Chapter 9

Exponential Synchronization for Stochastic Neutral Neural Networks with Impulses

9.1 Introduction

In the past decades, much attention has been paid to chaos synchronization due to its potential applications such as secure communications, biological systems, information science, etc. Since Pecora and Carroll [101] originally proposed the drive-response concept for achieving the synchronization of coupled chaotic systems, researchers have also proposed a variety of alternative schemes for ensuring the synchronization. As Haykin [43] pointed out that, in real nervous systems, synaptic transmission is a noisy process brought out by random fluctuations from the release of neurotransmitters and other probabilistic causes. Thus noise is unavoidable in actual applications of artificial neural networks and it should be taken into account in the mathematical model. On the other hand, noise may vary with time and cause important effects on the dynamic behavior of delayed system. Therefore it is of great importance to consider the synchronization problem of stochastic time-delayed chaotic neural networks.

In the optimal control of signal processing systems, computer networks, automatic control systems, flying object motions and telecommunications, many systems are characterized by abrupt changes at certain moments due to instantaneous perturbations which lead to impulsive effects. Mathematically, these systems with impulsive effects are described by the impulsive differential equations. Some qual-
Iterative properties such as oscillation, stability and periodicity have been investigated extensively by many authors (see Refs. [96, 115, 118, 129] and references cited therein). It is also worth noting that few results for synchronization of delayed neural networks with impulsive effects are reported. Impulsive control synchronization, as a type of synchronization, has been developed in [119, 130, 131, 152, 153].

Motivated by the above discussion, the aim of this chapter is to discuss the synchronization of the stochastic neutral neural networks with impulsive effects. By establishing two new integro-differential inequalities, some new sufficient conditions ensuring $p$th moment exponential synchronization are obtained. The remainder of the chapter is organized as follows: in Section 9.2, some necessary definitions and lemmas are given. In Section 9.3, $p$th moment exponential synchronization for stochastic neutral neural networks with impulses is investigated and some criteria for synchronization are obtained. In Section 9.4, one example is given to show the effectiveness of the proposed method.

\section{9.2 Preliminaries}

Consider the following chaotic neural network

\begin{equation}
\begin{aligned}
d(x(t) - L(x(t - \tau))) &= [-Cx(t) + Mg(x(t)) + Ng(x(t - \tau)) \\
&\quad + D \int_{-\infty}^{t} K(t - s)g(x(s))ds + I']dt, \quad t \neq t_k, \\
x(\theta) &= \phi(\theta), \quad \theta \in (-\infty, 0], \\
\Delta x(t_k) &= I_kx(t_k), \quad t = t_k,
\end{aligned}
\end{equation}

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is a real $n$-vector which denotes the state variable; $C = \text{diag}(c_1, c_2, \cdots, c_n), c_i > 0 (i = 1, 2, \cdots, n)$ represent the rate with which the $i$th unit will reset its potential to the resting state in isolation when disconnected from the network and the external inputs; $M = (m_{ij})_{n \times n}$ represents the connection weight matrix; $N = (n_{ij})_{n \times n}, D = (d_{ij})_{n \times n}, L = (l_{ij})_{n \times n}$ are the delayed connection weight matrices; $I'$ represents the external input; $g$ is the activation function; $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \cdots, g_n(x_n(t)))^T, g(x(t - \tau)) = (g_1(x_1(t - \tau)), g_2(x_2(t - \tau)), \cdots, g_n(x_n(t - \tau)))^T$. $K(t) = (k_{ij}(t))_{n \times n}$ is the kernel function matrix; $\phi(.)$ denotes the continuous initial function; $t_k, k = 1, 2, \cdots$, are the moments of impulsive perturbations and satisfy $t_0 < t_1 < t_2 < \cdots$ and $\lim_{k \to \infty} t_k = \infty$ and $I_k(x(t_k))$ represents the abrupt change of the state $x(t)$ at the
impulsive moment $t_k$.

In order to synchronize the system (9.2.1) via feedback control, the response system can be introduced as

$$
d(y(t) - L(y(t - \tau))) = [-Cy(t) + Mg(y(t)) + N g(y(t - \tau)) + D \int_{-\infty}^{t} K(t - s) g(y(s)) ds + I + u(t)] dt + \sigma(t, e(t), e(t - \tau)) dw(t), \quad t \neq t_k, \quad (9.2.2)
$$

$$
y(\theta) = \varphi(\theta), \quad \theta \in (-\infty, 0],
$$

$$
\Delta y(t_k) = I_k y(t_k), \quad t = t_k
$$

where $\sigma : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$; $u(t) = F_1[g(y(t)) - g(x(t))] + F_2[g(y(t - \tau)) - g(y(t - \tau))]$ is the state feedback controller used to achieve synchronization between the derive and response systems; $F_1, F_2$ are feedback gain parameters to be scheduled; $e(t) = y(t) - x(t)$ represents the error; $\varphi(.)$ denotes the continuous initial function.

Throughout this chapter, we assume that the following conditions hold:

(H1) Function $g_i(x)$ satisfies the Lipschitz condition. That is, for each $i = 1, 2, \ldots, n$, there exists constant $l_i > 0$ such that

$$
|g_i(x) - g_i(y)| \leq l_i |x - y|, \forall x, y \in \mathbb{R},
$$

where $l_i$ is Lipschitz constant.

(H2) There exist nonnegative constants $\mu_i, \nu_i (i = 1, 2, \ldots, n)$ such that

$$
\text{trace}[\sigma^T(t; x, y) \sigma(t; x, y)] \leq \sum_{i=1}^{n} (\mu_i x_i^2 + \nu_i y_i^2), \quad \forall (t; x, y) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n
$$

(H3) Kernel functions $k_{ij}(t) : [0, +\infty) \rightarrow [0, +\infty) (i, j = 1, 2, \ldots, n)$ are real-valued non-negative continuous functions.

(H4) $\int_{0}^{\infty} k_{ij}(s) ds = 1, \quad i, j = 1, 2, \ldots, n.$

(H5) There exists a $\epsilon > 0$ such that $\int_{0}^{\infty} e^{\epsilon t} k_{ij}(t) dt = \bar{k}_{ij} < \infty$; $k(t) = \sup_{1 \leq i, j \leq n} \{k_{ij}(t)\}$, $k' = \max_{1 \leq i, j \leq n} (\bar{k}_{ij}).$

(H6) There exists some positive number $q_k (k = 1, 2, \ldots)$ such that

$$
|I_k(x) - I_k(y)| \leq q_k |x - y| \quad \text{and} \quad \sum_{k=1}^{+\infty} q_k < +\infty.
$$
From (9.2.1) and (9.2.2), the error system can be written as
\[
d(e(t) - L(e(t - \tau))) = [-Ce(t) + (M + F_1)f(e(t)) + (N + F_2)f(e(t - \tau))
+ D \int_{-\infty}^{t} K(t - s)f(e(s))ds]dt
+ \sigma(t, e(t), e(t - \tau))dw(t), \quad t \neq t_k,
\]
\[
e(\theta) = \psi(\theta), \quad \theta \in (-\infty, 0],
\]
\[
\Delta e(t_k) = I_k e(t_k), \quad t = t_k,
\]
where \(f(e(t)) = g(y(t)) - g(x(t)), f(e(t - \tau)) = g(y(t - \tau)) - g(x(t - \tau)), \psi(.) = \varphi(.) - \phi(.)\).

Set \(A = M + F_1 = (a_{ij})_{n \times n}, B = N + F_2 = (b_{ij})_{n \times n}.\) The system (9.2.4) can be rewritten as
\[
d(e_i(t) - \sum_{j=1}^{n} l_{ij}(e(t - \tau))) = [-c_i e_i(t) + \sum_{j=1}^{n} a_{ij} f_j(e_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(e_j(t - \tau))
+ \sum_{j=1}^{n} d_{ij} \int_{-\infty}^{t} k_{ij}(t - s)f_j(e_j(s))ds]dt
+ \sum_{j=1}^{n} \sigma_{ij}(t, e_j(t), e_j(t - \tau))dw_j(t), \quad t \neq t_k,
\]
\[
e_i(\theta) = \psi_i(\theta), \quad i = 1, 2, \ldots, n, \quad \theta \in (-\infty, 0],
\]
\[
\Delta e_i(t_k) = I_k e_i(t_k), \quad t = t_k.
\]
(9.2.4)

For further discussion, we introduce the following definition and lemmas.

**Definition 9.2.1.** [89] The drive system (9.2.1) and the response system (9.2.2) are said to be \(p\)th moment exponentially synchronized if there exist a pair of positive constants \(\lambda\) and \(\alpha\) such that
\[
E\|e(t, t_0, \psi)\|^p \leq \alpha E\|\psi\|^p e^{-\lambda(t-t_0)}, \quad t \geq t_0,
\]
holds for any \(t_0\) and \(\psi \in L^p_{\mathcal{F}_{t_0}}((-\infty, 0], \mathbb{R}^n).\) Here \(L^p_{\mathcal{F}_{t_0}}((-\infty, 0], \mathbb{R}^n)\) is the family of all \(\mathcal{F}_0\) measurable \(C((-\infty, 0], \mathbb{R}^n)\)-valued random variables satisfying
\[
\sup_{-\infty < t \leq 0} E|\psi(t)|^p < \infty; C((-\infty, 0], \mathbb{R}^n)\) denotes the family of all continuous \(\mathbb{R}^n\)-valued functions \(\phi(t)\) on \((-\infty, 0]\) with the norm \(\|\phi\|^p = \sup_{-\infty < t \leq 0} |\phi(t)|^p.\) Especially, when \(p = 2,\) it is said to be exponentially synchronized in mean square.
Lemma 9.2.1. [89] Assume that there exist two continuous functions \( f(x), g(x) \) and a set \( \Omega, p \) and \( q \) satisfying \( \frac{1}{q} + \frac{1}{p} = 1 \), for any \( p > 0, q > 0 \), if \( p > 1 \). Then the following inequality holds:

\[
\int_{\Omega} |f(x)g(x)| \, dx \leq \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p} \left( \int_{\Omega} |g(x)|^q \, dx \right)^{1/q}.
\]

Lemma 9.2.2. [58] For any stochastic variable \( \xi_i, i = 1, 2, \cdots, n \)

\[
\mathbb{E} \left( \left| \sum_{i=1}^{n} \xi_i \right|^p \right) \leq C_p \sum_{i=1}^{n} \mathbb{E}(|\xi_i|^p),
\]

where

\[
C_p = \begin{cases} 
  n^{p-1}, & p > 1, \\
  1, & 0 < p \leq 1.
\end{cases}
\]

Lemma 9.2.3. [131] Let \( w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^T \) be an \( n \)-dimensional Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We have the following formula

\[
\mathbb{E} \left( \int_0^t f_i(s) \, dw_i(s) \int_0^t f_j(s) \, dw_j(s) \right) = \mathbb{E} \int_0^t f_i(s) f_j(s) \, d \langle w_i, w_j \rangle_s,
\]

where \( \langle w_i, w_j \rangle_s = \delta_{ij} t \) is cross-variations and \( \delta_{ij} \) is the correlation coefficient.

### 9.3 Exponential Synchronization

In this section, the synchronization problem of the derive and response systems (9.2.1) and (9.2.2) will be discussed. To proceed, we first generalize two important inequalities as follows:

**Theorem 9.3.1.** Assume that positive scalars \( c, \gamma_1 > 1, \gamma_2, \gamma_3, \gamma_4 \) satisfy

\[
\frac{\gamma_2 + \gamma_3 + c}{1 - \gamma_1} < c,
\]

\( m(t) \) is a nonnegative continuous function on \((-\infty, +\infty)\) and satisfies the following inequality on the interval \([t_0, +\infty)\)

\[
m(t) \leq e^{-ct} h + e^{-ct} \gamma_1 m(-\tau) + \gamma_1 m(t - \tau) + \gamma_2 \int_{t_0}^t e^{-c(t-s)} m(s) \, ds \\
+ \gamma_3 \int_{t_0}^t e^{-c(t-s)} m(s - \tau) \, d\tau + \gamma_4 \int_{t_0}^t e^{-c(t-s)} \int_{-\infty}^{s \xi} k(s - \xi) m(\xi) \, d\xi \, ds \\
+ \sum_{t_k < t} e^{-c(t-t_k)} q_k m(t_k),
\]
where \( t_0 \geq 0, k = \int_0^\infty e^{cs} k(s) ds, k(s) > 0, \tau, h \) are positive constants. Then as \( t \geq t_0, \) we have

\[
m(t) \leq he^{-\epsilon(t-t_0)},
\]

(9.3.1)

where \( \epsilon \) is the unique positive solution of the following equation

\[
\epsilon = \frac{1}{1 - \gamma_1} \left[ c - \gamma_2 - (c\gamma_1 + \gamma_3)e^{\tau} - k\gamma_4 \right].
\]

Proof. Set

\[
y(t) = e^{-ct}h + e^{-ct}\gamma_1 m(-\tau) + \gamma_1 m(t - \tau) + \gamma_2 \int_{t_0}^t e^{-c(t-s)} m(s) ds
\]

\[
+ \gamma_3 \int_{t_0}^t e^{-c(t-s)} m(s - \tau) d\tau + \gamma_4 \int_{t_0}^t e^{-c(t-s)} \int_{-\infty}^s k(s - \xi) m(\xi) d\xi ds
\]

\[
+ \sum_{t_k < t} e^{-c(t-t_k)} q_k m(t_k).
\]

In view of \( m(t) \leq y(t), \) we have

\[
y(t) = -ce^{-ct}h - ce^{-ct}\gamma_1 m(-\tau) + \gamma_1 m'(t - \tau) - c\gamma_2 \int_{t_0}^t e^{-c(t-s)} m(s) ds
\]

\[
- c\gamma_3 \int_{t_0}^t e^{-c(t-s)} m(s - \tau) d\tau - c\gamma_4 \int_{t_0}^t e^{-c(t-s)} \int_{-\infty}^s k(s - \xi) m(\xi) d\xi ds
\]

\[
- c \sum_{t_k < t} e^{-c(t-t_k)} q_k m(t_k) + \gamma_2 m(t) + \gamma_3 m(t - \tau) + \gamma_4 \int_{-\infty}^t k(t - s) m(s) ds
\]

\[
y(t) = -cy(t) + c\gamma_1 m(t - \tau) + \gamma_1 m'(t - \tau) + \gamma_2 m(t) + \gamma_3 m(t - \tau)
\]

\[
+ \gamma_4 \int_{-\infty}^t k(t - s) y(s) ds
\]

\[
\leq \frac{1}{1 - \gamma_1} \left[ (\gamma_2 - c)y(t) + (c\gamma_1 + \gamma_3)y(t - \tau) + \gamma_4 \int_{-\infty}^t k(t - s) m(s) ds \right](9.3.2)
\]

Set \( \tilde{y}(t) = \{\sup_{s < t \leq 0} h e^{c(t_0+s)}\} e^{-ct} = he^{-c(t-t_0)}. \) We first prove that \( y(t) \leq he^{-c(t-t_0)}. \) Let \( l > 1 \) be arbitrary. We claim that \( y(t) \leq lhe^{-c(t-t_0)}. \) If it's not true, since \( y(t) \leq he^{-c(t-t_0)} < lhe^{-c(t-t_0)} \) for all \( t \leq t_0, \) there must exist \( t^* > t_0 \) such that

\[
y(t) < l\tilde{y}(t), \forall t < t^*; \quad y(t^*) = l\tilde{y}(t^*).
\]

that is,

\[
y'(t^*) - l\tilde{y}'(t^*) \geq 0 \quad (9.3.3)
\]
On the other hand, from the inequality (9.3.2) and the conditions of Theorem 9.3.1, we have

\[
y'(t) \leq \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2)\tilde{y}(t^*) + (c\gamma_1 + \gamma_3)\tilde{y}(t^* - \tau) + \gamma_4 \int_{-\infty}^{t^*} k(t^* - s)\tilde{y}(s)ds \right] \\
= \frac{1}{1 - \gamma_1} \left[ -l(c - \gamma_2)\tilde{y}(t^*) + (c\gamma_1 + \gamma_3)\tilde{y}(t^* - \tau) + \gamma_4 \int_{-\infty}^{t^*} k(t^* - s)\tilde{y}(s)ds \right] \\
< \frac{1}{1 - \gamma_1} \left[ -l(c - \gamma_2)\tilde{y}(t^*) + (c\gamma_1 + \gamma_3)\tilde{y}(t^* - \tau) + \gamma_4 l \int_{-\infty}^{t^*} k(t^* - s)\tilde{y}(s)ds \right] \\
= \frac{1}{1 - \gamma_1} \left[ -l(c - \gamma_2)he^{-e(t^* - t_0)} + (c\gamma_1 + \gamma_3)lhe^{-e(t^* - \tau - t_0)} \right. \\
\left. + \gamma_4 l \int_{-\infty}^{t} k(t^* - s)he^{-e(s - t_0)}ds \right] \\
\leq \frac{1}{1 - \gamma_1} \left[ -l(c - \gamma_2)he^{-e(t^* - t_0)} + (c\gamma_1 + \gamma_3)lhe^{-e(t^* - \tau - t_0)} \right. \\
\left. + \gamma_4 lhe^{-e(t^* - t_0)} \int_{0}^{+\infty} e^{-es}k(s)ds \right] \\
= \frac{1}{1 - \gamma_1} \left[ -l(c - \gamma_2) + (c\gamma_1 + \gamma_3)e^\epsilon\tau + \gamma_4 lhe^{-e(t^* - t_0)} \right] \\
= -clhe^{-e(t^* - t_0)} = l\tilde{y}(t^*). \tag{9.3.4}
\]

This contradicts to inequality (9.3.3). Thus \( y(t) \leq lhe^{-e(t - t_0)} \). Let \( l \to 1 \). We conclude that \( y(t) \leq he^{-e(t - t_0)} \). Note that \( m(t) \leq y(t) \). We have \( m(t) \leq he^{-e(t - t_0)} \), which complete the proof. \( \square \)

**Theorem 9.3.2.** Assume that positive scalars \( c, \gamma_1 > 1, \gamma_2, \gamma_3, \gamma_4 \) satisfy \( \frac{\gamma_1 + \gamma_2 \gamma_3}{1 - \gamma_1} < c \), \( m(t) \) is a nonnegative continuous function on \( (-\infty, +\infty) \) and satisfies the following inequality on interval \( [t_0, +\infty) \):

\[
m(t) \leq e^{-ct}h + e^{-ct}\gamma_1 m(-\tau) + \gamma_1 m(t - \tau) + \gamma_2 \int_{t_0}^{t} e^{-c(t - s)}m(s)ds \\
+ \gamma_3 \int_{t_0}^{t} e^{-c(t - s)}m(s - \tau)d\tau + \gamma_4 \int_{t_0}^{t} e^{-c(t - s)} \int_{-\infty}^{s} k(s - \xi)m(\xi)d\xi ds \\
+ \sum_{k < t} e^{-c(t - t_k)}q_km(t_k),
\]

where \( t_0 \geq 0, \int_{0}^{\infty} k(s)ds = 1, k(s) > 0, \tau, h \) are positive constants. Then as \( t \geq t_0 \), we have

\[
m(t) \leq \left\{ \sup_{-\infty < \theta \leq 0} he^{-e(t_0 + \theta)} \right\} = he^{ct_0} and \lim_{t \to +\infty} m(t) = 0. \tag{9.3.5}
\]
Proof. We will complete the proof in two steps. In step 1, we prove that \( m(t) \leq y_0 = h e^{ct_0} = \sup_{-\infty < \theta \leq 0} h e^{c(t_0 + \theta)} \). In step 2, we prove that \( \lim_{t \rightarrow +\infty} m(t) = 0 \).

Step 1: we first prove that for any positive constant \( d > 1 \), the following inequality holds

\[
y(t) < d \cdot y_0, t \geq t_0,
\]

(9.3.6)

where \( y(t) \) is the same as that defined in Theorem 9.3.1. Since, for any \( t \in (-\infty, t_0) \), \( y(t) \leq \sup_{-\infty < \theta \leq 0} h e^{c(t_0 + \theta)} = y_0 \). If \( y_0 = 0 \), then we get \( 0 \leq y(t) \leq 0 \), namely, \( y(t) \equiv 0 \). Thus we always assume that \( y_0 > 0 \). When \( t \leq t_0 \), we have \( y(t) \leq y_0 < d \cdot y_0 \). If the inequality (9.3.6) does not hold, there must exist \( t_1 > t_0 \) such that

\[
y(t_1) = d \cdot y_0, y(t) < d \cdot y_0, \forall t < t_1,
\]

(9.3.7)

which implies that \( y'(t) \geq 0 \). From the inequality (9.3.2), we get

\[
y'(t_1) \leq \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2)y(t_1) + (c \gamma_1 + \gamma_3)y(t_1 - \tau) + \gamma_4 \int_{-\infty}^{t_1} k(t_1 - s)y(s)ds \right]
\]

\[
< \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2)d \cdot y_0 + (c \gamma_1 + \gamma_3)d \cdot y_0 + \gamma_4 \int_{-\infty}^{t_1} k(t_1 - s)y_0ds \right]
\]

\[
= \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2) + (c \gamma_1 + \gamma_3) + \gamma_4 \int_{-\infty}^{t_1} k(t_1 - s)ds \right] d \cdot y_0
\]

\[
= \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2) + (c \gamma_1 + \gamma_3) + \gamma_4 \int_{0}^{t_1} k(s)ds \right] d \cdot y_0 < 0.
\]

(9.3.8)

This contradicts \( y'(t_1) \geq 0 \). So inequality (9.3.6) holds. According to the arbitrary property of positive constant \( d \), we have \( y(t) \leq h e^{ct_0} \). In views of \( m(t) \leq y(t) \), we get

\[
m(t) \leq h e^{ct_0}, \forall t \geq t_0.
\]

(9.3.9)

Step 2: In what follows, we prove that \( \lim_{t \rightarrow +\infty} m(t) = 0 \). From the inequality (9.3.6), we know that \( y(t) \) is a bounded continuous function. Thus, when \( t \rightarrow +\infty \), the upper limit (noted by \( p \)) of \( y(t) \) exists, namely,

\[
\overline{\lim}_{t \rightarrow +\infty} y(t) = p, p \geq 0.
\]

(9.3.10)
The remaining part of the proof is to prove \( p = 0 \). If it is not true, there must exist arbitrary positive constant \( \epsilon > 0 \) and constant \( T_1 > t_0 \) such that

\[
y(t - \tau) < p + \epsilon, \quad y(t) < p + \epsilon, \quad \forall t > T_1.
\]

On the other hand, since \( \int_0^{+\infty} k(s)ds = 1 \), there must exist \( T_2 > t_0 \) such that

\[
\int_t^{+\infty} k(s)ds < \epsilon, \quad \forall t \geq T_2.
\] (9.3.11)

Set \( T = \max\{T_1, T_2\} \). When \( t \geq 2T \), we have

\[
y'(t) \leq \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2)y(t) + (c\gamma_1 + \gamma_3)y(t - \tau) + \gamma_4 \int_{-\infty}^{t} k(t - s)y(s)ds \right]
\]

\[
= \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2)y(t) + (c\gamma_1 + \gamma_3)y(t - \tau) + \gamma_4 \int_{-\infty}^{t-T} k(t - s)y(s)ds 
+ \gamma_4 \int_{t-T}^{t} k(t - s)y(s)ds \right]
\]

\[
< \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2)y(t) + (c\gamma_1 + \gamma_3)y(t - \tau) + \gamma_4 y_0 \int_{-\infty}^{t-T} k(t - s)ds 
+ \gamma_4 (p + \epsilon) \int_{t-T}^{t} k(t - s)ds \right]
\]

\[
= \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2)y(t) + (c\gamma_1 + \gamma_3)y(t - \tau) + \gamma_4 y_0 \int_{T}^{+\infty} k(s)ds 
+ \gamma_4 (p + \epsilon) \int_{0}^{T} k(s)ds \right]
\]

\[
< \frac{1}{1 - \gamma_1} \left[ -(c - \gamma_2)y(t) + (c\gamma_1 + \gamma_3)(p + \epsilon) + \gamma_4 \epsilon y_0 + \gamma_4 (p + \epsilon) \right]. \tag{9.3.12}
\]

By direct calculation, we get

\[
y(t) \leq y(t_0) \exp\left\{ \frac{-(c - \gamma_2)}{1 - \gamma_1} (t - t_0) \right\} + \frac{1}{1 - \gamma_2} [ (c\gamma_1 + \gamma_3)(p + \epsilon) + \gamma_4 \epsilon y_0 + \gamma_4 (p + \epsilon) ].
\]

From (9.3.10), we get

\[
p \leq \frac{1}{c - \gamma_2} \left[ c\gamma_1 p + \gamma_3 p + c\gamma_1 \epsilon + \gamma_3 \epsilon + \gamma_4 \epsilon y_0 + \gamma_4 p + \gamma_4 \epsilon \right].
\]

In view of the arbitrary property of \( \epsilon \), we have \( p \leq \frac{\gamma_1 p + \gamma_3 p + \gamma_4 p}{c - \gamma_2} \), namely, \( \frac{\gamma_1 + \gamma_3 + \gamma_4}{1 - \gamma_1} \geq c \), which contradicts to the assumption \( \frac{\gamma_1 + \gamma_3 + \gamma_4}{1 - \gamma_1} < c \). Thus \( \lim_{t \to +\infty} m(t) = 0 \). The proof is completed. \( \square \)
In what follows, based on the established inequalities in Theorem 9.3.1 and 9.3.2, some novel $p$th moment synchronization criteria for the concerned derive-response systems can be derived as follows:

**Theorem 9.3.3.** Under the hypotheses $(H1) - (H3)$ and $(H5) - (H6)$, the derive system (9.2.1) and the response system (9.2.2) are $p$th moment exponentially synchronized ($p \geq 2$), if

$$\frac{\gamma_2 + \gamma_3 + k' \gamma_4}{1 - \gamma_1} < c,$$

where

$$\gamma_1 = \sum_{j=1}^{n} |l_{ij}|^p,$$

$$\gamma_2 = \left[ c^{-\frac{p}{q}} \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} |a_{ji}|^q |l_i|^q \right]^\frac{p}{q} + n^p \left( \frac{1}{c} \right)^{p/2-1} \mu^{p/2} \right] g^{p-1},$$

$$\gamma_3 = \left[ c^{-\frac{p}{q}} \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} |b_{ji}|^q |l_i|^q \right]^\frac{p}{q} + c^{-\frac{p}{q}} \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} |b_{ji}|^q |l_i|^q \right]^\frac{p}{q} + n^p \left( \frac{1}{c} \right)^{p/2-1} \nu^{p/2} \right] g^{p-1},$$

$$\gamma_4 = g^{p-1} \left( \frac{c}{k'} \right)^{-\frac{p}{q}} \sum_{j=1}^{n} \left( \sum_{i=1}^{n} |d_{ji}|^q |l_i|^q \right)^\frac{p}{q},$$

$$c = \min\{c_1, c_2, \cdots, c_n\}, \quad \mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\},$$

$$\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}, \quad k' = \max\{k_{ij}\}, \quad q = \frac{p}{p-1}.\quad (9.3.14)$$

**Proof.** Since $c = \min\{c_1, c_2, \cdots, c_n\}$, from system (9.3.2), we have

$$|e_i(t)| = |e^{-c_i t} e_i(0) - \sum_{j=1}^{n} e^{-c_i t} l_{ij} e_j(-\tau) + \sum_{j=1}^{n} l_{ij} e_j(t - \tau)$$

$$+ \int_0^t e^{-c_i (t-s)} \sum_{j=1}^{n} l_{ij} e_j(s - \tau) ds + \int_0^t e^{-c_i (t-s)} \left( \sum_{j=1}^{n} a_{ij} f_j(e_j(s)) \right) ds$$

$$+ \int_0^t e^{-c_i (t-s)} \left( \sum_{j=1}^{n} b_{ij} f_j(e_j(s - \tau_j)) \right) ds$$
\[
\begin{align*}
&+ \int_0^t e^{-\alpha(t-s)} \left( \sum_{j=1}^n d_{ij} \int_{-\infty}^t k_{ij}(s-r_1)f_j(e_j(r_1)) \right) \, dr_1 \, ds \\
&+ \int_0^t e^{-\alpha(t-s)} \left( \sum_{j=1}^n \sigma_{ij}(s, e_j(s), e_j(s-\tau_j)) \right) \, dw_j(s) \\
&+ \sum_{t_k < t} e^{-\alpha(t-t_k)} I_k x(t_k) |I_k x(t_k)| \\
&\leq e^{-\alpha t} |e_i(0)| - \sum_{j=1}^n e^{-\alpha t} |l_{ij} e_j(-\tau)| + \sum_{j=1}^n |l_{ij} e_j(t-\tau)| \\
&+ \int_0^t e^{-\alpha(t-s)} \left| \sum_{j=1}^n l_{ij} e_j(s-\tau) \right| \, ds + \int_0^t e^{-\alpha(t-s)} \left| \sum_{j=1}^n a_{ij} f_j(e_j(s)) \right| \, ds \\
&+ \int_0^t e^{-\alpha(t-s)} \left| \sum_{j=1}^n b_{ij} f_j(e_j(s-\tau_j)) \right| \, ds \\
&+ \int_0^t e^{-\alpha(t-s)} \left| \sum_{j=1}^n d_{ij} \int_{-\infty}^t k_{ij}(s-r_1)f_j(e_j(r_1)) \right| \, dr_1 \, ds \\
&+ \int_0^t e^{-\alpha(t-s)} \left| \sum_{j=1}^n \sigma_{ij}(s, e_j(s), e_j(s-\tau_j)) \right| \, dw_j(s) \\
&+ \sum_{t_k < t} e^{-\alpha(t-t_k)} |I_k x(t_k)| \\
&= I_{1i} + I_{2i} + I_{3i} + I_{4i} + I_{5i} + I_{6i} + I_{7i} + I_{8i} + I_{9i}.
\end{align*}
\]

In view of Lemma 9.2.2, the following inequality holds

\[
\sum_{i=1}^n \mathbb{E}|e_i(t)|^p \leq 9^{p-1} \sum_{i=1}^n \left( \mathbb{E}|I_{1i}|^p + \mathbb{E}|I_{2i}|^p + \mathbb{E}|I_{3i}|^p + \mathbb{E}|I_{4i}|^p + \mathbb{E}|I_{5i}|^p + \mathbb{E}|I_{6i}|^p + \mathbb{E}|I_{7i}|^p + \mathbb{E}|I_{8i}|^p + \mathbb{E}|I_{9i}|^p \right).
\]

By direct calculation, we obtain

\[
\sum_{i=1}^n \mathbb{E}|I_{4i}|^p = \sum_{i=1}^n \mathbb{E} \left[ \int_0^t e^{-\alpha(t-s)} \left| \sum_{j=1}^n l_{ij} e_j(s-\tau) \right| ds \right]^p \\
\leq \sum_{i=1}^n \mathbb{E} \left[ \int_0^t e^{-\alpha(t-s)} \left( \sum_{j=1}^n |l_{ij}||e_j(s-\tau)| \right)^p ds \right] \\
= \sum_{i=1}^n \mathbb{E} \left[ \int_0^t e^{-\frac{\alpha(t-s)}{q}} \left( \sum_{j=1}^n |l_{ij}||e_j(s-\tau)| \right)^p ds \right].
\]
\[
\sum_{i=1}^{n} \mathbb{E} \left\{ \left[ \int_{0}^{t} e^{-c(t-s)} ds \right]^{\frac{p}{q}} \int_{0}^{t} e^{-c(t-s)} \left[ \sum_{j=1}^{n} |l_{ij}| \|e_{j}(s - \tau)\| \right] ds \right\} \\
\leq c^{-\frac{p}{q}} \sum_{i=1}^{n} \left\{ \mathbb{E} \left[ \int_{0}^{t} e^{-c(t-s)} \left( \sum_{j=1}^{n} |l_{ij}| \right)^{\frac{p}{q}} \int_{0}^{t} e^{-c(t-s)} \left( \sum_{j=1}^{n} |e_{j}(s - \tau)|^{p} \right) ds \right] \right\} \\
= c^{-\frac{p}{q}} \sum_{j=1}^{n} \left\{ \mathbb{E} \left[ \left( \sum_{i=1}^{n} |l_{ij}| \right)^{\frac{p}{q}} \int_{0}^{t} e^{-c(t-s)} \sum_{i=1}^{n} |e_{i}(s - \tau)|^{p} ds \right] \right\} \\
= c^{-\frac{p}{q}} \sum_{j=1}^{n} \mathbb{E} \left[ \left( \sum_{i=1}^{n} |l_{ij}| \right)^{\frac{p}{q}} \int_{0}^{t} e^{-c(t-s)} \sum_{i=1}^{n} |e_{i}(s - \tau)|^{p} ds \right], \quad (9.3.15) \\
\sum_{i=1}^{n} \mathbb{E} I_{bi}^{p} = \sum_{i=1}^{n} \mathbb{E} \left[ \int_{0}^{t} e^{-c(t-s)} \left( \sum_{j=1}^{n} a_{ij} f_{j}(e_{j}(s)) \right) ds \right]^{p} \\
\leq \sum_{i=1}^{n} \mathbb{E} \left[ \int_{0}^{t} e^{-c(t-s)} \left( \sum_{j=1}^{n} |a_{ij} f_{j}(e_{j}(s))| \right) ds \right]^{p} \\
= \sum_{i=1}^{n} \mathbb{E} \left[ \int_{0}^{t} e^{-c(t-s) - \frac{c(t-s)}{q}} \left( \sum_{j=1}^{n} |a_{ij} f_{j}(e_{j}(s))| \right) ds \right]^{p} \\
\leq \sum_{i=1}^{n} \mathbb{E} \left\{ \left[ \int_{0}^{t} e^{-c(t-s)} ds \right]^{\frac{p}{q}} \int_{0}^{t} e^{-c(t-s)} \left[ \sum_{j=1}^{n} |a_{ij} f_{j}(e_{j}(s))| \right]^{p} ds \right\} \\
\leq c^{-\frac{p}{q}} \sum_{i=1}^{n} \left\{ \mathbb{E} \int_{0}^{t} e^{-c(t-s)} \left[ \sum_{j=1}^{n} |a_{ij} f_{j}(e_{j}(s))| \right]^{p} ds \right\} \\
= c^{-\frac{p}{q}} \sum_{i=1}^{n} \left\{ \mathbb{E} \left[ \left( \sum_{j=1}^{n} |a_{ij} f_{j}(e_{j}(s))| \right)^{p} \right] ds \right\} \\
= c^{-\frac{p}{q}} \sum_{j=1}^{n} \left\{ \mathbb{E} \left[ \left( \sum_{i=1}^{n} |a_{ij} f_{j}(e_{j}(s))| \right)^{p} \right] ds \right\} \\
= c^{-\frac{p}{q}} \sum_{j=1}^{n} \mathbb{E} \left[ \left( \sum_{i=1}^{n} |a_{ij} f_{j}(e_{j}(s))| \right)^{p} \right] ds. \quad (9.3.16)
Similarly, for \( I_{6i}^p, I_{7i}^p, I_{8i}^p, I_{9i}^p \), we have

\[
\sum_{i=1}^{n} \mathbb{E} I_{6i}^p = c^{-\frac{p}{q}} \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} |b_{ji}|^q |l_i|^q \right]^\frac{p}{q} \int_0^t c^{-c(t-s)} \sum_{i=1}^{n} \mathbb{E} \left( e_i(s - \tau_i) \right)^p ds, \tag{9.3.17}
\]

\[
\sum_{i=1}^{n} \mathbb{E} I_{7i}^p = \sum_{i=1}^{n} \mathbb{E} \left\{ \int_0^t e^{-c(t-s)} \sum_{j=1}^{n} d_{ij} \int_0^s k_{ij}(s-r_1) f_j(e_j(r_1)) dr_1 ds \right\}^p
\]

\[
\leq \sum_{i=1}^{n} \mathbb{E} \left\{ \int_0^t e^{-c(t-s)} \left( \sum_{j=1}^{n} d_{ij} \int_\infty^s k_{ij}(s-r_1) f_j(e_j(r_1)) dr_1 \right) ds \right\}^p
\]

\[
\leq \sum_{i=1}^{n} \mathbb{E} \left\{ \left[ \int_0^t e^{-c(t-s)} ds \right]^{\frac{p}{q}} \left[ \int_0^t e^{-c(t-s)} \left( \sum_{j=1}^{n} d_{ij} \int_\infty^s k_{ij}(s-r_1) f_j(e_j(r_1)) dr_1 \right)^p ds \right] \right\}
\]

\[
= \sum_{i=1}^{n} \mathbb{E} \left\{ \left[ \frac{1 - e^{-ct}}{c} \right]^{\frac{p}{q}} \left[ \int_0^t e^{-c(t-s)} \left( \sum_{j=1}^{n} d_{ij} \int_\infty^s k_{ij}(s-r_1) f_j(e_j(r_1)) dr_1 \right)^p ds \right] \right\}
\]

\[
\leq c^{-\frac{p}{q}} \sum_{i=1}^{n} \mathbb{E} \left\{ \int_0^t e^{-c(t-s)} \left( \sum_{j=1}^{n} d_{ij} \int_\infty^s k_{ij}(s-r_1) f_j(e_j(r_1)) dr_1 \right)^p ds \right\}
\]

\[
\leq c^{-\frac{p}{q}} \sum_{i=1}^{n} \mathbb{E} \left\{ \int_0^t e^{-c(t-s)} \left( \sum_{j=1}^{n} d_{ij} \int_0^s k_{ij}(s-r_1) f_j(e_j(r_1)) dr_1 \right)^p ds \right\}
\]

\[
\leq c^{-\frac{p}{q}} \sum_{i=1}^{n} \mathbb{E} \left\{ \int_0^t e^{-c(t-s)} \left( \sum_{j=1}^{n} d_{ij} |l_j|^q \right)^{\frac{p}{q}} \times \sum_{j=1}^{n} \left( \int_\infty^s k_{ij}(s-r_1) f_j(e_j(r_1)) dr_1 \right)^p ds \right\}
\]

\[
= c^{-\frac{p}{q}} \sum_{i=1}^{n} \left\{ \left( \sum_{j=1}^{n} |d_{ij}|^q |l_j|^q \right)^{\frac{p}{q}} \times \mathbb{E} \left\{ \int_0^t e^{-c(t-s)} \sum_{j=1}^{n} \left( \int_\infty^s k_{ij}(s-r_1) f_j(e_j(r_1)) dr_1 \right)^p ds \right\} \right\}
\]
\[
= c^{\frac{p}{q}} \sum_{i=1}^{n} \left\{ \left( \sum_{j=1}^{n} |d_{ij}|^q |l_j|^q \right)^{\frac{p}{q}} \right. \\
\times \mathbb{E} \left\{ \int_0^t e^{-c(t-s)} \sum_{j=1}^{n} \left( \int_{-\infty}^{s} k^{\frac{1}{q}}_0 (s-r_1) k^{\frac{1}{q}}_0 (s-r_1) |e_j(r_1)| dr_1 \right)^{p} ds \right\} \\
\leq c^{\frac{p}{q}} \sum_{i=1}^{n} \left\{ \left( \sum_{j=1}^{n} |d_{ij}|^q |l_j|^q \right)^{\frac{p}{q}} \right. \\
\left. \times \left\{ \int_0^t e^{-c(t-s)} \sum_{j=1}^{n} k^{\frac{p}{q}} \int_{-\infty}^{s} k(s-r_1) \mathbb{E} |e_j(r_1)|^{p} dr_1 ds \right\} \right. \\
= \left( \frac{c}{k^q} \right)^{\frac{p}{q}} \left[ \sum_{j=1}^{n} \left( \sum_{i=1}^{n} |d_{ji}|^q |l_i|^q \right)^{\frac{p}{q}} \right. \\
\left. \left[ \int_0^t e^{-c(t-s)} \int_{-\infty}^{s} k(s-r_1) \sum_{i=1}^{n} \mathbb{E} |e_i(r_1)|^{p} dr_1 ds \right] \right]. \quad (9.3.18)
\]

\[
\sum_{i=1}^{n} \mathbb{E} I_{8i}^p = \sum_{i=1}^{n} \mathbb{E} \left[ \left| \int_0^t e^{-c(t-s)} \sum_{j=1}^{n} \sigma_{ij}(s, e_j, e_j(s-\tau_j)) |dw_j(s)|^p \right] \right. \\
\leq n^{p-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \left\{ \int_0^t e^{-c(t-s)} |\sigma_{ij}(s, e_j, e_j(s-\tau_j))| dw_j(s) \right. \\
= n^{p-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \left\{ \left[ \int_0^t e^{-c(t-s)} |\sigma_{ij}(s, e_j, e_j(s-\tau_j))| dw_j(s) \right]^2 \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
\[
\leq n^{p-1}2^{p/2-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \left\{ \left[ \int_{0}^{t} e^{-2c(t-s)} ds \right]^{p/2-1} \times \left[ \int_{0}^{t} e^{-2c(t-s)} \mu_j^{p/2} |e_j(s)|^p ds + \int_{0}^{t} e^{-2c(t-s)} \nu_j^{p/2} |e_j(s - \tau_j)|^p ds \right] \right\} \\
\leq n^{p-1} \left( \frac{1}{c} \right)^{p/2-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \left\{ \int_{0}^{t} e^{-2c(t-s)} \mu_j^{p/2} |e_j(s)|^p ds + \int_{0}^{t} e^{-2c(t-s)} \nu_j^{p/2} |e_j(s - \tau_j)|^p ds \right\} \\
\leq n^{p} \left( \frac{1}{c} \right)^{p/2-1} \left\{ \mu_j^{p/2} \int_{0}^{t} e^{-c(t-s)} \mathbb{E} |e_i(s)|^p ds + \nu_j^{p/2} \int_{0}^{t} e^{-c(t-s)} |e_i(s - \tau_j)|^p ds \right\}. \tag{9.3.19}
\]

From the inequalities (9.3.15)-(9.3.19) and Theorem 9.3.1, there exists an \( \epsilon > 0 \) such that \( \sum_{i=1}^{n} \mathbb{E} |e_i(t)|^p \leq \sum_{i=1}^{n} \mathbb{E} |\eta_i|^p e^{-\epsilon t} (t_0 = 0) \), namely,

\[
\mathbb{E} \|e(t)\|^p \leq \mathbb{E} \|\eta\|^p e^{-\epsilon t}, \tag{9.3.20}
\]

which completes the proof of Theorem 9.3.3.

\[\square\]

**Theorem 9.3.4.** Under the hypotheses (H1) – (H6), the derive system (9.2.1) and the response system (9.2.2) are \( p \)th moment exponentially synchronized (\( p \geq 2 \)), if

\[
\frac{\gamma_2 + \gamma_3 + \gamma_4}{1 - \gamma_1} < c, \tag{9.3.21}
\]

where

\[
\begin{align*}
\gamma_1 &= \sum_{j=1}^{n} |l_{ij}|^p, \\
\gamma_2 &= c^{-\frac{p}{2}} \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} |a_{ji}|^q |l_{ij}|^q \right]^{\frac{p}{q}} + n^{p-1} \left( \frac{1}{c} \right)^{p/2-1} \mu_j^{p/2} \right]^{g^{-1}}, \\
\gamma_3 &= c^{-\frac{p}{q}} \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} |l_{ij}|^q \right]^{\frac{p}{q}} + c^{-\frac{p}{q}} \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} |b_{ji}|^q |l_{ij}|^q \right]^{\frac{p}{q}} + n^{p-1} \left( \frac{1}{c} \right)^{p/2-1} \nu_j^{p/2} \right]^{g^{-1}}, \\
\gamma_4 &= g^{-1} c^{-\frac{p}{q}} \sum_{j=1}^{n} \left( \sum_{i=1}^{n} |d_{ji}|^q |l_{ij}|^q \right)^{\frac{p}{q}}, \\
c &= \min\{c_1, c_2, \ldots, c_n\}, \quad \mu = \max\{\mu_1, \mu_2, \ldots, \mu_n\}, \\
\nu &= \max\{\nu_1, \nu_2, \ldots, \nu_n\}, \quad q = \frac{p}{p-1}. \tag{9.3.22}
\end{align*}
\]
Proof. In view of (H4), similar to the proof of Theorem 9.3.3, inequality has the following form

\[
\sum_{i=1}^{n} \mathbb{E} I_{7i}^p = c^{-\frac{p}{q}} \left[ \sum_{j=1}^{n} \left( \sum_{i=1}^{n} |d_{ij}|^p |l_i|^q \right)^{\frac{2}{q}} \right] \times \left[ \int_{0}^{t} e^{-c(t-s)} \int_{s}^{\infty} k(s-r_1) \sum_{i=1}^{n} \mathbb{E}|e_i(r_1)|^p dr_1 ds \right]. \quad (9.3.23)
\]

From the inequalities (9.3.15)-(9.3.17),(9.3.19),(9.3.23) and Theorem 9.3.2, we conclude that the trivial solution of the error system (9.2.4) is \( p \)-th moment asymptotically stable, which completes the proof. \( \square \)

### 9.4 Example

Consider the following neutral neural network

\[
d(x(t) - L(x(t-\tau))) = [-Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + D \int_{-\infty}^{t} K(t-s)g(x(s))ds + I']dt, \quad t \neq t_k,
\]

\[
x(\theta) = \phi(\theta), \quad \theta \in (-\infty, 0],
\]

\[
\Delta x(t_k) = 0.2x(t_k), \quad t = t_k,
\]

where

\[
C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 3.0 \end{bmatrix},
\]

\[
B = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0.52 & 0 \\ 0 & 0.52 \end{bmatrix}, \quad L = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},
\]

where \( \tau = 1, I' = [0,0]^T, \phi(\theta) = [0.4, 0.6]^T, K(t) = 2e^{-2t} \). In order to synchronize the system (9.4.1) via feedback controller, the response system can be introduced as follows:

\[
d(y(t) - L(y(t-\tau))) = [-Cy(t) + Ag(y(t)) + Bg(y(t-\tau)) + D \int_{-\infty}^{t} K(t-s)g(y(s))ds + I' + u(t)]dt
\]

\[
+ \sigma(t, e(t), e(t-\tau))dw(t), \quad t \neq t_k,
\]

\[
y(\theta) = \varphi(\theta), \quad \theta \in (-\infty, 0],
\]

\[
\Delta y(t_k) = 0.2y(t_k), \quad t = t_k,
\]

(9.4.2)
\[
\sigma(t, e(t), e(t - \tau)) = \begin{bmatrix}
\sqrt{0.2}e_1(t) \\
\sqrt{0.3}e_1(t - \tau)
\end{bmatrix} \begin{bmatrix}
\sqrt{0.3}e_2(t - \tau) \\
\sqrt{0.2}e_2(t)
\end{bmatrix}.
\]

From (9.4.1) and (9.4.2), the error system can be written as

\[
d(e(t) - L(e(t - \tau))) = [-Ce(t) + (F_1 + A)g(e(t)) + (F_2 + B)g(e(t - \tau))
+ D \int_{-\infty}^{t} K(t - s)g(e(s))ds]dt
+ \sigma(t, e(t), e(t - \tau))dw(t), \quad t \neq t_k,
\]

\[
e(\theta) = \psi(\theta), \quad \theta \in (-\infty, 0],
\]

\[
\Delta e(t_k) = 0.2e(t_k), \quad t = t_k,
\]

(9.4.3)

where \(F_1, F_2\) are the gain matrices in the feedback controller \(u(t) = F_1[g(y(t)) - g(x(t))] + F_2[g(y(t - \tau)) - g(x(t - \tau))]\). Set \(p = 3\).

\[
F_1 = \begin{bmatrix}
-1.8 & 0.11 \\
5.1 & -3.1
\end{bmatrix},
F_2 = \begin{bmatrix}
1.54 & 0.100 \\
0.100 & 2.54
\end{bmatrix}.
\]

It is easy to verify that the conditions of Theorem 9.3.4 are satisfied. Thus the systems (9.4.1) and (9.4.2) are 3rd moment exponentially synchronized.

\[
\bullet \bullet \bullet \bullet
\]