Chapter 3

Global Asymptotic Stability of BAM Neural Networks with Mixed Delays and Impulses

3.1 Introduction

Recently a class of two-layer heteroassociative networks called bidirectional associative memory (BAM) networks [11, 14, 56] with or without axonal signal transmission delays has attracted the attention of many researchers and is used in many fields such as pattern recognition, automatic control, associative memory, parallel computation and optimization problems. Cao and Song [12] derived a set of sufficient conditions for the exponential stability of Cohen-Grossberg-type bidirectional associative memory neural networks with time-varying delays. Chen et al. [16] used some analytical techniques and derived several simple criteria for the global exponential stability of delayed BAM neural networks. Incorporating time delays in both neural processing and axonal transmission, Gopalsamy and He [38] studied bidirectional associative memory networks with transmission delays. Recently Park et al. [99] investigated the asymptotic stability of BAM neural networks of neutral-type by using the Lyapunov method.

Though the BAM non-impulsive systems have been well studied in theory and practice (for example see [51, 145, 100] and references cited therein), the theory of impulsive differential equations is not only being recognized to be richer than the corresponding theory of differential equations, but also represents a more natural framework for mathematical modelling of many real-world phenomena, such
as population dynamic and neural networks. However there are only few papers that considered the stability problems for the BAM networks with delays and impulsive terms. Li [62] established some sufficient conditions for the existence and the global exponential stability of a BAM network with Lipschitzian activation functions without assuming their boundedness, monotonicity and subjected to impulsive state displacements at fixed instants of time. The global asymptotic stability of delay bi-directional associative memory neural networks with impulses is established by constructing a suitable Lyapunov functional in [83]. More recently Lou and Cui [84] studied the global asymptotic stability of delay BAM neural networks with impulses.

The problem of global asymptotic stability of impulsive BAM neural networks with mixed delays have not been reported in the literature. In order to fill this gap, the purpose of this chapter is to give sufficient conditions for the global asymptotic stability of BAM impulsive neural networks with mixed delays. New sufficient conditions are derived by constructing a suitable Lyapunov functional. Further the advantage of the employed strategy is that the global asymptotic result is obtained efficiently with Lipschitzian activation functions without assuming their boundedness and differentiability.

### 3.2 Problem Formulation

In this chapter, we propose the following system of integro-differential equations as a model for BAM impulsive neural networks involving mixed distributed delays

\[
\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} h_{ji} f_j(y_j(t - \tau_{ji}(t))) + \sum_{j=1}^{n} l_{ji} \int_{0}^{\infty} k_{ji}(s) g_j(y_j(t - \tau_j(s)))ds + I_i,
\]

\[
\Delta x_i(t_k) = P_k(x_i(t_k)), \quad i = 1, 2, ..., m, \quad k = 1, 2, ..., (3.2.1)
\]

\[
\frac{dy_j(t)}{dt} = -d_j y_j(t) + \sum_{i=1}^{m} h_{ij} \bar{f}_i(x_i(t - \sigma_{ij}(t))) + \sum_{i=1}^{m} l_{ij} \int_{0}^{\infty} \bar{k}_{ij}(s) \bar{g}_i(x_i(t - \sigma_i(s)))ds + J_j,
\]

\[
\Delta y_j(t_k) = Q_k(y_j(t_k)), \quad j = 1, 2, ..., n, \quad k = 1, 2, ..., (3.2.2)
\]

where \(\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)\), \(\Delta y_j(t_k) = y_j(t_k^+) - y_j(t_k^-)\) are the impulses at moments \(t_k\), and \(t_1 < t_2 < ...\) is a strictly increasing sequence such that \(\lim_{k \to \infty} t_k = +\infty\); \(x_i(t)\) and \(y_j(t)\) are the activations of the \(i\)th neurons and the \(j\)th neurons.
respectively; $c_i, d_j$ are positive constants and denote the rate with which the cells $i$ and $j$ reset their potential to the resting state when isolated from the other cells and inputs; $f_j, g_j, \overline{f}_i, \overline{g}_i$ are the activation functions of the neurons; $h_{ji}, l_{ji}, \overline{h}_{ij}, \overline{l}_{ij}$ are constants and denote the connection weights at time $t$; $\tau_{ji}(t) \geq 0, \sigma_{ij}(t) \geq 0$ correspond to the transmission delay at time $t$; $\tau_{j}(s) \geq 0, \sigma_{i}(s) \geq 0$ correspond to the transmission delays at time $s$; $I_i, I_j$ are constants and denote the $i$th and the $j$th components of an external input source introduced from outside the network to the cell $i$ and $j$ respectively.

In this chapter, we assume that the following conditions hold:

(H1) For $i, j = 1, 2, \ldots, n$, $\tau_{ji}, \sigma_{ij}, \tau_j, \sigma_i$ are continuous functions such that

$$
\sigma = \max_{1 \leq i, j \leq m} \sup_{t \in \mathbb{R}} \sigma_{ij}(t), \quad \tau = \max_{1 \leq j \leq n} \sup_{t \in \mathbb{R}} \tau_{ji}(t), \quad \sigma_1 = \max_{1 \leq i \leq m} \sup_{s \in \mathbb{R}} \sigma_i(s), \\
\tau_1 = \max_{1 \leq j \leq n} \sup_{s \in \mathbb{R}} \tau_j(s).
$$

(H2) The neuron activation functions $f_j, g_j, \overline{f}_i, \overline{g}_i$ are Lipchitz-continuous, that is, there exist constants $F_j > 0, G_j > 0, \overline{F}_i > 0, \overline{G}_i > 0$ such that

$$
|f_j(\xi_1) - f_j(\xi_2)| \leq F_j|\xi_1 - \xi_2|, \quad |g_j(\xi_1) - g_j(\xi_2)| \leq G_j|\xi_1 - \xi_2|, \\
|\overline{f}_i(\xi_1) - \overline{f}_i(\xi_2)| \leq \overline{F}_i|\xi_1 - \xi_2|, \quad |\overline{g}_i(\xi_1) - \overline{g}_i(\xi_2)| \leq \overline{G}_i|\xi_1 - \xi_2|,
$$

for all $\xi_1, \xi_2 \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

(H3) The kernels $k_{ji}, \overline{k}_{ij} : [0, \infty) \to [0, \infty)$ are bounded, continuous and satisfy

$$
\int_0^\infty k_{ji}(s)ds = 1, \quad \int_0^\infty \overline{k}_{ij}(s)ds = 1.
$$

The initial functions associated with the system (3.2.1) are given by

$$
x_i(t) = \varphi_{x_i}(t), \quad t \in (-\infty, 0], \quad i = 1, 2, \ldots, m, \\
y_j(t) = \varphi_{y_j}(t), \quad t \in (-\infty, 0], \quad j = 1, 2, \ldots, n,
$$

where $\varphi_{x_i}$ and $\varphi_{y_j}$ are assumed to be bounded and continuous on $(-\infty, 0]$.

As usual in the theory of impulsive differential equations, at the points of discontinuity $t_k$ of the solution $t \mapsto [x_1(t), x_2(t), \ldots, x_m(t), y_1(t), y_2(t), \ldots, y_n(t)]^T$, we assume that

$$
[x_1(t_k), x_2(t_k), \ldots, x_m(t_k), y_1(t_k), y_2(t_k), \ldots, y_n(t_k)]^T \\
\equiv [x_1(t_k - 0), x_2(t_k - 0), \ldots, x_m(t_k - 0), y_1(t_k - 0), y_2(t_k - 0), \ldots, y_n(t_k - 0)]^T.
$$
It is clear that, in general, the derivatives \(x'_i(t_k)\) and \(y'_j(t_k)\) do not exist. On the other hand, according to the system (3.2.1), there exist the limits \(x'_i(t_k \mp 0)\) and \(y'_j(t_k \mp 0)\). In view of the above convention, we assume \(x'_i(t_k) \equiv x'_i(t_k - 0)\) and \(y'_j(t_k) \equiv y'_j(t_k - 0)\).

Let \(x(t) = [x_1(t), x_2(t), ..., x_m(t)]^T, y(t) = [y_1(t), y_2(t), ..., y_n(t)]^T\) be an arbitrary solution of (3.2.1). Then we have

\[
\frac{d^+|x_i(t) - x^*_i|}{dt} \leq -c_i|x_i(t) - x^*_i| + \sum_{j=1}^{n} |h_{ji}||F_j||y_j(t - \tau_{ji}(t)) - y^*_j| \\
+ \sum_{j=1}^{n} |l_{ji}| \int_0^\infty |k_{ji}(s)||G_j||y_j(t - \tau_j(s)) - y^*_j|ds, \quad i = 1, 2, ..., m,
\]

and

\[
\frac{d^+|y_j(t) - y^*_j|}{dt} \leq -d_j|y_j(t) - y^*_j| + \sum_{i=1}^{m} |\bar{h}_{ij}||\bar{F}_i||x_i(t - \sigma_{ij}(t)) - x^*_i| \\
+ \sum_{i=1}^{m} |\bar{l}_{ij}| \int_0^\infty |\bar{k}_{ij}(s)||\bar{G}_i||x_i(t - \sigma_i(s)) - x^*_i|ds, \quad (3.2.3)
\]

for \(t > 0, t \neq t_k, \ k \in Z^+, \) where \((x_1^*, x_2^*, ..., x_m^*, y_1^*, y_2^*, ..., y_n^*)^T \in \mathbb{R}^{m+n}\).

**Definition 3.2.1.** A constant vector \((x_1^*, x_2^*, ..., x_m^*, y_1^*, y_2^*, ..., y_n^*)^T \in \mathbb{R}^{m+n}\) is said to be an equilibrium point of the system (3.2.2), if it satisfies

\[
c_i x^*_i = \sum_{j=1}^{n} h_{ji} f_j(y^*_j) + \sum_{j=1}^{n} l_{ji} \int_0^\infty k_{ji}(s)g_j(y^*_j)ds + I_i, \quad i = 1, 2, ..., m,
\]

\[
d_j y^*_j = \sum_{i=1}^{m} \bar{h}_{ij} f_i(x^*_i) + \sum_{i=1}^{m} \bar{l}_{ij} \int_0^\infty \bar{k}_{ij}(s)\bar{g}_i(x^*_i)ds + J_j, \quad i = 1, 2, ..., n,
\]

when the impulsive jumps \(P_k(.), Q_k(.)\) as assumed to satisfy \(P_k(x^*_i) = 0, Q_k(y^*_j) = 0, i = 1, 2, ..., m, j = 1, 2, ..., n, k \in Z^+, \) where \(Z^+\) denotes the set of all positive integers.

Let us define \(u_i = x_i(t) - x^*_i, v_j = y_j(t) - y^*_j, i = 1, 2, ..., m, j = 1, 2, ..., n.\) Then it follows, from (3.2.3) that

\[
\frac{d^+u_i(t)}{dt} \leq -c_i u_i(t) + \sum_{j=1}^{n} |h_{ji}||F_j||v_j(t - \tau_{ji}(t))| \\
+ \sum_{j=1}^{n} |l_{ji}| \int_0^\infty |k_{ji}(s)||G_j||v_j(t - \tau_j(s))|ds,
\]
\[
\frac{d^+ v_j(t)}{dt} \leq -d_j v_j(t) + \sum_{i=1}^m |\overline{h}_{ij}||\overline{F}_i||u_i(t - \sigma_{ij}(t))| + \sum_{i=1}^m |\overline{l}_{ij}| \int_0^\infty |\overline{k}_{ij}(s)||\overline{G}_i||u_i(t - \sigma_i(s))|ds,
\] (3.2.4)
for \( t > 0, t \neq t_k, k \in Z^+ \).

Changing the first and third equalities of (3.2.1) to vector form,
\[
\frac{dx(t)}{dt} = -Cx(t) + Hf(y(t - \tau)) + L \int_0^\infty K(s)g(y(t - \tau_1))ds + I,
\]
and
\[
\frac{dy(t)}{dt} = -Dy(t) + \overline{Hf}(x(t - \sigma)) + \overline{L} \int_0^\infty \overline{K}(s)\overline{g}(x(t - \sigma_1))ds + J,
\] (3.2.5)
where \( x = (x_1, x_2, ..., x_m)^T, y = (y_1, y_2, ..., y_n)^T, C = \text{diag}(c_1, c_2, ..., c_m), \)
\( D = \text{diag}(d_1, d_2, ..., d_n), H = ((h_{ij})_{n \times n})^T, L = ((l_{ij})_{n \times n})^T, K = ((k_{ij})_{n \times n})^T, \)
\( \overline{H} = ((h_{ij})_{m \times n})^T, L = ((l_{ij})_{m \times n})^T, \overline{K} = ((k_{ij})_{m \times n})^T, I = (I_1, I_2, ..., I_m), \)
\( J = (J_1, J_2, ..., J_n), \)
\[
f(y(t - \tau)) = (f_j(y_j(t - \tau_j)))_{n \times 1}, g(y(t - \tau_1)) = (g_j(y_j(t - \tau_j))_{n \times 1}, \)
\[
\overline{f}(x(t - \sigma)) = (\overline{f}_i(x_i(t - \sigma_i)))_{m \times 1}, \overline{g}(x(t - \sigma_1)) = (\overline{g}_i(x_i(t - \sigma_i)))_{m \times 1}.
\]

By shifting the nonlinear active functions states in (3.2.5) to the origin, (3.2.5) can be written as
\[
\frac{du(t)}{dt} = -Cu(t) + HM(v(t - \tau)) + L \int_0^\infty K(s)N(v(t - \tau_1))ds,
\]
and
\[
\frac{dv(t)}{dt} = -Dv(t) + \overline{H} \overline{M}(u(t - \sigma)) + \overline{L} \int_0^\infty \overline{K}(s)\overline{N}(u(t - \sigma_1))ds,
\] (3.2.6)
where \( M(v(t)) = f(v(t) + y^*) - f(y^*), N(v(t)) = g(v(t) + y^*) - g(y^*), \overline{M}(u(t)) = \overline{f}(u(t) + x^*) - f(x^*), \overline{N}(u(t)) = \overline{g}(u(t) + x^*) - \overline{g}(x^*); \)
then we have
\[
|M(v(t))| \leq A|v(t)|, |N(v(t))| \leq B|v(t)|,
\]
\[
|\overline{M}(u(t))| \leq \overline{A}|u(t)|, |\overline{N}(u(t))| \leq \overline{B}|u(t)|,
\] (3.2.7)
where \( A = \text{diag}\{F_1, F_2, ..., F_n\}, B = \text{diag}\{G_1, G_2, ..., G_n\}, \overline{A} = \text{diag}\{\overline{F}_1, \overline{F}_2, ..., \overline{F}_m\}, \)
\( \overline{B} = \text{diag}\{\overline{G}_1, \overline{G}_2, ..., \overline{G}_m\}. \)

In this chapter, we use the following norm of \( \mathbb{R}^{m+n} : ||h|| = \sum_{p=1}^{m+n} |h_p|, \) for
\( h = (h_1, h_2, ..., h_{m+n})^T \in \mathbb{R}^{m+n}. \)
Definition 3.2.2. The equilibrium point \( x^* = (x_1^*, x_2^*, \ldots, x_m^*), y^* = (y_1^*, y_2^*, \ldots, y_n^*) \) of the system (3.2.1) is said to be globally asymptotically stable, if there exist constants \( K \geq 1 \) and \( L \geq 1 \) such that
\[
\sum_{i=1}^{m} |(x_i(t) - x_i^*)|^2 + \sum_{j=1}^{n} |(y_j(t) - y_j^*)|^2 \leq K\|\varphi_x - x^*\|^2 + L\|\varphi_y - y^*\|^2, \tag{3.2.8}
\]
for all \( t > 0 \), where
\[
\|\varphi_x - x^*\| = \left\{ \sum_{i=1}^{m} \sup_{s \in (-\infty, 0]} |\varphi_{x_i}(s) - x^*)|^2 \right\}^{1/2}
\]
and
\[
\|\varphi_y - y^*\| = \left\{ \sum_{j=1}^{n} \sup_{s \in (-\infty, 0]} |(\varphi_{y_j}(s) - y^*)|^2 \right\}^{1/2}.
\]

From Theorem 3.1 in [162], we can easily obtain the following Lemma.

Lemma 3.2.1. [162] Suppose that the system (3.2.1) satisfies hypotheses \((H_1) - (H_3)\) and
\[
c_i > \sum_{j=1}^{n} (F_i|\bar{h}_{ij}| + G_i|\bar{l}_{ij}|), \quad i = 1, 2, \ldots, m,
\]
\[
d_j > \sum_{i=1}^{m} (F_j|h_{ji}| + G_j|l_{ji}|), \quad j = 1, 2, \ldots, n,
\]
then the system (3.2.1) has a unique solution.

3.3 Global Asymptotic Stability

In this section, we present the global asymptotic stability of the system (3.2.1) using Lyapunov function and matrix theory.

Theorem 3.3.1. Assume that \((H1) - (H3)\) hold for the impulsive system (3.2.1). Furthermore suppose that the impulsive operators \( P_k(x_i(t)) \) and \( Q_k(y_j(t)) \) satisfy
\[
P_k(x_i(t_k)) = -\gamma_{ik}(x_i(t_k) - x_i^*), \quad 0 < \gamma_{ik} < 2, \quad i = 1, 2, \ldots, m, \quad k \in Z^+,
\]
\[
Q_k(y_j(t_k)) = -\delta_{jk}(y_j(t_k) - y_j^*), \quad 0 < \delta_{jk} < 2, \quad i = 1, 2, \ldots, m, \quad k \in Z^+, \tag{3.3.1}
\]
and if there exist positive symmetrical matrices $P, Q$ and positive diagonal matrices $R = \text{diag}\{r_1, r_2, \ldots, r_n\}, \ S = \text{diag}\{s_1, s_2, \ldots, s_m\}$ such that
\[
-(PC + CP) + PHR^2H^TP + PLR^2L^TP + \overline{A}^T S^{-2} \overline{A} + \overline{B}^T S^{-2} \overline{B} < 0,
-(QD + DQ) + Q\overline{H} S^{-2} \overline{H}^T Q + Q\overline{L} S^{-2} \overline{L}^T Q + A^T R^{-2} A + B^T R^{-2} B < 0,
\]
then the BAM system given in (3.2.1) has a unique equilibrium solution $[x_1^*, x_2^*, \cdots, x_m^*, y_1^*, y_2^*, \cdots, y_n^*]^T$ and moreover the equilibrium solution $x^*, y^*$ of the system (3.2.1) is globally asymptotically stable.

**Proof.** In view of (3.3.1), if $P_k(x_i^*) = 0, Q_k(y_i^*) = 0$, the existence of a unique equilibrium of (3.2.1) follows from Lemma 3.2.1. Consider the Lyapunov functional defined as follows
\[
V(t) = u^T(t)Pu(t) + \int_{t-\tau}^{t} M^T(v(s))R^{-2}M(v(s))ds
+ \int_{0}^{\infty} K(s) \int_{t-\tau_1}^{t} N^T(v(r))R^{-2}N(v(r))drds
+ v^T(t)Qv(t) + \int_{t-\sigma}^{t} M^T(u(s))S^{-2}\overline{M}(u(s))ds
+ \int_{0}^{\infty} \overline{K}(s) \int_{t-\sigma_1}^{t} \overline{N}^T(u(r))S^{-2}\overline{N}(u(r))drds.
\]

Now we can calculate the upper right derivative of $V$ along the trajectories of the system (3.2.6). Then we have
\[
D^+ V = -u^T(t)(PC + CP)u(t) + u^T(t)PHM(v(t - \tau)) + u^T(t)PLN(v(t - \tau_1))
+ M^T(v(t - \tau))H^TPu(t) + N^T(v(t - \tau_1))L^TPu(t) + M^T(v(t))R^{-2}M(v(t))
- M^T(v(t - \tau))R^{-2}M(v(t - \tau)) + N^T(v(t))R^{-2}N(v(t))
- N^T(v(t - \tau_1))R^{-2}N(v(t - \tau_1)) - v^T(t)(QD + DQ) + v^T(t)Q\overline{H} \overline{M}(v(t - \sigma))
+ v^T(t)Q\overline{L} \overline{N}(u(t - \sigma_1)) + \overline{M}^T(u(t - \sigma))\overline{H}^TQv(t) + \overline{N}^T(u(t - \sigma_1))\overline{L}^TQv(t)
+ \overline{M}^T(u(t))S^{-2}\overline{M}(u(t)) - \overline{M}^T(u(t - \sigma))S^{-2}\overline{M}(u(t - \sigma))
+ \overline{N}^T(u(t))S^{-2}\overline{N}(u(t)) - \overline{N}^T(u(t - \sigma_1))S^{-2}\overline{N}(u(t - \sigma_1)).
\]

\[
u^T(t)PHM(v(t - \tau)) + M^T(v(t - \tau))H^TPu(t) - M^T(v(t - \tau))R^{-2}M(v(t - \tau))
= -[R^{-1}M(v(t - \tau)) - RH^TPu(t)]^T \times [R^{-1}M(v(t - \tau)) - RH^TPu(t)]
+ u^T PHR^2H^TPu(t),
\]
\[ u^T(t) PLN(v(t - \tau_1)) + N^T(v(t - \tau_1))L^T P u(t) - N^T(v(t - \tau_1))R^{-2} N(v(t - \tau_1)) \]
\[ = -[R^{-1} N(v(t - \tau_1)) - RL^T P u(t)]^T \times [R^{-1} N(v(t - \tau_1)) - RL^T P u(t)] \]
\[ + u^T(t) PLR^2 L^T P u(t), \] (3.3.6)

\[ v^T(t) Q \bar{M}(u(t - \sigma)) + \bar{M}^T(u(t - \sigma)) \bar{H}^T Q v(t) - \bar{M}^T(u(t - \sigma))S^{-2} \bar{M}(u(t - \sigma)) \]
\[ = -[S^{-1} \bar{M}(u(t - \sigma)) - SH^T Q v(t)]^T \times [S^{-1} \bar{M}(u(t - \sigma)) - SH^T Q v(t)] \]
\[ + v^T Q \bar{H} S^2 \bar{H}^T Q v(t), \] (3.3.7)

\[ v^T(t) Q \bar{L} N(u(t - \sigma_1)) + N^T(u(t - \sigma_1))L^T Q v(t) - N^T(u(t - \sigma_1))S^{-2} N(u(t - \sigma_1)) \]
\[ = -[S^{-1} N(u(t - \sigma_1)) - SL^T Q v(t)]^T \times [S^{-1} N(u(t - \sigma_1)) - SL^T Q v(t)] \]
\[ + v^T(t) Q \bar{L} S^2 \bar{L}^T Q v(t). \] (3.3.8)

Then we obtain

\[ u^T(t) PHM(v(t - \tau)) + M^T(v(t - \tau))H^T P u(t) - M^T(v(t - \tau))R^{-2} M(v(t - \tau)) \leq u^T(t) PHR^2 H^T P u(t), \]

\[ v^T(t) PLN(v(t - \tau_1)) + N^T(v(t - \tau_1))L^T P u(t) - N^T(v(t - \tau_1))R^{-2} N(v(t - \tau_1)) \leq v^T(t) PLR^2 L^T P u(t), \]

\[ v^T(t) Q \bar{H} \bar{M}(u(t - \sigma)) + \bar{M}^T(u(t - \sigma)) \bar{H}^T Q v(t) - \bar{M}^T(u(t - \sigma))S^{-2} \bar{M}(u(t - \sigma)) \leq v^T Q \bar{H} S^2 \bar{H}^T Q v(t), \]

\[ v^T(t) Q \bar{L} N(u(t - \sigma_1)) + N^T(u(t - \sigma_1))L^T Q v(t) - N^T(u(t - \sigma_1))S^{-2} N(u(t - \sigma_1)) \leq v^T(t) Q \bar{L} S^2 \bar{L}^T Q v(t). \] (3.3.9)

Substituting (3.3.5)-(3.3.9) in (3.3.4), we get

\[ D^T V \leq -u^T(t)(PC + CP)u(t) + v^T(t)A^T R^{-2} A v(t) + u^T(t) PHR^2 H^T P u(t) \]
\[ + v^T(t) B R^{-2} B v(t) + u^T(t) PLR^2 L^T P u(t) - v^T(t) (QD + DQ) v(t) \]
\[ + u^T(t) \bar{A} S^{-2} \bar{A} u(t) + v^T(t) Q \bar{H} S^2 \bar{H}^T Q v(t) + u^T(t) \bar{B} S^{-2} \bar{B} u(t) \]
\[ + v^T(t) Q \bar{L} S^2 \bar{L}^T Q v(t) \]
\[ = u^T(t) Z_1 u(t) + v^T Z_2 v(t) \leq 0, \] (3.3.10)

where

\[ Z_1 = -(PC + CP) + PHR^2 H^T P + PLR^2 L^T P + \bar{A}^T S^{-2} \bar{A} + \bar{B}^T S^{-2} \bar{B}, \]
\[ Z_2 = -(QD + DQ) + Q \bar{H} S^2 \bar{H}^T Q + Q \bar{L} S^2 \bar{L}^T Q + A^T R^{-2} A + B^T R^{-2} B. \]
So we have

$$V(t) \leq V(0), \\ t \geq 0, t \neq t_k. \quad (3.3.11)$$

Let us define $\min(\lambda_P)$, $\min(\lambda_Q)$ as the minimal eigenvalues of matrix $P$ and $Q$. Then we obtain

$$\min(\lambda_P)u^T(t)u(t) + \min(\lambda_Q)v^T(t)v(t) \leq u^T(t)Pu(t) + v^TQv(t) \leq V(t) \leq V(0). \quad (3.3.12)$$

$$V(0) = u^T(0)Pu(0) + \int_{-\tau}^{0} M^T(v(s))R^{-2}M(v(s))ds$$

$$+ \int_{0}^{\sigma} K(s) \int_{-\sigma_1}^{0} N^T(v(r))R^{-2}N(v(r))drds$$

$$+ v^T(0)Qv(0) + \int_{-\sigma}^{0} \overline{M}^T(u(s))S^{-2}\overline{M}(u(s))ds$$

$$+ \int_{0}^{\sigma} \overline{K}(s) \int_{-\sigma_1}^{0} \overline{N}^T(u(r))S^{-2}\overline{N}(u(r))drds$$

$$\leq \max(\lambda_P) \sum_{i=1}^{m} u_i^2(0) + \int_{-\tau}^{0} \sum_{j=1}^{n} \frac{F_j^2}{r_j^2}v_j^2(s)ds + \int_{-\tau_1}^{0} \sum_{j=1}^{n} \frac{G_j^2}{r_j^2}v_j^2(r)dr$$

$$+ \max(\lambda_Q) \sum_{j=1}^{n} v_j^2(0) + \int_{-\sigma}^{0} \sum_{i=1}^{m} \frac{F_i^2}{s_i^2}u_i^2(s)ds + \int_{-\sigma_1}^{0} \sum_{i=1}^{m} \frac{G_i^2}{s_i^2}u_i^2(r)dr$$

$$\leq \max(\lambda_P) \sum_{i=1}^{m} u_i^2(0) + \sum_{j=1}^{n} \left( \max_{1 \leq j \leq n} \frac{F_j^2}{r_j^2} \right) \int_{-\tau}^{0} \sum_{j=1}^{n} v_j^2(s)ds$$

$$+ \sum_{j=1}^{n} \left( \max_{1 \leq j \leq n} \frac{G_j^2}{r_j^2} \right) \int_{-\tau_1}^{0} \sum_{j=1}^{n} v_j^2(r)dr + \max(\lambda_Q) \sum_{j=1}^{n} v_j^2(0)$$

$$+ \sum_{i=1}^{m} \left( \max_{1 \leq i \leq m} \frac{F_i^2}{s_i^2} \right) \int_{-\sigma}^{0} \sum_{i=1}^{m} u_i^2(s)ds + \sum_{i=1}^{m} \left( \max_{1 \leq i \leq m} \frac{G_i^2}{s_i^2} \right) \int_{-\sigma_1}^{0} \sum_{i=1}^{m} u_i^2(r)dr$$

$$\leq \left[ \max(\lambda_P) + \sigma \sum_{i=1}^{m} \left( \max_{1 \leq i \leq m} \frac{F_i^2}{s_i^2} \right) + \tau_1 \sum_{i=1}^{m} \left( \max_{1 \leq i \leq m} \frac{G_i^2}{s_i^2} \right) \right]$$

$$\times \sum_{i=1}^{m} \left[ \max_{t \in (-\sigma,0)} \left( \varphi_{xi}(t) - x_i^* \right)^2 \right]$$

$$+ \left[ \max(\lambda_Q) + \tau \sum_{j=1}^{n} \left( \max_{1 \leq j \leq n} \frac{F_j^2}{r_j^2} \right) + \tau_1 \sum_{j=1}^{n} \left( \max_{1 \leq j \leq n} \frac{G_j^2}{r_j^2} \right) \right]$$

$$\times \sum_{j=1}^{n} \left[ \max_{t \in (-\tau,0)} \left( \varphi_{yj}(t) - y_j^* \right)^2 \right], \quad (3.3.13)$$
where max($\lambda_P$), max($\lambda_Q$) represent the maximal eigenvalues of matrix $P$ and $Q$.

On the other hand, from (3.3.1), we have

$$
x_i(t_k + 0) - x_i^* = x_i(t_k) + P_k(x_i(t_k)) - x_i^* = (1 - \gamma_{ik})(x_i(t_k) - x_i^*),
$$

$$
y_j(t_k + 0) - y_j^* = y_j(t_k) + Q_k(y_j(t_k)) - y_j^* = (1 - \delta_{jk})(y_j(t_k) - y_j^*). \tag{3.3.14}
$$

Hence we have

$$
|x_i(t_k + 0) - x_i^*| = |(1 - \gamma_{ik})||x_i(t_k) - x_i^*)| \leq |(x_i(t_k) - x_i^*)|,
$$

$$
|y_j(t_k + 0) - y_j^*| = |(1 - \delta_{jk})||y_j(t_k) - y_j^*)| \leq |(y_j(t_k) - y_j^*)|. \tag{3.3.15}
$$

Also

$$
u_i(t_k + 0) = |(1 - \gamma_{ik})|u_i(t_k) \leq u_i(t_k), \quad k \in \mathbb{Z}^+, i = 1, 2, ..., m,
$$

$$
v_j(t_k + 0) = |(1 - \delta_{jk})|v_j(t_k) \leq v_j(t_k), \quad k \in \mathbb{Z}^+, j = 1, 2, ..., n. \tag{3.3.16}
$$

and we get

$$
u^T(t_k + 0)Pu(t_k + 0) \leq u^T(t_k)Pu(t_k),
$$

$$
v^T(t_k + 0)Qv(t_k + 0) \leq v^T(t_k)Qv(t_k). \tag{3.3.17}
$$

Thus we obtain

$$
V(t_k + 0) = u^T(t_k + 0)Pu(t_k + 0) + \int_{t_k + 0 - \tau}^{t_k + 0} M^T(v(s))R^{-2}M(v(s))ds
$$

$$
+ \int_0^\infty K(s) \int_{t_k + 0 - \tau}^{t_k + 0} N^T(v(r))R^{-2}N(v(r))dr ds
$$

$$
+ v^T(t_k + 0)Qv(t_k + 0) + \int_{t_k + 0 - \sigma}^{t_k + 0} \overline{M}^T(u(s))S^{-2}\overline{M}(u(s))ds
$$

$$
+ \int_0^\infty \overline{K}(s) \int_{t_k + 0 - \sigma}^{t_k + 0} \overline{N}^T(u(r))S^{-2}\overline{N}(u(r))dr ds
$$

$$
\leq u^T(t_k)Pu(t_k) + \int_{t_k - \tau}^{t_k} M^T(v(s))R^{-2}M(v(s))ds
$$

$$
+ \int_0^\infty K(s) \int_{t_k - \tau}^{t_k} N^T(v(r))R^{-2}N(v(r))dr ds
$$

$$
+ v^T(t_k)Qv(t_k) + \int_{t_k - \sigma}^{t_k} \overline{M}^T(u(s))S^{-2}\overline{M}(u(s))ds
$$

$$
+ \int_0^\infty \overline{K}(s) \int_{t_k - \sigma}^{t_k} \overline{N}^T(u(r))S^{-2}\overline{N}(u(r))dr ds \leq V(t_k). \tag{3.3.18}
$$
From (3.3.12), (3.3.13) and (3.3.18) we get

\[
\sum_{i=1}^{m} u_i^2(t) + \sum_{j=1}^{n} v_j^2(t) \leq \left\{ \left[ \max(\lambda_P) + \sigma \left( \max_{1 \leq i \leq m} \frac{F_i}{s_i^2} \right) \right] + \sigma_1 \left( \max_{1 \leq i \leq m} \frac{G_i}{s_i^2} \right) / \min(\lambda_P) \right\} \sum_{i=1}^{m} \left[ \max_{t \in (-\sigma,0)} \left( \varphi_{ix}(t) - x_i^* \right) \right]^2 \\
+ \left\{ \left[ \max(\lambda_Q) + \tau \left( \max_{1 \leq j \leq n} \frac{F_j}{r_j^2} \right) + \tau_1 \left( \max_{1 \leq j \leq n} \frac{G_j}{r_j^2} \right) \right] / \min(\lambda_Q) \right\} \sum_{j=1}^{n} \left[ \max_{t \in (-\tau,0)} \left( \varphi_{yx}(t) - y_j^* \right) \right]^2,
\]

that is,

\[
\sum_{i=1}^{m} (x_i(t) - x_i^*)^2 + \sum_{j=1}^{n} (y_j(t) - y_j^*)^2 \leq K \|\varphi_x - x_i^*\|^2 + L \|\varphi_y - y_j^*\|^2, \quad (3.3.20)
\]

where

\[
K = \left[ \max(\lambda_P) + \sigma \left( \max_{1 \leq i \leq m} \frac{F_i}{s_i^2} \right) + \sigma_1 \left( \max_{1 \leq i \leq m} \frac{G_i}{s_i^2} \right) \right] / \min(\lambda_P) \geq 1,
\]

\[
L = \left[ \max(\lambda_Q) + \tau \left( \max_{1 \leq j \leq n} \frac{F_j}{r_j^2} \right) + \tau_1 \left( \max_{1 \leq j \leq n} \frac{G_j}{r_j^2} \right) \right] / \min(\lambda_Q) \geq 1.
\]

According to Definition 3.2.1, the equilibrium solution \( x^* = (x_1^*, x_2^*, \ldots, x_m^*), \)
\( y^* = (y_1^*, y_2^*, \ldots, y_n^*) \) of the impulsive system (3.2.1) is globally asymptotically stable. This completes the proof. \( \square \)

**Remark 3.3.1.** It should be noted that when the impulsive jumps are absent, the results reduce to those of the non-impulsive systems. Also the results appearing in this chapter are new in the case of non-impulsive BAM neural networks with mixed delays.

**Remark 3.3.2.** In [83, 84], the authors studied global asymptotic stability of delay BAM neural networks with impulses. However the global asymptotic stability of impulsive BAM neural networks with mixed delays have not been taken into account in those papers. This implies that the results of this chapter are essentially new.
3.4 Example

Consider the BAM neutral networks with the following parameters in (3.2.6)

\[
C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad H = \begin{bmatrix} 0.6 & -0.12 \\ -0.6 & 0.3 \end{bmatrix},
\]

\[
L = \begin{bmatrix} 0.2 & -0.1 \\ -0.2 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},
\]

\[
\bar{H} = \begin{bmatrix} 0.8 & -0.14 \\ -0.8 & 0.5 \end{bmatrix}, \quad \bar{L} = \begin{bmatrix} 0.4 & -0.12 \\ -0.4 & 0.12 \end{bmatrix},
\]

\[
A = B = \bar{A} = \bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

Take \(\Delta x(t_k) = -\gamma_{1k}(x(t_k) - 1)\), \(\Delta y(t_k) = -\delta_{1k}(y(t_k) - 1)\), here \(\gamma_{1k} = 1 + \frac{1}{2} \sin(1 + k), \delta_{1k} = 1 + \frac{2}{3} \cos(2k)\). It is easy to verify that \(P_k(x_i(t_k)) = -\gamma_{ik}(x_i(t_k) - x_i^*)\), \(0 < \gamma_{ik} < 2\) and \(Q_k(y_j(t_k)) = -\delta_{jk}(y_j(t_k) - y_j^*)\), \(0 < \delta_{jk} < 2\). Solving the LMI in Theorem 3.3.1 using MATLAB LMI toolbox, we obtain a feasible solution as

\[
P = \begin{bmatrix} 7.5739 & 2.3092 \\ 2.3092 & 1.4179 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.1473 & 0.1028 \\ 0.1028 & 0.0749 \end{bmatrix},
\]

\[
R = \begin{bmatrix} 11.9963 & 0 \\ 0 & 11.9963 \end{bmatrix}, \quad S = \begin{bmatrix} 11.9963 & 0 \\ 0 & 11.9963 \end{bmatrix},
\]

The above results show that all the conditions stated in Theorem 3.3.1 are satisfied and hence the system (3.2.5) is globally asymptotically stable.

\*\*\*\*