\ k \end{array} \right) \frac{k-1}{b} \frac{\gamma a}{(1+\beta + \mu a)} \frac{n-k-1}{\mu b} \frac{(1-\beta + \mu a)}{(1+\gamma a)}.
\]

We have dealt with approximation of functions on an unbounded interval and established certain results similar to that earlier studied by Chlodovsky [1] for Bernstein operators for our generalized Bernstein operators \( w_n^f(x; a, b) \).

1.9. **Approximation by Boral exponential means**

The problem of approximation of continuous functions by finite-summability operators has been discussed by Alexits [1], Sahney and Geel [1], Holland et al. [1], Wafi etc. For the sake of completeness we state only their results.
Alexits [1] in 1928 proved the following theorem on the
degree of approximation of function \( f \in \text{Lip } \alpha \), by \((C, \delta)\)-means
of its Fourier series.

**Theorem (1.10):** If a periodic function \( f \in \text{Lip } \alpha \) for
\( 0 < \alpha \leq 1 \), the degree of approximation of the \((C, \delta)\)-means of its
Fourier series for \( 0 < \alpha < \delta \leq 1 \) is given by

\[
\max_{0 \leq k \leq n^2} | t_n^{(\delta)}(x) - f(x) | = O\left( \frac{1}{n^\alpha} \right),
\]

and for \( 0 < \alpha < \delta \leq 1 \), is given by

\[
\max_{0 \leq k \leq n^2} | t_n^{(\delta)}(x) - f(x) | = O\left( \frac{\log n}{n^\alpha} \right),
\]

where \( t_n^{(\delta)} \) are the \((C, \delta)\) - means of the partial sums of \( f \).

Later on Sahney and Goel [1] and Holland et al. [1] extended
**Theorem (1.18)** for the degree of approximation to a function \( f \)
by Nörlund means of its Fourier series belonging to \text{Lip } \alpha \) and
\( C^\infty[0, 2\pi] \), the class of all continuous functions on \([0, 2\pi] \), periodic
and of period \( 2\pi \).
Theorem (1.19) (Sahney et al. [1]): The degree of approximation of a periodic function \( f \) with period \( 2\pi \) and belonging to Lip \( \alpha, 0 < \alpha \leq 1 \) is given by

\[
\max_{0 \leq x \leq 2\pi} | T_n(x) - f(x) | = o \left( \frac{1}{P_n} \sum_{k=1}^{n} \frac{p_k}{k^{1+\alpha}} \right),
\]

where \( T_n(x) \) are the \( (N,p_n) \) - means of its Fourier series for \( f \) provided the sequence \( \{p_n\} \) is positive and non-increasing.

Theorem (1.20) (Holland et al. [1]): If \( \omega(t) \) is the modulus of continuity of \( f \in C^\alpha[0,2\pi] \), then the degree of approximation of \( f \) by Nörlund means of the Fourier series of \( f \) is given by

\[
\max_{0 \leq t \leq 2\pi} | T_n(t) - f(t) | = o \left( \frac{1}{P_n} \sum_{k=1}^{n} \frac{p_k \omega \left( \frac{1}{k} \right)}{k} \right),
\]

where \( T_n \) are the \( (N,p_n) \) - means of the Fourier series of \( f \).

Recently, Wafi [1] generalized the above results by replacing the operators \( (N,p_n) \) - means by matrix operators and proved the following theorem which includes theorems (1.18), (1.19) and (1.20) as special cases.

Theorem (1.21) (Wafi [1]): If \( \left\{ D_{a_{nk}} \right\}_{k=0}^{n} \) is a non-negative and non-decreasing sequence and if \( \omega(t) \) is the modulus of
continuity of \( f \in C^2[0,2\pi] \), then the degree of approximation of \( f \) by matrix-means of the Fourier series is given by

\[
\max_{0 \leq x \leq 2\pi} | t_n(f \ast x) - f(x) | = O\left( \sum_{k=1}^{n} D^k a_n, n-k \omega\left( \frac{1}{k} \right) \right),
\]

where \( t_n(f \ast x) \) are the matrix-means of the Fourier series of \( f \) and

\[
D^k a_n, n-k = a_n, k - a_n, k-1.
\]

In Chapter VI we determine the degree of approximation of functions for infinite summability operators viz. Borel exponential means using a wider class of functions \( \text{Lip}_k(t) \) of which \( \text{Lip} \alpha \) is the special case.