CHAPTER II
\(\lambda\)-STRONG CONVERGENT SEQUENCE DEFINED BY ORLICZ FUNCTION

2.1. PRELIMINARIES AND INTRODUCTION.

Let \(\lambda = \lambda_n\) be a non decreasing sequence of positive reals tending to infinity and \(\lambda_1 = 1\) and \(\lambda_{n+1} \leq \lambda_n + 1\). The generalized da la vallee-Pousin means is defined by;

\[
l_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k.
\]

Where \(I_n = [n - \lambda_{n+1}, n]\). A sequence \(x = (x_k)\) is said to be \((V, \lambda)\)-summable to a number \(l\) if \(t_n(x) \to l\) as \(n \to \infty\). Define:

\[
[V, \lambda]_0 = \left\{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0 \right\},
\]

\[
[V, \lambda] = \left\{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - l| = 0 \text{ for some } l \in \mathbb{C} \right\}
\]

and

\[
[V, \lambda]_\infty = \left\{ x = x_k : \sum_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| < \infty \right\},
\]

for the sets of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallee-Pousin method. In the special case where \(\lambda_n = n\), for \(n = 1, 2, 3, \ldots\), the sets \([V, \lambda]_0, [V, \lambda] \) and \([V, \lambda]_\infty\) reduces to the sets \(W_0, W\) and \(W_\infty\) introduced and studied by Maddox[20].

Let \(M\) be an Orlicz function and \(p = (p_k)\) be any sequence of strictly positive real numbers. E.Savas and R.Savas defined the following sequence spaces:

\[
[V, M, p] = \left\{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} M\left(\frac{|x_k - l|}{\rho}\right)^{p_k} = 0 \text{ for some } l \text{ and } \rho > 0 \right\},
\]
\( [V, M, p]_0 = \left\{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in \mathbb{I}_n} \left[ M\left( \frac{|x_k|}{\rho} \right) \right]^{p_k} = 0 \text{ for some } \rho > 0 \right\} \)

and

\( [V, M, p]_{\infty} = \left\{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sup_{k \in \mathbb{I}_n} \left[ M\left( \frac{|x_k|}{\rho} \right) \right]^{p_k} < \infty \text{ for some } \rho > 0 \right\} \).

**Remark 2.1.1.** When \( p_k = 1 \) for all \( k \), then denote \([V, M, p], [V, M, p]_0 \) and \([V, M, p]_{\infty} \) as \([V, M], [V, M]_0 \) and \([V, M]_{\infty} \). If \( x \in [V, M] \) we say that \( x \) is \( \lambda \)-strongly convergent with respect to the Orlicz function \( M \).

**Remark 2.1.2.** If \( M(x) = x, \ p_k = 1 \) for all \( k \), then \([V, M, p] = [V, \lambda], [V, M, p] = [V, \lambda]_0 \) and \([V, M, p]_{\infty} = [V, \lambda]_{\infty} \).

### 2.2. MAIN RESULTS.

**Theorem 2.2.1.** For any Orlicz function \( M \) and any sequence \( p = \{p_k\} \) of strictly positive real numbers, \([V, M, p], [V, M, p]_0 \) and \([V, M, p]_{\infty} \) are linear spaces over the set of complex numbers.

**Proof:** Let \( x, y \in [V, M, p]_0 \) and \( \alpha, \beta \in \mathbb{C} \).

In order to prove the result we need to find \( \rho_3 > 0 \) such that

\[
\lim_{n} \frac{1}{\lambda_n} \sum_{k \in \mathbb{I}_n} \left[ M\left( \frac{|\alpha x_k + \beta y_k|}{\rho_3} \right) \right]^{p_k} = 0.
\]

Since \( x, y \in [V, M, p]_0 \), there exists some positive \( \rho_1 \) and \( \rho_2 \) such that

\[
\lim_{n} \frac{1}{\lambda_n} \sum_{k \in \mathbb{I}_n} \left[ M\left( \frac{|x_k|}{\rho_1} \right) \right]^{p_k} = 0
\]

and

\[
\lim_{n} \frac{1}{\lambda_n} \sum_{k \in \mathbb{I}_n} \left[ M\left( \frac{|y_k|}{\rho_2} \right) \right]^{p_k} = 0.
\]
Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$.

Since $M$ is non decreasing and convex,

$$
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\alpha x_k + \beta y_k|}{\rho_3} \right) \right]^{p_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\alpha x_k|}{\rho_3} + \frac{|\beta y_k|}{\rho_3} \right) \right]^{p_k}
$$

$$
\leq \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} \left[ M \left( \frac{|x_k|}{\rho_1} + M \left( \frac{|y_k|}{\rho_2} \right) \right) \right]^{p_k}
$$

$$
\leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho_1} \right) + M \left( \frac{|y_k|}{\rho_2} \right) \right]^{p_k}
$$

$$
\leq D \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho_1} \right) \right]^{p_k} + D \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|y_k|}{\rho_2} \right) \right]^{p_k}
$$

$$
\rightarrow 0 \text{ as } n \rightarrow \infty
$$

(from equation [1.3.2])

Where $D = \max(1, 2^{G^{-1}})$, $G = \sup p_k$, so that $\alpha x + \beta y \in [V, M, p]_0$.

Let $x, y \in [V, M, p]$, and $\alpha, \beta \in \mathbb{C}$. In order to show that $[V, M, p]$ is a linear space we need to find some $\rho_3 > 0$ such that

$$
\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\alpha x_k + \beta y_k - l|}{\rho_3} \right) \right]^{p_k} = 0 \text{ for some } l \text{ and } \rho > 0.
$$

Since $x, y \in [V, M, p]$, there exists some positive $\rho_1$ and $\rho_2$ such that

$$
\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k - l_1|}{\rho_1} \right) \right]^{p_k} = 0 \text{ and } l_1 > 0
$$

$$
\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|y_k - l_2|}{\rho_1} \right) \right]^{p_k} = 0 \text{ and } l_2 > 0.
$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$.

Since $M$ is non decreasing and convex, $l = \max(l_1, l_2)$

$$
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\alpha x_k + \beta y_k - l|}{\rho_3} \right) \right]^{p_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\alpha x_k - l_1|}{\rho_3} + \frac{|\beta y_k - l_2|}{\rho_3} \right) \right]^{p_k}
$$
\[
\leq \frac{1}{\lambda_n} \sum_{k \in I_n} 2^{pk} \left[ M\left( \frac{|x_k - l_1|}{\rho_1} + M\left( \frac{|y_k - l_2|}{\rho_2} \right) \right)^{pk} \right]
\]

\[
\leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M\left( \frac{|x_k - l_1|}{\rho_1} \right)^{pk} \right]
\]

\[
\leq D \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M\left( \frac{|x_k - l_1|}{\rho_1} \right)^{pk} \right]
\]

\[
+ D \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M\left( \frac{|y_k - l_2|}{\rho_2} \right)^{pk} \right]
\]

\[\to 0 \text{ as } n \to \infty\]

(from equation [1.3.2])

Where \( D = \max(1, 2^G - 1) \), \( G = \sup p_k \), so that \( \alpha x + \beta y \in [V, M, p] \).

Similarly we can show that \([V, M, p]_\infty\) is a linear space over the set of complex numbers.

**Theorem 2.2.2.** For any Orlicz function \( M \) and a bounded sequence \( p = (p_k) \) of strictly positive real numbers, \([V, M, p]_0\) is a total paranormed spaces with

\[
g(x) = \inf\left\{ \rho^{\frac{n}{\rho}} : \left( \sum_k \left[ M\left( \frac{|x_k|}{\rho} \right)^{pk} \right] \right)^{\frac{1}{n}} \leq 1, \quad n = 1, 2, ... \right\}.
\]

**Proof:** \( g(-x) = \inf\left\{ \rho^{\frac{n}{\rho}} : \left( \sum_k \left[ M\left( \frac{|x_k|}{\rho} \right)^{pk} \right] \right)^{\frac{1}{n}} \leq 1, \quad n = 1, 2, ... \right\} \)

\[= \inf\left\{ \rho^{\frac{n}{\rho}} : \left( \sum_k \left[ M\left( \frac{|x_k|}{\rho} \right)^{pk} \right] \right)^{\frac{1}{n}} \leq 1, \quad n = 1, 2, ... \right\} \]

\[= g(x). \]

By using theorem (2.2.1), for \( \alpha = \beta = 1 \), we get

\[g(x + y) \leq g(x) + g(y).\]

Suppose that \( x = 0 \), then since \( M(0) = 0 \), we get

\[\inf \left\{ \rho^{p_n} H \right\} = 0 \Rightarrow g(x) = 0 \text{ for } x = 0. \]
Conversely, suppose \( g(x) = 0 \), then

\[
\inf \{ \rho^n : \left( \sum_k \left[ M \left( \frac{|x_k|}{\rho} \right) \right]^{pk} \right)^{\frac{1}{p}} \leq 1 \} = 0.
\]

This implies that for \( \epsilon > 0 \), there exists some \( \rho_\epsilon \) (\( 0 < \rho_\epsilon < \epsilon \)) such that

\[
\left( \frac{1}{\lambda_n} \sum_k \left[ M \left( \frac{|x_k|}{\epsilon} \right) \right]^{pk} \right)^{\frac{1}{p}} \leq \left( \frac{1}{\lambda_n} \sum_k \left[ M \left( \frac{|x_k|}{\rho_\epsilon} \right) \right]^{pk} \right)^{\frac{1}{p}} \leq 1.
\]

Suppose \( x_{nm} \neq 0 \) for some \( m \in I_n \). Let \( \epsilon \to 0 \), then \( \left( \frac{|x_{nm}|}{\epsilon} \right) \to \infty \) it follows that

\[
\left( \frac{1}{\lambda_n} \sum_k \left[ M \left( \frac{|x_{nm}|}{\epsilon} \right) \right]^{pk} \right)^{\frac{1}{p}} \to \infty.
\]

Which is a contradiction. Therefore \( x_{nm} = 0 \) for each \( m \). Finally we show that scalar multiplication is continuous. Let \( \mu \) be any complex number. Then by definition we have;

\[
g(\mu x) = \inf \left\{ \rho^n : \left( \sum_k \left[ M \left( \frac{|\mu x_k|}{\rho} \right) \right]^{pk} \right)^{\frac{1}{p}} \leq 1, \ n = 1, 2, \ldots \right\}
\]

\[
= \inf \left\{ (|\mu|s)^n : \left( \sum_k \left[ M \left( \frac{|x_k|}{s} \right) \right]^{pk} \right)^{\frac{1}{p}} \leq 1, \ n = 1, 2, \ldots \right\}.
\]

Where \( s = \frac{\rho}{|\mu|} \).

Since \( |\mu|^p \leq \max(1, |\mu|^{\sup H}) \), we have

\[
g(\mu x) \leq \max(1, \lambda^{\sup H})^{\frac{1}{p}} \inf \left\{ s^n : \left( \sum_k \left[ M \left( \frac{|x_k|}{s} \right) \right]^{pk} \right)^{\frac{1}{p}} \leq 1, \ n = 1, 2, \ldots \right\}
\]

\[
\leq \max(1, |\lambda|^H)^{\frac{1}{p}} \cdot g(\mu x).
\]

Which converges to zero as \( x \) converges to zero in \([V, M, p]_0\).

Now suppose \( \mu_m \to 0 \) and \( x \) be a fixed element in \([V, M, p]_0\).

For arbitrary \( \epsilon > 0 \), let \( N \) be a positive integer such that

\[
\frac{1}{\lambda_n} \sum_k \left[ M \left( \frac{|x_k|}{\rho} \right) \right]^{pk} < \left( \frac{\epsilon}{2} \right)^H \text{ for some } \rho > 0 \text{ and all } n > N.
\]
This implies that

\[ \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M\left( \frac{|x_k|}{\rho} \right) \right]^{p_k} < \frac{\epsilon}{2} \text{ for some } \rho > 0 \text{ and all } n > N. \]

Let \(0 < |\mu| < 1\), using convexity of \(M\), for \(n > N\), we get

\[ \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M\left( \frac{|\mu x_k|}{\rho} \right) \right]^{p_k} < \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ |\mu| M\left( \frac{|x_k|}{\rho} \right) \right]^{p_k} < \left( \frac{\epsilon}{2} \right)^{H}. \]

Since \(M\) is continuous everywhere in \([0, \infty]\), then for \(n \leq N\)

\[ f(t) = \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M\left( \frac{|\mu x_k|}{\rho} \right) \right]^{p_k} \text{ is continuous at } 0. \]

So there exists \(1 > \delta > 0\) such that \(|f(t)| < \left( \frac{\delta}{2} \right)^{H}\) for \(0 < t < \delta\).

Let \(K\) be such that \(|\mu_m| \leq \infty\) for \(m > K\), then for \(m > K\) and \(n \leq N\)

\[ \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M\left( \frac{|\mu x_k|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{H}} \leq \frac{\epsilon}{2}. \]

Thus

\[ \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M\left( \frac{|\mu x_k|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{H}} < \epsilon. \]

for \(m > K\) and all \(n\), so that \(g(\mu x) \to 0 \ (\mu \to 0)\).

**Theorem 2.2.3.** For any Orlicz function \(M\) which satisfies \(\Delta_2\)-condition, we have \([V, \lambda] \subseteq [V, M]\).

**Proof:** Let \(x \in [V, \lambda]\) then

\[ T_{n} = \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - l| \to 0 \text{ as } n \to \infty \text{ for some } l. \]

\[ T_{n} \lambda_{n} = \sum_{k \in I_n} |x_k - l| \to 0 \text{ as } n \to \infty. \]
Let $\varepsilon > 0$ and choose $\delta$ such that $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t < \delta$.

Let $y_k = |x_k - l|$ and consider

$$
\frac{1}{\lambda_n} \sum_{k \in I_n} M(|y_k|) = \sum_{1} + \sum_{2}
$$

Where the first summation is over $y_k \leq \delta$ and the second summation $y_k > \delta$.

Since $M$ is continuous

$$
\sum 1 < \lambda_n \varepsilon \quad \text{that is} \quad \sum_{y_k \leq \delta} M(|y_k|) < \lambda_n \varepsilon.
$$

Now for $y_k > \delta$, we use the fact that

$$
y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.
$$

Since $M$ is non decreasing and convex

$$
M(y_k) < M(1 + \delta^{-1}y_k) < M(1) + M(\delta^{-1}y_k)
$$

$$
< M\left(\frac{1}{2} \cdot 2\right) + M\left(\frac{1}{2} \cdot 2\delta^{-1}y_k\right).
$$

Since $M$ satisfies $\Delta_2$-condition.

$$
M(y_k) < \frac{1}{2} K M(2) + \frac{1}{2} K \delta^{-1} y_k M(2)
$$

$$
< M(y_k) < \frac{1}{2} K y_k \delta^{-1} M(2) + \frac{1}{2} K \delta^{-1} y_k M(2)
$$

(since $y_k > \delta \Rightarrow y_k \delta^{-1} > 1$)

$$
< K \delta^{-1} y_k M(2).
$$

Hence

$$
\sum 2M(y_k) \leq K \delta^{-1} M(2) \lambda_n T_n
$$

This implies that

$$
\frac{1}{\lambda_n} \sum_{k \in I_n} M(|y_k|) = \sum_{1} + \sum_{2} \rightarrow 0 \text{ as } n \rightarrow \infty
$$
yielding thereby $[V, \lambda] \subseteq [V, M]$.

**Remark 2.2.1.** The method of the proof of theorem (2.2.3) shows that for any Orlicz function $M$ which satisfies $\Delta_2$-condition, we have $[V, \lambda]_0 \subseteq [V, M]_0$ and $[V, \lambda]_\infty \subseteq [V, M]_\infty$.

**Theorem 2.2.4.** Let $0 \leq p_k \leq q_k$ and $\left( \frac{p_k}{q_k} \right)$ be bounded. Then $[V, M, q] \subseteq [V, M, p]$.

**Proof:** Let $x \in [V, M, q]$ Write $t_k = \left[ M \left( \frac{|x_k - \ell|}{\rho} \right) \right]^{q_k}$ and $\lambda_k = \frac{p_k}{q_k}$.

Since $p_k < q_k$, therefore $0 < \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$. Define;

$$u_k = \begin{cases} t_k & t_k \geq 1 \\ 0 & t_k < 1. \end{cases}$$

$$v_k = \begin{cases} 0 & t_k \geq 1 \\ t_k & t_k < 1. \end{cases}$$

So $t_k = u_k + v_k$ and $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$.

Now it follows that $u_k^{\lambda_k} \leq t_k$, and $v_k^{\lambda_k} \leq v_k^\lambda$.

Therefore

$$\frac{1}{\lambda_k} \sum_{k \in I_n} t_k^{\lambda_k} \leq t_k + \frac{1}{\lambda_k} \sum_{k \in I_n} t_k + \left( \frac{1}{\rho} \sum_{k \in I_n} v_k \right)^\lambda.$$ 

This implies that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k - \ell|}{\rho} \right) \right]^{q_k} \frac{p_k}{q_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k - \ell|}{\rho} \right) \right]^{q_k} + \left( \frac{1}{\lambda_n} \sum_{k \in I_n} v_k \right)^\lambda \to 0 \text{ as } n \to \infty$$

Since $x \in [V, M, q]$ and $v_k = 0$, $t_k \geq 1$.

This implies that
This implies that \( x \in [V, M, p] \).

### 2.3. \( \lambda \)-Statistical Convergence.

A sequence \( x = (x_k) \) is said to be \( \lambda \)-Statistically Convergent or \( s_\lambda \)-Statistically Convergent to \( L \) if for every \( \epsilon > 0 \)

\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \left| \{ k \in I_n : |x_k - L| \geq \epsilon \} \right| = 0,
\]

where the vertical bars indicate the number of elements in the enclosed set. In this case we write \( s_\lambda - \lim x = L \) or \( x_k \to L(s_\lambda) \) and \( s_\lambda = \{ x : \exists L \in \mathbb{R} : s_\lambda - \lim x = L \} \).

**Theorem 2.3.1.** For any Orlicz function \( M, [V, M] \subset s_\lambda \).

**Proof:** Let \( x \in [V, M] \) and \( \epsilon > 0 \). Then

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} M \left( \frac{|x_k - l|}{\rho} \right) \geq \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - L| \geq \epsilon} M \left( \frac{|x_k - l|}{\rho} \right) \\
\geq \frac{1}{\lambda_n} M \left( \frac{\epsilon}{\rho} \right) \left| \left\{ k \in I_n : |x_k - L| \geq \epsilon \right\} \right|.
\]

Hence \( x \in s_\lambda \).

This implies that \( [V, M] \subset s_\lambda \).