CHAPTER IV

Multifractal Analysis of Multiplicity Fluctuations in $^{28}\text{Si}-\text{AgBr}$ Collisions at 14.6A GeV
4.1 Introduction

The study of non-statistical fluctuations in the density of charged particles produced in high-energy collisions is considered an important tool in the understanding of the dynamics of multiparticle production. Among various methods of studying the non-statistical fluctuations, the most commonly used one is the method of studying the behaviour of the scaled factorial moments $F_q$ as a function of the size of phase space interval $\delta \eta$. Bialas and Peschanski [1] first suggested this method. This method removes statistical fluctuations due to the finite number of charged particles in an event. A power law growth of the scaled factorial moments with decreasing size of phase space interval $\delta \eta$ ($F_q \propto \delta \eta^{-\phi_q}$) signals the presence of dynamical fluctuations in the data. Being similar in nature to the power law variation of moments due to the intermittent structure that appears in a turbulent fluid hydro-dynamics, the phenomenon of power law variation of the scaled $F_q$-moments is termed as intermittency. $\phi_q$ are called intermittency indices. Evidence of intermittency has been found in $e^+ e^-$ annihilation [2-4], hadron-nucleus [5,6] and nucleus-nucleus collisions [7-9].

The power law behaviour of the scaled factorial moments indicates the existence of self-similarity. The concept of self-similarity is closely related to multifractality, which is a consequence of cascade mechanism prevailing in the multiparticle production. Therefore, the necessity of using the multifractal technique to study the multiplicity fluctuations has arisen. In the multifractal technique, the nuclear interactions can be treated as geometrical objects with non-integer dimensions. The technique has been successfully used to study the intermittent behavior of turbulent fluids and other chaotic systems [10-12]. Chiu and Hwa [13] developed formalism for investigating the fractal structure in multiparticle production. According to this formalism, the empty bins in the pseudorapidity distribution are treated as holes and the set of non-empty bins at any stage of binning constitutes one of the fractal sets of multifractal structure in multiparticle production. Using this concept of multifractality, Chiu and Hwa [13] defined $G_q$-moments and evaluated the parameters that characterize the fractal properties. The $G_q$-moments share with the scaled factorial moments the property that the self-similar density fluctuations lead to the power law behavior: $G_q \propto \delta \eta^{-\tau_q}$ as $\delta \eta \to 0$, where $\tau_q$ are known as fractal indices. This method has been used to study multifractality in $e^+ e^-$ annihilation...
The $G_q$-moments have the advantage that not only peaks of the rapidity distribution are included in the analysis but also the non-empty valleys ($\eta<0$). The disadvantages are that the $G_q$-moments saturate as $S/\pi$. Further it has been found that for low multiplicity events the $G_q$-moments are dominated by statistical fluctuations [15]. In order to suppress the statistical contribution to the observed fluctuations, Hwa and Pan [17] suggested a modified form of $G_q$-moments with the help of a step function.

Although many attempts have been made to study the fractal properties of $\mu p$, $p\bar{p}$ and $e^+e^-$ data [18,19] using the modified $G_q$-moments, there have been only a few attempts to study the fractal behavior of relativistic nucleus-nucleus collisions using this method [20,21]. In this chapter, we therefore investigate the fractal behavior of multiplicity fluctuations in $^{28}\text{Si-AgBr}$ collisions at 14.6A GeV using the modified $G_q$-moments in both the pseudorapidity ($\eta$) and azimuthal angle ($\phi$) spaces. The dynamical component of $<G_q>$ has been determined using the method suggested by Hwa and Pan [17].

As already mentioned above, the power law behaviour of $F_q$-moment indicates the existence of self-similarity, which is closely related to multifractality. We have therefore studied the connection between $F_q$-moments and $G_q$-moments. Using a relation between the intermittency indices $\phi_q$ and fractal indices $\tau_q$, multifractality in our data has also been studied through $F_q$-moment method in both the pseudorapidity ($\eta$) and azimuthal angle ($\phi$) spaces. And we have calculated the values of the generalized fractal dimensions $D_q^{\delta_m}$ and the multifractal specific heat ($c$) for our data.

Apart from the modified $G_q$-moment and $F_q$-moment methods, Takagi method [22] has also been used to study multifractality in multiparticle production. This method has been successfully applied to study multifractality in UA5 data on $p\bar{p}$ collisions [23] and TASSO and DELPHI data on $e^+e^-$ annihilations [24,25]. This method has also been used to study the multifractal behaviour of shower particles produced in hadron-nucleus [26,27] and nucleus-nucleus collisions [27-29] at relativistic energies. Ghosh et al. [28,29] have used Takagi method to study multifractal behaviour of target evaporated slow particles (black particles) and medium energy target fragmented protons (grey particles) emitted in $^{32}\text{S-AgBr}$ collisions at 200A GeV and $^{16}\text{O-AgBr}$ collisions at
60A GeV. We have also used Takagi method to study the multifractal behavior of shower particles produced in $^{28}$Si-AgBr collisions at 14.6A GeV in the pseudorapidity ($\eta$) and azimuthal angle ($\phi$) spaces. In both the spaces the multiplicity moments are found to have a power law dependence on the mean multiplicity with decreasing bin size.

Several models of multiparticle production in relativistic nucleus-nucleus collisions have been developed. A comparison of our results with the corresponding results for events generated using these models would provide valuable inputs to these models. We have therefore simulated $^{28}$Si-AgBr collisions using the string hadronic model UrQMD and found that our results deviate significantly from the corresponding results for UrQMD events.

4.2 Results and Discussion

Various methods have been proposed to investigate multifractality in multiparticle production. We have studied the multifractality in our data using the following three methods: (i) modified $G_q$-moment method [17], (ii) $F_q$-moment method [1] and (iii) Takagi method [22]. The analysis has been done for $^{28}$Si-AgBr collisions only. Out of the total 784 inelastic collisions collected in the present experiment, 360 were $^{28}$Si-AgBr collisions. To minimize the contribution of statistical fluctuations due to low multiplicity, only collisions with $N_\gamma \geq 8$ were considered for the analysis. Thus 297 events representing $^{28}$Si-AgBr collisions were selected for the present analysis. Further the analysis was restricted to shower particles lying in the central region of pseudorapidity with $\eta_{peak} - 2 \leq \eta \leq \eta_{peak} + 2$. This range of pseudorapidity covers almost all produced particles.

The single particle density distribution in the pseudorapidity space is not flat due to the energy and momentum constraints [30]. Although some models [31] predict a flat central region in the density distribution in the pseudorapidity space at ultrarelativistic energies, the experimental distributions are found to have some shape even in the central region [30]. To avoid the effect of non-flat particle density distribution $A(\eta)$ on the investigation of the dynamical fluctuations, Bialas and Gazdzicki [32] have introduced a new scaled cumulative variable $X$ related to the single particle density distribution $A(\eta)$ as
where $\eta_1$ and $\eta_2$ are the minimum and maximum values of the pseudorapidity in the interval $\Delta \eta$. In $X$ space, the single particle density distribution is uniformly distributed from 0 to 1. Pseudorapidity values of shower particles emitted in all $^{28}\text{Si}-\text{AgBr}$ collisions were converted to the corresponding $X$-values using Equation 4.1.

4.2.1 Modified $G_q$-Moment Method

In order to suppress the contribution of statistical fluctuations, Pan and Hwa [17] defined a modified form of $G_q$-moment as

$$G_q = \sum_{m=1}^{M} \left( \frac{n_m}{N} \right)^q \theta(n_m - q),$$

where $q$ is a positive integer, $M$ is the total number of bins in which the $X$-interval (0-1) has been divided, $n_m$ is the multiplicity of particles in the $m$th bin, $N = \sum_{m=1}^{M} n_m$ denotes the total multiplicity in the interval $X = 0$-1 and $\theta$ is the step function. It is equal to 1 for $n_m \geq q$ and 0 otherwise. This definition of $G_q$-moments differs from the earlier definition [13] only by the $\theta$ function. When the multiplicity $N$ is very large so that $N/M \gg q$, the two definitions of $G_q$ give the same result. However, for low multiplicity events, the step function exerts a crucial influence on the values of $G_q$-moments. For an ensemble of events, the vertical average of $G_q$-moments can be calculated from the following relation.

$$\langle G_q \rangle = \frac{1}{N_{ev}} \sum_{1}^{N} G_q,$$

where $N_{ev}$ is the total number of events in the ensemble. Values of $\langle G_q \rangle$ for our data were calculated using Equations 4.2 and 4.3 for $q = 2$-6. $M$ was varied from 2 to 35. The $\ln \langle G_q \rangle$ versus $\ln M$ graphs for $^{28}\text{Si}-\text{AgBr}$ collisions with $N_e \geq 8$ are shown in Figure 4.1(a). $\ln \langle G_q \rangle$ exhibits a linear dependence on $\ln M$, indicating that $G_q$-moments have a power law dependence on $M$ of the form

$$\langle G_q \rangle \propto M^{-\gamma_q},$$

where $\gamma_q$ is the exponent.
Figure 4.1: (a) $\ln <G>_{q}$ versus $\ln M$ plots for $^{28}$Si-AgBr collisions at 14.6 $A$ GeV in $\eta$-space.
(b) $\ln <G>_{q}$ versus $\ln M$ plots for randomly generated events in $\eta$-space.
(c) $\ln <G>_{q}$ versus $\ln M$ plots for UrQMD events in $\eta$-space.
This is an evidence of self-similarity in the multiparticle production in our data. Straight lines in Figure 4.1(a) show the linear fits to the data points. Slopes of the lines \( r_q \) for different values of \( q \) are given in Table 4.1.

To calculate the statistical contribution to \( G_q \), random events were generated in \( X \)-space according to the following criteria:

(i) \( N \) particles in each event are distributed randomly in \( X \)-space from 0-1.

(ii) The randomly generated events have the same multiplicity distribution as that of the experimental events.

\( \ln < G_q >^{stat} \) versus \( \ln M \) graphs for the randomly generated events for \( q = 2-6 \) are shown in Figure 4.1(b). As can be seen from the figure, \( < G_q >^{stat} \) exhibits a linear dependence on \( \ln M \) indicating that \( < G_q >^{stat} \), like \( < G_q > \), also exhibits power law dependence on \( M \) as

\[
< G_q >^{stat} \propto M^{-r_q^{stat}}
\]

Values of \( r_q^{stat} \), obtained from straight line fits to the plots, shown in Figure 4.1(b), are listed in Table 4.1. The dynamical component of \( < G_q > \) can now be determined from the following relation

\[
< G_q >^{dyn} = \frac{< G_q >}{< G_q >^{stat}} M^{1-q}
\]

It is clear from Figures 4.1 (a) and (b) that both \( < G_q > \) and \( < G_q >^{stat} \) exhibit a power law dependence on \( M \). If \( < G_q >^{dyn} \) also exhibits a power law dependence on \( M \) as

\[
< G_q >^{dyn} \propto M^{-r_q^{dyn}}
\]

then we would have

\[
r_q^{dyn} = r_q - r_q^{stat} + q - 1
\]

If \( < G_q > \) is purely statistical, that is, \( r_q = r_q^{stat} \) then according to Equation 4.7, \( r_q^{dyn} = (q-1) \). Therefore, any deviation of \( r_q^{dyn} \) from \( (q-1) \) would signal the presence of dynamical fluctuations. Values of \( r_q^{dyn} \) for our data for \( q = 2-6 \) calculated from Equation 4.7 are also
Table 4.1: Values of the slopes $\tau_q$ along with the statistical and dynamical components for $^{28}\text{Si}-\text{AgBr}$ collisions at 14.64 GeV for different values of $q$ in $\eta$-space.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\tau_q$</th>
<th>$\tau_{q,\text{stat}}$</th>
<th>$\tau_{q,\text{dyn}}$</th>
<th>$(q-1)$- $\tau_{q,\text{dyn}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.753 ± 0.017</td>
<td>0.833 ± 0.002</td>
<td>0.920 ± 0.017</td>
<td>0.080 ± 0.017</td>
</tr>
<tr>
<td>3</td>
<td>1.380 ± 0.028</td>
<td>1.641 ± 0.004</td>
<td>1.739 ± 0.028</td>
<td>0.261 ± 0.028</td>
</tr>
<tr>
<td>4</td>
<td>1.912 ± 0.035</td>
<td>2.424 ± 0.008</td>
<td>2.488 ± 0.036</td>
<td>0.512 ± 0.036</td>
</tr>
<tr>
<td>5</td>
<td>2.374 ± 0.045</td>
<td>3.289 ± 0.026</td>
<td>3.085 ± 0.052</td>
<td>0.915 ± 0.052</td>
</tr>
<tr>
<td>6</td>
<td>2.791 ± 0.067</td>
<td>4.196 ± 0.059</td>
<td>3.595 ± 0.089</td>
<td>1.605 ± 0.089</td>
</tr>
</tbody>
</table>
Table 4.2: Values of slopes $\tau_q$ along with the statistical and dynamical components for the events generated using $UrQMD$ model for different values of $q$ in $\eta$-space.

<table>
<thead>
<tr>
<th>q</th>
<th>$\tau_q$</th>
<th>$\tau_q^{\text{stat}}$</th>
<th>$\tau_q^{\text{dyn}}$</th>
<th>$(q-1) - \tau_q^{\text{dyn}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.824 \pm 0.003$</td>
<td>$0.833 \pm 0.002$</td>
<td>$0.991 \pm 0.004$</td>
<td>$0.009 \pm 0.004$</td>
</tr>
<tr>
<td>3</td>
<td>$1.617 \pm 0.004$</td>
<td>$1.641 \pm 0.004$</td>
<td>$1.976 \pm 0.006$</td>
<td>$0.024 \pm 0.006$</td>
</tr>
<tr>
<td>4</td>
<td>$2.408 \pm 0.007$</td>
<td>$2.424 \pm 0.008$</td>
<td>$2.984 \pm 0.011$</td>
<td>$0.016 \pm 0.011$</td>
</tr>
<tr>
<td>5</td>
<td>$3.215 \pm 0.016$</td>
<td>$3.289 \pm 0.026$</td>
<td>$3.926 \pm 0.031$</td>
<td>$0.074 \pm 0.031$</td>
</tr>
<tr>
<td>6</td>
<td>$4.084 \pm 0.043$</td>
<td>$4.196 \pm 0.059$</td>
<td>$4.888 \pm 0.073$</td>
<td>$0.112 \pm 0.073$</td>
</tr>
</tbody>
</table>
listed in Table 4.1. As can be seen from the table, these values of $\tau_{q}^{\text{dyn}}$ deviate significantly from $(q-1)$. Thus fluctuations of dynamical origin are present in our data.

We have also carried out $G_{q}$-moment analysis of our data in the azimuthal angle ($\phi$) space. Like the pseudorapidity distribution, the azimuthal angle distribution of all shower particles lying in the central region of pseudorapidity $\eta_{\text{peak}}-2 \leq \eta \leq \eta_{\text{peak}}+2$ has also been converted into the corresponding $X$-distribution using the relation

$$
X(\phi) = \frac{\int_{0}^{\pi} \rho(\phi) \, d\phi}{\int_{0}^{\pi} \rho(\phi) \, \phi}.
$$

We divided the interval $\Delta X (=1.0)$ into $M$ bins each of size $\delta X = 1.0/M$ and values of modified $<G_{q}>$ moments were calculated using Equations 4.1 and 4.2 for $q=2-6$ for $^{28}\text{Si-ArgBr}$ collisions with $N_s \geq 8$. $M$ was varied from 2-35. Figure 4.2(a) represents the plots of $\ln <G_{q}>$ versus $\ln M$ in $\phi$-space. It is clear from the figure that $\ln <G_{q}>$ exhibits a linear dependence on $\ln M$. Thus in azimuthal angle ($\phi$) space also $G_{q}$-moments for our data have a power law dependence on $M$, which again indicates self-similarity in particle production process. Straight lines in Figure 4.2(a) show linear fits to the data points. Slopes of the lines $\tau_{q}$ for different values of $q$ are given in Table 4.3.

To calculate the statistical contribution to $G_{q}$, random events were generated in the azimuthal angle space according to the following criteria:

(i) $N$ particles in each event are distributed randomly in $X$-space from $X=0$ to $X=1$.

(ii) The randomly generated events have the same multiplicity distribution as that of the experimental events.

Again we divided the interval $\Delta X (=1.0)$ into $M$ bins each of size $\delta X = 1.0/M$ and values of the modified $<G_{q}>^{\text{stat}}$ moments were calculated using Equations 4.1 and 4.2 for $q=2-6$ for $^{28}\text{Si-ArgBr}$ collisions with $N_s \geq 8$. $M$ was varied from 2-35. Figure 4.2 (b) shows plots of $\ln <G_{q}>^{\text{stat}}$ versus $\ln M$ for different values of $q$ in $\phi$-space for the generated events. The straight lines shown in the graph are the linear fits to the plots. The slopes of the fitted lines give the values of $\tau_{q}^{\text{stat}}$ in $\phi$-space. Values of $\tau_{q}^{\text{dyn}}$ were calculated using
Figure 4.2: (a) $\ln\langle G \rangle$ versus $\ln M$ plots for $^{28}$Si-AgBr collisions at 14.64 GeV in $\varphi$-space.
(b) $\ln\langle G \rangle$ versus $\ln M$ plots for randomly generated events in $\varphi$-space.
(c) $\ln\langle G \rangle$ versus $\ln M$ plots for UrQMD events in $\varphi$-space.
Table 4.3: Values of slopes $\tau_q$ along with the statistical and dynamical components for $^{28}\text{Si}-\text{AgBr}$ collisions at 14.6$^4\text{ GeV}$ for different values of $q$ in $\varphi$-space.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\tau_q$</th>
<th>$\tau_q^{\text{stat}}$</th>
<th>$\tau_q^{\text{dyn}}$</th>
<th>$(q-1) - \tau_q^{\text{dyn}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.770 \pm 0.003$</td>
<td>$0.834 \pm 0.002$</td>
<td>$0.936 \pm 0.004$</td>
<td>$0.064 \pm 0.004$</td>
</tr>
<tr>
<td>3</td>
<td>$1.434 \pm 0.009$</td>
<td>$1.643 \pm 0.004$</td>
<td>$1.791 \pm 0.010$</td>
<td>$0.209 \pm 0.010$</td>
</tr>
<tr>
<td>4</td>
<td>$2.018 \pm 0.017$</td>
<td>$2.427 \pm 0.009$</td>
<td>$2.591 \pm 0.019$</td>
<td>$0.409 \pm 0.019$</td>
</tr>
<tr>
<td>5</td>
<td>$2.473 \pm 0.033$</td>
<td>$3.299 \pm 0.022$</td>
<td>$3.174 \pm 0.040$</td>
<td>$0.826 \pm 0.040$</td>
</tr>
<tr>
<td>6</td>
<td>$3.075 \pm 0.071$</td>
<td>$4.182 \pm 0.062$</td>
<td>$3.893 \pm 0.094$</td>
<td>$1.107 \pm 0.094$</td>
</tr>
</tbody>
</table>

Table 4.4: Values of slopes $\tau_q$ along with the statistical and dynamical components for $^{28}\text{Si}-\text{AgBr}$ collisions generated using $\text{UrQMD}$ model for different values of $q$ in $\varphi$-space.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\tau_q$</th>
<th>$\tau_q^{\text{stat}}$</th>
<th>$\tau_q^{\text{dyn}}$</th>
<th>$(q-1) - \tau_q^{\text{dyn}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.825 \pm 0.003$</td>
<td>$0.834 \pm 0.002$</td>
<td>$0.991 \pm 0.004$</td>
<td>$0.009 \pm 0.004$</td>
</tr>
<tr>
<td>3</td>
<td>$1.621 \pm 0.004$</td>
<td>$1.643 \pm 0.004$</td>
<td>$1.978 \pm 0.006$</td>
<td>$0.022 \pm 0.006$</td>
</tr>
<tr>
<td>4</td>
<td>$2.423 \pm 0.008$</td>
<td>$2.427 \pm 0.009$</td>
<td>$2.996 \pm 0.012$</td>
<td>$0.004 \pm 0.012$</td>
</tr>
<tr>
<td>5</td>
<td>$3.241 \pm 0.014$</td>
<td>$3.299 \pm 0.022$</td>
<td>$3.942 \pm 0.026$</td>
<td>$0.058 \pm 0.026$</td>
</tr>
<tr>
<td>6</td>
<td>$4.116 \pm 0.031$</td>
<td>$4.182 \pm 0.062$</td>
<td>$4.934 \pm 0.069$</td>
<td>$0.066 \pm 0.069$</td>
</tr>
</tbody>
</table>
Equation 4.7. Values of $\tau_q$, $\tau_*^{stat}$ and $\tau_*^{dyn}$ for $q = 2-6$ in $\varphi$-space are listed in Table 4.3. As can be seen from the table, $\tau_*^{dyn}$ values in $\varphi$-space also differ significantly from $(q-1)$, thus giving another evidence of the dynamical origin of multiplicity fluctuations in our data.

In order to study whether the multifractality observed in our data could be explained by the standard models of particle production, we simulated 14000 $^{28}$Si-$^{208}$AgBr events using the string hadronic model UrQMD. A similar $G_q$-moment analysis was done for UrQMD events also in both the pseudorapidity ($\eta$) and azimuthal ($\varphi$) angle spaces. Figures 4.1 (c) and 4.2 (c) show $\ln<G_q>$ versus $\ln M$ plots for UrQMD events for $q = 2-6$ in $\eta$ and $\varphi$ spaces respectively. $M$ was again varied from 2 to 35. As can be seen from Figures 4.1 (c) and 4.2 (c), $\ln<G_q>$ for UrQMD events also exhibits a linear dependence on $\ln M$ in both the spaces. Table 4.2 shows the slopes $\tau_q$ of $\ln<G_q>$ versus $\ln M$ plots for $q = 2-6$ for UrQMD events in $\eta$-space. $\tau_*^{stat}$ are the slopes of the corresponding plots for random events generated as mentioned earlier and are listed in Table 4.2. Values of $\tau_*^{dyn}$ have been calculated using Equation 4.7 and are also listed in the Table 4.2. From Table 4.2 we notice that the slopes of $\ln<G_q>$ versus $\ln M$ plots $\tau_q$ for UrQMD events are greater than the slopes of the corresponding plots for experimental events (Table 4.1). In fact these slopes $\tau_q$ are about the same as $\tau_*^{stat}$, the slopes of the corresponding plots for random events. We also notice from Table 4.2 that for UrQMD events values of $(q-1) - \tau_*^{dyn}$, which quantifies the amount of dynamical fluctuations are negligible as compared to the corresponding values for experimental events given in Table 4.1. Therefore, UrQMD model fails to explain the observed dynamical fluctuations and multifractality in our data. A similar result is observed for $\varphi$-space also (Table 4.4).

4.2.2 $F_q$-Moment Method

The scaled factorial moments $F_q$ that have been studied in emulsion experiments are defined as

$$\langle F_q \rangle = M^{q-1} \frac{\sum_{m=1}^{M} n_m (n_m - 1) \ldots (n_m - q + 1)}{\langle N \rangle^q}$$

4.9
Figure 4.3: (a) $\ln<F_q>$ versus $\ln M$ plots for $^{28}$Si-AgBr collisions at 14.6 A GeV in $\eta$-space.
(b) $\ln<F_q>$ versus $\ln M$ plots for randomly generated events in $\eta$-space.
(c) $\ln<F_q>$ versus $\ln M$ plots for UrQMD events in $\eta$-space.
Figure 4.4: (a) $\ln <F_q>$ versus $\ln M$ plots for $^{28}\text{Si-AgBr}$ collisions at 14.6$A$ GeV in $\varphi$-space.
(b) $\ln <F_q>$ versus $\ln M$ plots for randomly generated events in $\varphi$-space.
(c) $\ln <F_q>$ versus $\ln M$ plots for UrOMD events in $\varphi$-space.
Table 4.5: Values of the slopes $\phi_q$ of $\ln\langle F_q \rangle$ versus $\ln M$ plots for $^{28}\text{Si- AgBr}$ collisions at 14.6 GeV along with those for the randomly generated and UrQMD events in $\eta$-space.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\phi_q$</th>
<th>$\phi_q$</th>
<th>$\phi_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Experimental Events</strong></td>
<td><strong>Randomly Generated Events</strong></td>
<td><strong>UrQMD Events</strong></td>
</tr>
<tr>
<td>2</td>
<td>0.020 ± 0.002</td>
<td>0.0000 ± 0.0002</td>
<td>0.0007 ± 0.0002</td>
</tr>
<tr>
<td>3</td>
<td>0.060 ± 0.007</td>
<td>0.0005 ± 0.0006</td>
<td>0.0007 ± 0.0008</td>
</tr>
<tr>
<td>4</td>
<td>0.124 ± 0.017</td>
<td>0.0059 ± 0.0021</td>
<td>-0.0034 ± 0.0025</td>
</tr>
<tr>
<td>5</td>
<td>0.195 ± 0.044</td>
<td>0.0231 ± 0.0059</td>
<td>-0.0202 ± 0.0074</td>
</tr>
<tr>
<td>6</td>
<td>0.290 ± 0.096</td>
<td>0.0543 ± 0.0142</td>
<td>-0.0761 ± 0.0019</td>
</tr>
</tbody>
</table>

Table 4.6: Values of the slopes $\phi_q$ of $\ln\langle F_q \rangle$ versus $\ln M$ plots for $^{28}\text{Si- AgBr}$ collisions at 14.6 GeV along with those for the randomly generated and UrQMD events in $\varphi$-space.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\phi_q$</th>
<th>$\phi_q$</th>
<th>$\phi_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Experimental Events</strong></td>
<td><strong>Randomly Generated Events</strong></td>
<td><strong>UrQMD Events</strong></td>
</tr>
<tr>
<td>2</td>
<td>0.069 ± 0.003</td>
<td>-0.0001 ± 0.0002</td>
<td>-0.0020 ± 0.0004</td>
</tr>
<tr>
<td>3</td>
<td>0.194 ± 0.010</td>
<td>0.0008 ± 0.0007</td>
<td>-0.0059 ± 0.0012</td>
</tr>
<tr>
<td>4</td>
<td>0.362 ± 0.024</td>
<td>0.0030 ± 0.0022</td>
<td>-0.0127 ± 0.0029</td>
</tr>
<tr>
<td>5</td>
<td>0.580 ± 0.056</td>
<td>0.0077 ± 0.0065</td>
<td>-0.0317 ± 0.0085</td>
</tr>
<tr>
<td>6</td>
<td>0.842 ± 0.125</td>
<td>0.0188 ± 0.0168</td>
<td>-0.0831 ± 0.0233</td>
</tr>
</tbody>
</table>
where \(<\) denotes the average over the number of events analysed. \(M\) is the total number of bins in which the \(X\)-interval (0-1) has been divided, \(n_m\) is the multiplicity of particles in \(m\)th bin, \(<N>\) denotes the average multiplicity of particles in the pseudorapidity interval \(\eta_{\text{peak}} - 2 \leq \eta \leq \eta_{\text{peak}} + 2\). Figures 4.3 (a) and 4.4(a) show \(\ln <F_q>\) versus \(\ln M\) plots for our data in \(\eta\) and \(\varphi\) spaces. The values of \(<F_q>\) were calculated from Equation 4.9. A linear dependence of \(\ln <F_q>\) on \(\ln M\), indicating the power law dependence of \(F_q\) on \(M\), is observed. The values of the slopes \(\phi_q\) in both the \(\eta\) and \(\varphi\) spaces, determined from the least square fitting of the data points, are listed in Tables 4.5 and 4.6 respectively. Deviation of \(\phi_q\) from 0 is a measure of the non-statistical or dynamical fluctuations in our data and points to a self-similar mechanism of multiparticle production.

Figures 4.3 (b and c) and 4.4 (b and c) show \(\ln <F_q>\) versus \(\ln M\) plots for random and \(UrQMD\) events in \(\eta\) and \(\varphi\) spaces respectively. Values of the slopes of these plots are given in Tables 4.5 and 4.6 respectively. We observe from these tables that the slopes \(\phi_q\) for \(UrQMD\) and random events are negligible as compared to the \(\phi_q\) values for the experimental events. In fact in the absence of dynamical fluctuations \(\phi_q\) values should all be zero. Therefore, \(UrQMD\) model fails to explain the observed power law growth of the scaled factorial moments with decreasing bin size also.

4.2.3 Connection Between \(F_q\) and \(G_q\)

The normalized factorial moments \(F_q\) are used to investigate the self-similar properties of multiparticle production whereas \(G_q\)-moments are defined as a mean of studying the multifractal properties of a self-similar process. Hwa and Pan [17] have investigated the relation between \(F_q\) and \(G_q\) by expanding both in terms of basic functions \(B_{q,k}(M)\) and have been able to establish a relationship between the fractal behavior of \(F_q\) and \(G_q\). In the following, we use their method to investigate the relationship between \(F_q\) and \(G_q\) for our data.

The normalized factorial moments \(F_q\) are defined for integer values of \(q \geq 1\) whereas \(G_q\) are defined for all real values of \(q\). However, to establish a connection between them \(q\) should be restricted to positive integer values \((q \geq 1)\) for \(G_q\)-moments also. Let us define a function \(B_{q,k}(M)\) as
\[ B_{q,k}(M) = \left( \frac{Q_{q+k}(M,N)}{N^q} \right), \]

where \( Q_{q+k}(M,N) \) are the number of bins containing \((q+k)\) particles in an event of multiplicity \( N \) in the pseudorapidity range \( \Delta \eta \), \( k = 0,1,2, \ldots \). The angular brackets indicate the average over all events. The functions \( B_{q,k}(M) \) express the basic fractal structure of the data. In terms of these functions \( G_q \)-moments can be expressed as

\[ \langle G_q(M) \rangle = \sum_{k=0}^{\infty} B_{q,k}(M)(q+k)^q \]

Writing the first few terms of the above expression

\[ \langle G_q(M) \rangle = B_{q,0}(M) q^q \left[ 1 + \frac{B_{q,1}(M)}{B_{q,0}(M)} \left( \frac{q+1}{q} \right)^q + \ldots \right] . \]

In the above expression, only the first few terms make significant contribution at large \( M \) and the sum of these terms determines \( \tau_q \).

However, to express \( F_q \) in terms of \( B_{q,k}(M), \langle N \rangle \) in Equation 4.9 has to be replaced by \( N \).

In terms of \( B_{q,k}(M), \langle F_q \rangle \) can be written as

\[ \langle F_q \rangle = M^{q-1} \sum_{k=0}^{\infty} B_{q,k}(M) \frac{(q+k)!}{k!} \]

The first few terms of the above expression are

\[ \langle F_q \rangle = M^{q-1} B_{q,0}(M) q! \left[ 1 + \frac{B_{q,1}(M)}{B_{q,0}(M)} (q+1) + \ldots \right] \]

It is clear from Equations 4.12 and 4.14 that \( \langle F_q \rangle \) and \( \langle G_q \rangle \) do not have identical \( M \) dependences. However, considering only the leading terms, we would have

\[ \phi_q \equiv q - 1 - \tau_q \]

If we use the modified \( G_q \)-moments, the above relation changes to

\[ \phi_q \equiv q - 1 - \tau^{\text{dyn}}_q \]

To compare \( \phi_q \) with the deviations of \( \tau^{\text{dyn}}_q \) from \((q-1)\), a plot of \( \phi_q \) and \( q-1-\tau^{\text{dyn}}_q \) as a function of \( q \) is shown in Figure 4.5. As can be seen from the figure, for \( q > 3 \) the values of
Figure 4.5: (a) Plots of $\phi_q$ and $q-1-\tau_{q}^{dyn}$ versus $q$ for $^{28}\text{Si-AgBr}$ collisions at 14.6$A$ $GeV$ in $\eta$-space.

(b) Plots of $\phi_q$ and $q-1-\tau_{q}^{dyn}$ versus $q$ for $^{28}\text{Si-AgBr}$ collisions at 14.6$A$ $GeV$ in $\Phi$-space.
\( \phi_q \) and \( q-1-\tau_q^{dyn} \) differ significantly for our data. A similar result was observed by Shiptu et al. for \( p\text{-Emulsion} \) collisions at 800 GeV/c \([33]\), by Jain et al. for nucleus-nucleus collisions at different energies \([20]\) and by Derado et al in EMC data on \( \mu p \) and \( \mu d \) collisions \([18]\). \( \phi_q \) values obtained in these investigations and also in the present one are smaller than the corresponding values \( q-1-\tau_q^{dyn} \). Recently Ghosh et al. \([21]\) have studied intermittency and multifractality in \( ^{16}O\text{-Emulsion} \) collisions at 60A GeV. The values of the intermittency indices \( \phi_q \) obtained by them are also very different from the corresponding values of \( q-1-\tau_q^{dyn} \). But unlike ours, \( \phi_q \) values obtained by Ghosh et al. are larger than the corresponding values of \( q-1-\tau_q^{dyn} \). The reason for the deviation of \( \phi_q \) from \( q-1-\tau_q^{dyn} \) in our data could be the fact that Equations 4.12 and 4.14 have different \( M \) dependences and that the relation \( \phi_q = q-1-\tau_q^{dyn} \) was obtained by considering only the leading terms of Equations 4.12 and 4.14. Another reason for the deviation could be the fact that the modified \( G_q \)-moments, although defined to suppress statistical fluctuations, are not able to eliminate them completely.

### 4.2.4 Takagi Method

Apart from \( G_q \)-moment and \( F_q \)-moment methods, Takagi method \([22]\) is also used to study the multifractality in multiparticle production. In the following we discuss Takagi method briefly. A single event contains \( n \) shower particles distributed in the interval \( \Delta \eta = \eta_{max} - \eta_{min} \) in the pseudorapidity space. The multiplicity of shower particles changes from event to event according to the distribution \( P_n(\Delta \eta) \). The interval \( \Delta \eta \) is divided into \( M \) bins having size of each bin \( \delta \eta = \Delta \eta / M \).

Let \( P_n(\delta \eta) \) denotes the multiplicity distribution for a single bin. It is assumed that the inclusive rapidity distribution \( dn/d \eta \) is constant and \( P_n(\delta \eta) \) is independent of the location of the bin. For particles produced in \( \Omega \) independent events, there are \( \Omega \times M \) bins and thus produced particles are distributed in these bins, each of size \( \delta \eta \). Let \( K \) be the total number of particles produced in \( \Omega \) events and \( n_p \) the multiplicity of particles in the \( i \)th bin of the \( j \)th event. The theory of multifractal motivates us to consider the normalized density \( P_{ij} \), defined by
and to consider if the quantity

\[ T_q(\delta \eta) = \ln \sum_{j=1}^{\Omega} \sum_{i=1}^{M} P_{ji}^q \quad \text{for } q > 0 \]

behaves like a linear function of logarithm of the resolution \( R(\delta \eta) \)

\[ T_q(\delta \eta) = A_q + B_q \ln R(\delta \eta), \]

where \( A_q \) and \( B_q \) are constants independent of \( \delta \eta \). If the linear behavior is observed over a large range of \( R(\delta \eta) \), a generalized dimension may be defined as

\[ D_q = \frac{B_q}{(q-1)} \]

Now we evaluate the double sum of \( P_{ji}^q \) when the number of events is very large.

\[ \sum_{j=1}^{\Omega} \sum_{i=1}^{M} P_{ji}^q = \sum_{n=0}^{\infty} \Omega M \langle P(n) \rangle \left[ \frac{n}{K} \right]^q = \frac{\langle n^q \rangle}{K^{q-1} \langle n \rangle} \]

where a generic notation

\[ \langle f(n) \rangle = \sum_{n=0}^{\infty} f(n) P_{n}(\delta \eta) \]

and the relation \( \langle n \rangle = K/\Omega M \) have been used. Here, \( \langle f(n) \rangle \) is a function of \( \delta \eta \) in general but the \( \delta \eta \) dependence is suppressed for brevity. Since we have

\[ \delta \eta = \frac{\Delta \eta}{M} \]

i.e.

\[ M = \Delta \eta / \delta \eta, \]

we can write

\[ \langle n \rangle = K/\Omega M = K. \delta \eta / \Omega. \Delta \eta. \]

Using Equations 4.18, 4.19 and 4.21, for the simplest choice \( R(\delta \eta) = \delta \eta \), we can obtain the relation

\[ \ln \langle n^q \rangle = A_q + \{B_q + 1\} \ln \delta \eta \]

By using Equation 4.20, the above equation can be written as
\[ \ln \langle n^q \rangle = A_q + (q-1) D_q + 1 \ln \delta \eta \]  

4.25

To check the validity of the above relation, Takagi [22] plotted \( \ln \langle n^q \rangle \) versus \( \ln \eta_c \) for UA5 data on \( p\overline{p} \) collisions at \( \sqrt{s} = 200 \text{GeV} \). Deviation from the linear behaviour was observed in large \( \eta_c \) region. It is reasonable to expect that this deviation may be due to the non-flat behaviour of \( dn/d\eta \) in projectile and target fragmentation regions where \( \eta \) is large. Therefore, \( <n> \) is considered a better choice of \( R(\delta \eta) \) because \( dn/d<n> \) is flat by definition. Thus Equation 4.25 can be written as

\[ \ln \langle n^q \rangle = A_q + K_q \ln \langle n \rangle \]  

4.26

where

\[ K_q = (q-1) D_q + 1 \]

A linear behavior over a considerable range of \( \langle n \rangle \) points towards the fractal structure in multiparticle production. The generalized dimensions \( D_q \) for \( q \geq 2 \) can then be easily obtained from the slopes of \( \ln \langle n^q \rangle \) versus \( \ln \langle n \rangle \) plots.

We now use the Takagi methodology to study multifractality in the multiplicity distribution of shower particles produced in \(^{28}\text{Si}-\text{AgBr} \) collisions at 14.6 \( A \text{ GeV} \) in both \( \eta \) and \( \phi \) spaces. The analysis was performed on the same data sample as was used for the \( G_q \)-moment analysis. As done earlier, the analysis was restricted to the central region of pseudorapidity with \( \eta_{\text{peak}} - 2 \leq \eta \leq \eta_{\text{peak}} + 2 \). As mentioned in the beginning of this section, the single particle density distribution is assumed constant in the Takagi method. Therefore, to avoid the effect of non-flat density distribution, the observed pseudorapidity and azimuthal angle distributions have been converted into the corresponding \( X \)-distributions using Equations 4.1 and 4.8. As mentioned earlier, in \( X \)-space single particle density distribution is flat and \( X \) values are uniformly distributed between \( X = 0.0 \) and \( X = 1.0 \).

Values of \( \eta \) and \( \phi \) converted into the corresponding cumulative \( X \)-variables (\( X_\eta \) and \( X_\phi \)) were used for the analysis. The initial cumulative \( X \) variable interval \( \Delta X \) becomes 1.0 corresponding to the initial rapidity interval \( \Delta \eta = 4.0 \). The \( \Delta X \) was subsequently decreased in steps of 0.025 and the values of \( \ln \langle n^q \rangle \) and \( \ln <n> \) were
computed for each interval. Here $n$ is the multiplicity of shower particles in an event in the given interval $\Delta X$. Figure 4.6 (a) shows the plots of $\ln\langle n^q \rangle$ versus $\ln\langle n \rangle$ for $q=2-6$ in $\eta$ space. From the figure it is clear that the behavior of $\ln\langle n^q \rangle$ versus $\ln\langle n \rangle$ plots is linear. Further, in the case of azimuthal angle space, the initial cumulative $X$ variable interval $\Delta X$ becomes 1.0 corresponding to the azimuthal interval $\Delta \phi = 360.0$. The $\Delta X$ was subsequently decreased in steps of 0.025 and $\ln\langle n^q \rangle$ and $\ln\langle n \rangle$ were computed for each interval. Here $n$ is the multiplicity of shower particles in an event in the given interval $\Delta X$. Figure 4.7 (a) shows the plots of $\ln\langle n^q \rangle$ versus $\ln\langle n \rangle$ for $q=2-6$ in $\phi$-space. From the figure we observe that the dependence of $\ln\langle n^q \rangle$ on $\ln\langle n \rangle$ is linear in $\phi$-space also.

In order to find whether the observed dependence of $\ln\langle n^q \rangle$ on $\ln\langle n \rangle$ could be explained by the string hadronic model $UrQMD$ a similar analysis was performed for the random and $UrQMD$ events in both the $\eta$ and $\phi$ spaces. Figures 4.6 (b) and 4.6 (c) show the $\ln\langle n^q \rangle$ versus $\ln\langle n \rangle$ plots for the random and $UrQMD$ events respectively in $\eta$-space. We observe from Table 4.7 that the slopes of the plots for the random and $UrQMD$ events are about the same as those for the corresponding plots for the experimental events in $\eta$-space. Similarly in $\phi$-space Figures 4.7 (b) and 4.7 (c) show the $\ln\langle n^q \rangle$ versus $\ln\langle n \rangle$ plots for the random and $UrQMD$ events respectively. We again observe from Table 4.8 that the slopes of the plots for the random and $UrQMD$ events are about the same as those for the corresponding plots for the experimental events in $\phi$-space. Whereas the statistical fluctuations are completely eliminated in $F_q$-moment method [1] and are suppressed in the modified $G_q$-moment method [17], no attempt is made in the Takagi method to eliminate or suppress the statistical fluctuations. This is the reason as to why we get about the same values of the slopes of $\ln\langle n^q \rangle$ versus $\ln\langle n \rangle$ plots for the experimental, random and $UrQMD$ events. Thus no meaningful conclusion regarding multifractality in the data could be drawn from the Takagi method. We therefore suggest that only $F_q$-moment or $G_q$-moment method should be used for the study of multifractality in multiparticle production as the multiplicity moments calculated in the Takagi method are dominated by statistical fluctuations.
Figure 4.6: (a) ln<\eta^q> versus ln<\eta> plots for $^{28}$Si-AgBr collisions at 14.64 GeV in \eta-space.
(b) ln<\eta^q> versus ln<\eta> plots for randomly generated events in \eta-space.
(c) ln<\eta^q> versus ln<\eta> plots for UrQMD events in \eta-space.
Figure 4.7:  
(a) $\ln< n^q >$ versus $\ln< n >$ plots for $^{28}$Si-$^{79}$AgBr collisions at 14.6 A GeV in $\varphi$-space. 
(b) $\ln< n^q >$ versus $\ln< n >$ plots for randomly generated events in $\varphi$-space. 
(c) $\ln< n^q >$ versus $\ln< n >$ plots for UrQMD events in $\varphi$-space.
Table 4.7: Values of the slopes of $\ln<n^q>$ versus $\ln<n>$ plots for $^{28}\text{Si}-\text{AgBr}$ collisions at 14.6$A$ GeV along with those for the randomly generated and UrQMD events for different values of $q$ in $\eta$-space.

<table>
<thead>
<tr>
<th>$q$</th>
<th>Experimental Events</th>
<th>Randomly Generated Events</th>
<th>UrQMD Events</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1.920 \pm 0.002$</td>
<td>$1.934 \pm 0.004$</td>
<td>$1.919 \pm 0.004$</td>
</tr>
<tr>
<td>3</td>
<td>$2.810 \pm 0.005$</td>
<td>$2.838 \pm 0.009$</td>
<td>$2.790 \pm 0.010$</td>
</tr>
<tr>
<td>4</td>
<td>$3.677 \pm 0.010$</td>
<td>$3.720 \pm 0.016$</td>
<td>$3.624 \pm 0.016$</td>
</tr>
<tr>
<td>5</td>
<td>$4.527 \pm 0.016$</td>
<td>$4.586 \pm 0.023$</td>
<td>$4.428 \pm 0.022$</td>
</tr>
<tr>
<td>6</td>
<td>$5.366 \pm 0.025$</td>
<td>$5.437 \pm 0.030$</td>
<td>$5.207 \pm 0.029$</td>
</tr>
</tbody>
</table>

Table 4.8: Values of the slopes of $\ln<n^q>$ versus $\ln<n>$ plots for $^{28}\text{Si}-\text{AgBr}$ collisions at 14.6$A$ GeV along with those for the randomly generated and UrQMD events for different values of $q$ in $\varphi$-space.

<table>
<thead>
<tr>
<th>$q$</th>
<th>Experimental Events</th>
<th>Randomly Generated Events</th>
<th>UrQMD Events</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1.873 \pm 0.007$</td>
<td>$1.934 \pm 0.004$</td>
<td>$1.925 \pm 0.004$</td>
</tr>
<tr>
<td>3</td>
<td>$2.698 \pm 0.017$</td>
<td>$2.836 \pm 0.009$</td>
<td>$2.803 \pm 0.011$</td>
</tr>
<tr>
<td>4</td>
<td>$3.505 \pm 0.029$</td>
<td>$3.713 \pm 0.014$</td>
<td>$3.648 \pm 0.018$</td>
</tr>
<tr>
<td>5</td>
<td>$4.308 \pm 0.043$</td>
<td>$4.570 \pm 0.020$</td>
<td>$4.465 \pm 0.026$</td>
</tr>
<tr>
<td>6</td>
<td>$5.113 \pm 0.060$</td>
<td>$5.410 \pm 0.026$</td>
<td>$5.259 \pm 0.034$</td>
</tr>
</tbody>
</table>
4.3 Generalized Dimensions

Power law behavior is a characteristic of fractals, that is, self-similar objects. According to the theory of fractals, self-similar systems are characterized by an infinite spectrum of non-integer generalized dimensions \( D_q \). It has been observed in sections 4.2.1 and 4.2.2 that both \( \langle F_q \rangle \) and \( \langle G_q \rangle \) exhibit power law dependences on \( M \), thereby indicating the self-similar nature of the multiparticle production process. Therefore, the generalized dimensions that characterize the multiparticle production process can be obtained from both the \( F_q \)-moment and \( G_q \)-moment analyses, using the following relations

\[
D_q = \frac{\xi_q^{\text{dyn}}}{(q - 1)}
\]  

4.27

Substitution of \( \xi_q^{\text{dyn}} \) from Equation 4.16 gives \( D_q \) in terms of intermittency indices \( \phi_q \)

\[
D_q = 1 - \frac{\phi_q}{(q - 1)}
\]  

4.28

\( D_q \) values for experimental and \( UrQMD \) events in both the \( \eta \) and \( \varphi \) spaces were calculated using Equation 4.27 from \( \xi_q^{\text{dyn}} \) values obtained from the \( G_q \)-moment analysis and are listed in Table 4.9. \( D_q \) values for experimental and \( UrQMD \) events in both the \( \eta \) and \( \varphi \) spaces were also calculated using Equation 4.28 from \( \phi_q \) values obtained from the \( F_q \)-moment analysis and are listed in Tables 4.10. It is clear from Tables 4.9 and 4.10 that for experimental events \( D_q \) values obtained from the \( F_q \)-moment method in \( \eta \)-space are greater than the corresponding values in \( \varphi \)-space. However, values of \( D_q \) obtained from the \( G_q \)-moment method in \( \eta \)-space are smaller than the corresponding values in \( \varphi \) space. This is a well-known result. \( D_q \) values not only depend on the space in which the analysis is done but also on the method of analysis. The reason is the difference between the definitions of the two moments. Whereas \( F_q \)-moments eliminate statistical fluctuations completely, \( G_q \)-moments, as pointed out in section 4.2.3 are not able to eliminate them completely. However, as can be seen from Tables 4.9 and 4.10, \( D_q \) values in each case for our experimental events decrease with increasing \( q \). This is an evidence of the presence of multifractality in our data. It can be observed from Tables 4.9 and 4.10 that values of the generalized dimensions \( D_q \) for \( UrQMD \) events calculated from the \( F_q \)-moment and \( G_q \)-moment methods are almost independent of \( q \). \( D_q \) values for \( q=2-6 \) are all equal to one.
Table 4.9: Values of the generalized dimensions $D_q$ for $^{28}$Si-$^{129}$I collisions at 14.6 $AGeV$ determined from the $G_q$-moment method in $\eta$ and $\phi$ spaces.

<table>
<thead>
<tr>
<th>$q$</th>
<th>Generalized Dimensions ($D_q$)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\eta$-space</td>
<td>$\phi$-space</td>
</tr>
<tr>
<td></td>
<td>Experimental Events</td>
<td>UrQMD Events</td>
</tr>
<tr>
<td>2</td>
<td>0.920 ± 0.017</td>
<td>0.991 ± 0.017</td>
</tr>
<tr>
<td>3</td>
<td>0.869 ± 0.014</td>
<td>0.988 ± 0.014</td>
</tr>
<tr>
<td>4</td>
<td>0.829 ± 0.012</td>
<td>0.994 ± 0.012</td>
</tr>
<tr>
<td>5</td>
<td>0.771 ± 0.013</td>
<td>0.982 ± 0.013</td>
</tr>
<tr>
<td>6</td>
<td>0.719 ± 0.018</td>
<td>0.977 ± 0.018</td>
</tr>
</tbody>
</table>

Table 4.10: Values of the generalized dimensions $D_q$ for $^{28}$Si-$^{129}$I collisions at 14.6 $AGeV$ determined from the $F_q$-moment method in $\eta$ and $\phi$ spaces.

<table>
<thead>
<tr>
<th>$q$</th>
<th>Generalized Dimensions ($D_q$)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\eta$-space</td>
<td>$\phi$-space</td>
</tr>
<tr>
<td></td>
<td>Experimental Events</td>
<td>UrQMD Events</td>
</tr>
<tr>
<td>2</td>
<td>0.980 ± 0.002</td>
<td>0.9993 ± 0.0002</td>
</tr>
<tr>
<td>3</td>
<td>0.970 ± 0.003</td>
<td>0.9996 ± 0.0004</td>
</tr>
<tr>
<td>4</td>
<td>0.960 ± 0.005</td>
<td>1.0011 ± 0.0008</td>
</tr>
<tr>
<td>5</td>
<td>0.951 ± 0.011</td>
<td>1.0050 ± 0.0018</td>
</tr>
<tr>
<td>6</td>
<td>0.942 ± 0.019</td>
<td>1.0152 ± 0.0004</td>
</tr>
</tbody>
</table>
another within errors. This points to the absence of dynamical fluctuation and multifractality in UrQMD events. This again establishes that the hadronic string model UrQMD fails to explain the observed dynamical fluctuations and multifractality in our data.

4.4 Multifractal Specific Heat

Bershadskii [34] showed that Bernoulli distribution appears in a natural way when transition from mono-fractality to multifractality is studied. Starting from the definition of $G_q$-moment, he derived the following relation

$$D_q = D_a + c \frac{\ln q}{(q-1)}$$

for the multifractal Bernoulli fluctuations. In the above relation, $D_q$ are the generalized dimensions. If we use the thermodynamic interpretation of multifractality [35], then the constant $c$ can be interpreted as multifractal specific heat of the system. Bershadskii analyzed the data on nucleus-nucleus collisions at various energies and found good agreement between the data and the multifractal Bernoulli representation (Equation 4.29).

We also determine the multifractal specific heat for our data using different sets of $D_q$ values given in Tables 4.9 and 4.10. Figures 4.8 (c) and (d) show the plots of $D_q$ obtained from the $F_q$-moment method as a function of $\ln q/(q-1)$ for $^{28}\text{Si}-\text{AgBr}$ collisions at 14.6 $AGeV$ in both the $\eta$ and $\phi$ phase spaces. Straight lines are the linear fits to the data, indicating good agreement between our data and the multifractal Bernoulli representation. The slopes of the fitted lines, which give the values of the multifractal specific heat $c$ for our data, are $0.11 \pm 0.01$ and $0.29 \pm 0.03$ for $\eta$-space and $\phi$-space respectively.

Figures 4.8 (a) and (b) show the plots of $D_q$, obtained from the $G_q$-moment method, as a function of $\ln q/(q-1)$ for $^{28}\text{Si}-\text{AgBr}$ collisions at 14.6 $AGeV$ in both the $\eta$ and $\phi$ phase spaces. The values of $c$, the slopes of the fitted lines, in $\eta$ and $\phi$ phase spaces in this case are $0.57 \pm 0.09$ and $0.48 \pm 0.08$ respectively.
Figure 4.8: The generalized dimensions $D_q$ versus $\ln q/(q-1)$ plots for $^{28}\text{Si}-\text{AgBr}$ collisions at 14.6A GeV (a) in $\eta$-space, $D_q$ obtained from $G_q$-moment method, (b) in $\varphi$-space, $D_q$ obtained from $G_q$-moment method, (c) in $\eta$-space, $D_q$ obtained from $F_q$-moment method and (d) in $\varphi$-space, $D_q$ obtained from $F_q$-moment method.
References