Chapter 3

APPROXIMATE METHODS FOR EVALUATING THE S-MATRIX
In this chapter we will briefly review the various approximate methods that have been used in the literature to evaluate Glauber model S-matrix for nucleus-nucleus scattering. The purpose is to provide necessary background for the present study and to bring to focus the importance of our investigation.

3.1 THE APPROACH OF CZYZ AND MAXIMON

Czyz and Maximon\(^1\) were perhaps the first to study nucleus-nucleus scattering in detail in the Glauber model. To evaluate the S-matrix element (eq. 2.6.5) they assume that the ground state many body densities of the target as well as projectile may be written as the product of the single particle model densities \(\rho^{(M)}_A\) and \(\rho^{(M)}_B\) (each normalized to unity):

\[
|\psi_A(r_1, r_2, \ldots, r_A)|^2 = \prod_{i=1}^{A} \rho^{(M)}_A (r_i)
\]

3.1.1

and

\[
|\psi_B(r'_1, r'_2, \ldots, r'_B)|^2 = \prod_{j=1}^{B} \rho^{(M)}_B (r'_j)
\]

3.1.2

These authors further assume Gaussian model for the single particle model densities which enable them to evaluate the multidimensional integrals analytically, after applying the
well known prescription for accounting for the c.m. correlation. In this way they derive a closed though lengthy and fairly complicated expression for the nucleus-nucleus scattering amplitude.

Application of Czyz and Maximon's approach to nucleus-nucleus scattering calculation presents two major problems. First, except for very light nuclei nuclear densities cannot be approximated by the Gaussian model and second, when the multiple scattering series is truncated at some point, which one is forced to do for non Gaussian models, the c.m. correlation correction factor which multiplies the independent particle model amplitude makes the calculated amplitude diverge at large momentum transfers $^{2,3}$.

The above remarks become obvious by considering the so called optical limit expression for the elastic scattering amplitude which Czyz and Maximon obtain by neglecting certain class of terms in the multiple scattering series. The optical limit result reads as

\[ F_{BA}(q) = \Theta_A(q) \Theta_B(q) iK \int d^2 b \ e^{i q \cdot b} \times \]

\[-AB \int \rho_A^{(M)}(r) \rho_B^{(M)}(r') \rho_{NN}(b-s+s') dr dr' \]

\[ [1-e^{-\frac{b^2}{2\sigma^2}}] \]

where $\Theta_A(B)$ are the c.m. correlation factors for target(projectile) nucleus:
\[ \theta_{A(B)} = e^{q^2 R_{A(B)}^2 / 4A(B)} \]

where \( R_{A(B)} \) is the parameter.

It is because of \( \theta_A(q) \) and \( \theta_B(q) \) that the calculated amplitude diverges when optical limit is considered or the multiple scattering series is truncated at some point.

3.2 THE PHASE EXPANSION APPROACH

Franco and Varma have studied the problem of medium and high energy nucleus-nucleus scattering in some detail. These authors adopt a symmetric approach in that the two colliding nuclei are treated on the same footing. The main idea is to define the total elastic phase function \( \chi_T(b) \) through the relation:

\[ S_{BA}(b) = e^{i\chi_T(b)} \]

and expand \( \chi_T(b) \) in terms of an infinite series

\[ i\chi_T(b) = \sum_{j=1}^{\infty} \chi_j(b) = i\chi_1(b) + i\chi_2(b) + i\chi_3(b) + \ldots \]

In this, the first term depends upon the ground-state densities, and the other involve the two-, three- and higher order many-body densities of the two colliding nuclei. We omit all mathematical details and describe some notable
features of this approach. The first two terms of the expansion involve one- and two- body intrinsic densities. For the independent particle model the leading terms which survive in \( i\chi_2(b) \) (let us assume \( A = B = A \) for simplicity ) are proportional to \( A^3 \), whereas \( i\chi_1 \) is proportional to \( A^2 \). Therefore, \( i\chi_2 \) is not necessarily small as compared to \( i\chi_1 \). Other higher order terms which involve three-, four- and many-body densities, are much cumbersome to evaluate. Moreover, at present our knowledge of higher order densities is rather poor.

Working within the framework of the independent particle model, these authors have shown that term-by-term accounting for the c.m. correlation is important for the convergence of the series at least for large impact parameters. The importance of the c.m. correlation in the double scattering term has also been shown explicitly by Varma\(^3\) and Ahmad\(^4\).

One of the major shortcomings of the phase expansion approach is that the series contains infinite number of terms.

3.3 ALKHAZOV’S EXPANSION

Alkhazov\(^5\) has proposed another approach for evaluating \( S_{BA}(b) \) which is a generalization of his correlation,
expansion formalism for N-nucleus scattering. In this approach $S_{BA}(b)$ is written as

$$S_{BA}(b) = (1 - G_o(b))^{AB}$$

3.3.1

where $G_o$ assumes the form

$$G_o(b) = \sum_{N>1} G_o^{(N)}(b)$$

3.3.2

Here again the quantities $G_o^{(1)}$, $G_o^{(2)}$, ... which depend upon the two-body, three-body, .... densities have similar structure and hence same kind of shortcomings as is the case with the phase expansion approach discussed in the previous section. Thus, one encounters an infinite series with rather slow convergence which makes it difficult to provide a direct interpretation to the expansion in terms of the multiple scattering processes.

3.4 THE RIGID PROJECTILE MODEL

The rigid projectile model (RPM) which treats the composite projectile as a 'rigid' single entity has been used in the low energy domain from the early days. However, at medium and high energies it was first used by Tekou to describe d-d scattering. The most attractive feature of RPM is that it reduces the nucleus-nucleus scattering problem to
N-nucleus scattering one with the difference that the elementary NN amplitude in the later is replaced by the projectile-N amplitude in the former.

It is instructive to see how the RPM can be obtained from the Glauber model. For this, we take the S matrix operator, \( \hat{S}(b) \), as the starting point and group together the factors in the operator which contains the same target nucleon coordinates \( \theta \):

\[
\hat{S}(b) = \prod_{i=1}^{A} Z_1
\]

where

\[
Z_1 = \frac{B}{\sum_{j=1}^{B} [1 - \Gamma_{NN}(\overline{p-S_1+S_j})]}
\]

so that application of the closure over the projectile states between each group gives:

\[
S_{BA}(b) = (\Psi_A \Psi_B | \prod_{i=1}^{A} Z_1 | \Psi_B \Psi_A)
\]

\[
= \sum_{m_1} \cdots \sum_{m_{A-1}} (\Psi_A [ (\Psi_B | Z_1 \Psi_{m_1} ) (\Psi_{m_1} | Z_2 \Psi_{m_2} ) \cdots (\Psi_{m_{A-1}} | Z_A \Psi_B ) ] \Psi_A).
\]

Thus, if all but the ground states of the projectile is neglected, one obtains the following expression for \( S_{BA}(b) \)

\[
S_{BA}(b) \approx (\Psi_A | \prod_{i=1}^{A} (\Psi_B | Z_1 \Psi_B ) | \Psi_A).
\]

Next, it is noted that the expression for the elastic
scattering of a nucleon on a nucleus $B$ reads as

$$f_{NB}(q) = \frac{ik_N}{2\pi} \int d^2b \ e^{iq\cdot b} \ [1 - \psi_B^* \prod_{j=1}^2 [1 - \Gamma_{NN}(b-s_j)] \psi_B], \quad 3.4.5$$

so that

$$\langle \psi_A | Z_i | \psi_B \rangle = 1 - \Gamma_{NB}(b-s_i). \quad 3.4.6$$

Thus

$$S_{BA}(b) = \langle \psi_A^* \prod_{i=1}^A [1 - \Gamma_{NB}(b-s_i)] \psi_A \rangle \quad 3.4.7$$

with

$$\Gamma_{NB}(b) = \frac{1}{2\pi ik_N} \int d^2q \ e^{iq\cdot b} f_{NB}(q). \quad 3.4.8$$

It may be recognized that the quantity $\Gamma_{NB}(b)$ is just the profile function for the elastic $N$-projectile scattering at the incident nucleon kinetic energy $E_N = E/B$ ($E$ is the kinetic energy of the incident projectile).

Now, particularising to $\alpha$-nucleus scattering we note that $S_{\alpha A}(b)$ may be expanded in a correlation series as in the case of $N$-nucleus elastic scattering:

$$S_{\alpha A}(b) = [1 - \Gamma_{\alpha}(b)]^A + \frac{A(A-1)}{2} [1 - \Gamma_{\alpha}(b)]^{A-2} \times$$

$$\int C(c_1, c_2) \ \Gamma_{\alpha}(b-s_1) \ \Gamma_{\alpha}(b-s_2) \ dc_1 \ dc_2 + \ldots$$

where

$$\Gamma_{\alpha}(b) = \int \rho_{\alpha}(c) \ \Gamma_{\alpha}(b-s_1) \ dc, \quad 3.4.9$$

and $C(c_1, c_2)$ is the ground-state pair correlation function of
the target:

\[ C(r_1, r_2) = \rho(2)(r_1, r_2) - \rho(r_1) \rho(r_2) \]  \hspace{1cm} (3.4.11)

Further, eq. 3.4.10 may be written in terms of the NoC scattering amplitude \( f_{N^A}(q) \) using the relation (3.4.8)

\[ P_\alpha(b) = \frac{1}{K_N} \int dq \left[ q J_0(qb) F_A(q) f_{N^A}(q) \right] \]  \hspace{1cm} (3.4.12)

where \( F_A(q) \) is the form factor of the target nucleus:

\[ F_A(q) = \int e^{-i q \cdot r} \rho_A(r) \, dr \]  \hspace{1cm} (3.4.13)

On the basis of the estimate by Ahmad and Auger\(^8\) regarding the contribution of the higher order terms in medium energy p-nucleus scattering, it may be said that the series 3.4.9 converges rather rapidly and that the contribution of the pair correlation term for medium and heavy weight nuclei is rather small. One expects the same for \( \alpha^- \) nucleus, scattering.

The RPM is found to give much better description\(^9\) of the nucleus-nucleus elastic scattering data than the optical limit description of Czyz and Maximon. It has been applied to analyze the elastic and inelastic scattering data of 1.37 GeV \( \alpha \) particles on nuclei with fair degree of success though it suffers from the following weaknesses. First, the excitations of the projectile between successive collisions with the
target nucleons are neglected, and second the treatment of the projectile in the target is rather unsymmetrical.

3.5 THE SWARM PROJECTILE MODEL

The Swarm Projectile Model (SPM) is an extension of the method used by Fälldt and Pilkuhn in which the projectile is treated as a swarm of nucleons moving with equal velocities and distributed according to the projectile wavefunctions. With the help of Eqs. 2.6.2 and 2.6.4 we have:

\[ S_{\alpha A}(b) = \int \prod_{i=1}^{A} \prod_{j=1}^{B} dr_i dr_j \rho_A(r_i) \rho_{\alpha}(r_j) [1 - \rho_{NN}(b-s_i+s_j)] \]  

3.5.1

which can be written in the standard target thickness approximation

\[ S_{\alpha A}(b) = \exp\left(\sum_{n=1}^{A} t_n\right) \int \prod_{j=1}^{B} \rho_{\alpha}(r_j) dr_j \]  

3.5.2

The term

\[ t_1 = -2\pi \gamma \sum_{l=1}^{A} T(b+s_l) \]  

3.5.3

do not involve the mutual shadowing of the nucleons of the \( \alpha \) particle while the terms \( t_2, t_3, \) and \( t_4 \) correct for the mutual shadowing of two, three, and four nucleons:

\[ t_2 = \pi \gamma^2 \sum_{1 < m} T\left(b + (s_1 + s_m) / 2\right) \exp\left(-\left(s_1 - s_m\right)^2 / 4a\right) \]  

3.5.4

\[ t_3 = -\frac{2}{3} \pi \gamma^3 \sum_{1 < m < n} T\left(b + (s_1 + s_m + s_n) / 3\right) \times \]
\[ x \exp[-((s_1-s_m)^2 +(s_1-s_n)^2 +(s_n-s_m)^2)/6a] \]  \hspace{1cm} 3.5.5

and

\[ t_4=(1/2)\pi a^4 T(b) \exp(-\sum_{i=1}^{4} s_i^2/2a) \]  \hspace{1cm} 3.5.6

where

\[ T(b) = A \int dz \rho_A(b,z) \]  \hspace{1cm} 3.5.7

\( S_{\alpha A} \) is expanded in such a way that the leading term involves only \( t_1, t_2, t_3 \) and \( t_4 \) appear in successively smaller terms. This is achieved by using the identity:

\[ \exp\left(\sum_{n=1}^{4} t_n\right) = e^{t_1} \left(1 + \sum_{i=2}^{4} t_i^{i-1} + \sum_{1<i<j} (e^{t_i-1})(e^{t_j-1}) + \prod_{i=2}^{4} (e^{t_i-1})\right) . \]  \hspace{1cm} 3.5.8

Using this expression Eq. 3.5.2 reduces to a finite expansion of the form

\[ S_{\alpha A} = \sum_{i=1}^{8} S^{(1)}_{\alpha A}(b) \]  \hspace{1cm} 3.5.9

where first dominant term is

\[ S^{(1)}_{\alpha A}(b) = \int e^{t_1} \prod_{j=1}^{4} \rho_{\alpha}(r_j) d^3r_j \]

\[ = \left[ \frac{2sds}{R^2} e^{-(s-b)^2/R^2} e^{-2a T(s)} I_0\left(\frac{2sb}{R^2}\right)\right]^4 \]  \hspace{1cm} 3.5.10

where \( I_0 \) is related to the modified bessel function.
The authors have also extended this formalism to the inelastic $\alpha$-nucleus scattering. In the analyses of $\alpha$-particle scattering on $^{12}\text{C}$ and Ca isotopes, the SPM gives fairly better results but deviates from the data at large momentum transfers.

Fälldt and Hulthage\textsuperscript{10} have claimed the relevance of their model over all other existing models by then such as RPM\textsuperscript{9,12-14}, folding model(optical model of Czyz and Maximon\textsuperscript{1}) and method of Franco and Varma\textsuperscript{2}.

3.6 THE EFFECTIVE PROFILE APPROACH

To have a symmetrical as well as workable description of nucleus-nucleus scattering it is useful to adopt the effective profile approach as developed by Ahmad\textsuperscript{4} a few years ago. The approach is a direct generalization of the effective profile approach for N-nucleus scattering discussed in ref.8.

The basic idea of the effective profile approach is to expand the $S$-matrix operator $\hat{S}$ in terms of the effective profile $\gamma_{ij}$ defined as

$$\gamma_{ij} = \Gamma_{00}(b) - \Gamma_{NN}(b-s_i+s_j'), \quad 3.6.1$$

where

$$\Gamma_{00}(b) = \int \rho_A(r) \rho_B(r') \Gamma_{NN}(b-s_i+s_j') \, dr \, dr', \quad 3.6.2$$
with $\rho_A(r)$ and $\rho_B(r)$ as the intrinsic one-body densities of the target and the projectile respectively. These densities differ from the model densities used by the Czyz and Maximon in that they can be identified with the empirical densities as measured in hadron-nucleus scattering.

The effective profile expansion is obtained by writing the $S$-matrix operator in the form:

$$S(b) = \prod_{i=1}^{A} \prod_{j=1}^{B} \left[ 1 - \Gamma_{ij}(b) + \gamma_{ij} \right], \quad 3.6.3$$

and expanding the double product as

$$S(b) = S_{\omega \omega}(b) + \sum_{l=1}^{AB} S_1(b) \quad 3.6.4$$

$$S_{\omega \omega}(b) = (1 - \Gamma_{\omega \omega}(b))^{AB} \quad 3.6.5$$

$$...$$

$$S_1(b) = \frac{1}{1!} (1 - \Gamma_{\omega \omega}(b))^{AB-1} \sum_{i_1 \neq j_1}^{'} \sum_{i_2 \neq j_2}^{'} \gamma_{i_1 j_1} \gamma_{i_2 j_2} \quad 3.6.6$$

The primes on the summation sign indicate that the two pairs of indices cannot be equal at the same time (i.e., if $i_1 = i_2$ then $j_1 \neq j_2$ and vice versa).

The above expansion leads to the following expression for the scattering amplitude:

$$F_{BA}(q) = F_{\omega \omega}(q) + \sum_{l=2}^{AB} F_1(q) \quad 3.6.7$$
where

\[
F_{oo}(q) = \frac{iK}{2\pi} \int d^2 b e^{i\mathbf{q} \cdot \mathbf{b}} \langle \Psi_A \Psi_B | 1 - \Gamma_{oo}(b) | \Psi_B \Psi_A \rangle.
\]

and

\[
F_1(q) = \frac{iK}{2\pi} \int d^2 b e^{i\mathbf{q} \cdot \mathbf{b}} \langle \Psi_A \Psi_B | \hat{S}_1(b) | \Psi_B \Psi_A \rangle.
\]

Eq. 3.6.8 may be simplified as in the case of nucleon-nucleus scattering. Evaluation of eq. 3.6.9 still presents formidable problem. So far it has been treated only for \( l=2 \), in which case it may be expressed as

\[
\langle \Psi_A \Psi_B | \hat{S}_1(b) | \Psi_B \Psi_A \rangle = \left( \frac{1}{2\pi i k_N} \right)^2 [1 - \Gamma_{oo}(b)]^{AB} \]

\[
\times \frac{AB}{2} [ (A-1)(B-1)(G_{22} - G_{oo}) + (B-1)(G_{12} - G_{oo}) + (A-1)(G_{21} - G_{oo}) ],
\]

where \( G_{22}, G_{12}, G_{21} \) and \( G_{oo} \) are four dimensional integrals which depend upon the one- and two-body form factors of the colliding nuclei.  

For the intrinsic two-body form factors, the evaluation of integrals presents a computationally difficult problem if realistic nuclear form factors are to be used. Therefore, required nucleon form factors are parametrised as a sum of Gaussians:

\[
F(q) = \sum_j a_j \exp(-b_j q^2),
\]

where \( a_j \) and \( b_j \) are parameters.
Calculations have been performed using Eqs. 3.6.5 and 3.6.10 for the scattering of $\alpha$ projectiles at 1.37 GeV/c on $^{40}$Ca and $^{12}$C targets in ref.4 where the above formulation is described in greater detail. It is found that consideration of eq. 3.6.10 in the calculation improves the theoretical situation greatly (see, Figs. 3.1 and 3.2). Therefore, it may be safely concluded that the effective profile approach not only avoids the divergence present in the optical limit approach but it also provides a rather converging expansion for the Glauber model $S$-matrix for nucleus-nucleus scattering. Relatively better results given by the effective profile expansion demand that the approach should be studied in greater detail by considering the next few higher order terms and by applying it to inelastic scattering as well.

This approach seems to be promising but it has not been thoroughly investigated, as yet. In the next sections, we will discuss $\alpha$-nucleus scattering as a very special.

3.7 THE $\alpha$-NUCLEUS SCATTERING

Almost all the approaches for evaluating nucleus-nucleus $S$-matrix element discussed so far, have been applied to analyze $\alpha$-nucleus scattering at intermediate energies. The main point of attraction in the application of $\alpha$ particle as a nuclear projectile for investigating nuclear
structure is its simple spin and i-spin structure. That is why a great deal of interest was shown soon after appearence of the Scalay nucleus scattering data on $^{12}\text{C}$ and Ca isotopes at 1.37 GeV. A number of theoretical analyses of the data were made notably by Saclay group itself\textsuperscript{13}. Alkhazov et al\textsuperscript{6}, Ahmad\textsuperscript{9} and Faldt and Hulthage\textsuperscript{10}. They all agree on a point that the optical limit approximation of the multiple scattering theory as proposed by Czyz and Maximon\textsuperscript{1} fails seriously to account for the data even at low momentum transfers.

The scattering of the projectile by a target nucleon can however be regarded as a measurement of the instantaneous positions of the nucleons in the projectile, so that in repeated interactions the target nucleons do not encounter a ground-state projectile with randomly distributed constituents. Indeed, since the internal nuclear motion is negligible at high energies, the instantaneous state of the projectile changes very little in successive collisions. Hence, there exist a correlation between successive scatterings that is neglected in the RPM. Therefore, attempts have also been made towards a complete microscopic interpretation of the data by going beyond the optical limit approximation of the multiple scattering model.

The phase expansion method of the Franco and Varma\textsuperscript{2} and the S-matrix expansion formalism of Ahmad\textsuperscript{4} in which higher
order correlations are included, seem to have been motivated from this consideration. These methods have been applied in a variety of situations. The results in addition to providing physical insight, strengthen confidence in the applicability of the Glauber model.

In the effective profile approach, the first and leading term of the expansion correspond to the so called optical limit calculations while the other terms represent respectively, the double, triple and higher order multiple scatterings between nucleons of the two colliding nuclei which involve virtual excitations only. The analyses of the $\alpha^{-12}$C and $\alpha^{-40}$Ca elastic scatterings at 1.37 GeV, retaining up to the two-body density term of the expansion, show that the inclusion of the two-body density term fairly improves the calculation.

3.8 THE SEMIPHENOMENOLOGICAL APPROACH

Since a realistic calculation of the higher order terms of the expansion into the calculation is not only mathematically complex but is also beset with some other limitations and ambiguities. In view of this, Ahmad and Alvi have proposed a semiphenomenological method for analyzing the $\alpha$-nucleus scattering data which avoids the complexities of the problem and is very successful at
In the semiphenomenological method of analysis for intermediate energy nucleus elastic scattering experiments, the RPM S matrix series is written in the following form

\[
S_{\alpha A}(b) = \left[1 - \frac{1}{i k} \int dq \, q \, J_0(qb) \, f_{\text{eff}}(q) \, F_\alpha(q) \right] \hat{A} + \ldots 3.8.1
\]

where \( F_\alpha(q) \) is the form factor of the target nucleus. An effective \( N \alpha \) amplitude, \( f_{\text{eff}}(q) \), is invoked such that the actual Glauber model S matrix element \( S_{\alpha A}(b) \) can be approximated as

\[
S_{\alpha A}(b) = \left[1 - \frac{1}{i k} \int dq \, q \, J_0(qb) \, f_{\text{eff}}(q) \, F_\alpha(q) \right] \hat{A} 3.8.2
\]

Further, effective \( N \alpha \) amplitude is parametrized as

\[
f_{\text{eff}}(q) = \frac{i k \sigma_\alpha (1 - i f_\alpha)}{4 \pi} \exp(-\beta_\alpha q^2/2) \left(1 + \xi(q)\right) 3.8.3
\]

where \( \sigma_\alpha \) is the \( N \alpha \) total cross section, \( f_\alpha \) is the ratio of real to the imaginary parts of the forward scattering amplitude, \( \beta_\alpha \) is the slope parameter. The following ansatz for \( \xi(q) \) is found to be successful

\[
\xi(q) = \lambda_R q^4 3.8.4
\]

which ensures that \( f_{\text{eff}}(q) \) reduces to \( f_{N \alpha}(q) \) at smaller \( q \) values; \( \lambda_R \) is a real parameter.

Using the ground state density for the target as
obtained from intermediate energy proton scattering experiments, expression 3.8.2 and $f_{\text{eff}}(q)$ as given by eq. 3.8.3 with the above proposed ansatz for $\xi(q)$. The data on Ca isotopes can be fitted by varying the parameter $\lambda_R$ (ref.15). The maxima and minima that are not accounted for by the RPM calculations, are now fairly reproduced (Fig. 3.3). Similarly, $^{12}$C elastic scattering data at 1.37 GeV are fitted reasonably well (Fig. 3.4). Although the agreement with the data in this case is not as good as for the Ca isotopes, yet it is very satisfying to find that the height of the maxima and positions of the minima are predicted nicely with the same value for $\lambda_R$ as for Ca. Thus subject to certain limitations, the method of analysis seems potentially good for studying the ground state densities of target nuclei.

3.9 THE FULL SERIES CALCULATION

Y. Yin et al.\textsuperscript{17} have proposed a simple method for evaluating complete expansion of the Glauber amplitude which can be handled without much difficulty for the multiple gaussian-type density functions. Moreover this approach suffers from the weakness that except for the simple Gaussian model the c.m. correlation is accounted for only approximately.
They authors expand the nuclear S-matrix element as:

\[
\langle \psi_A\psi_B\|1-\prod_{i=1}^{A}\prod_{j=1}^{B}[1-\Gamma_{ij}(b-g_{i1}+g'_{ij})]\|\psi_A\psi_B\rangle
\]

\[
=\langle \psi_A\psi_B\|\Gamma_{11}+\Gamma_{12}+\ldots+\Gamma_{AB}\rangle - (\Gamma_{11}\Gamma_{12}+\ldots) + (3-\Gamma \text{ terms})
\]

\[\ldots\ldots+ (AB-\Gamma \text{ term})\langle \psi_A\psi_B\rangle \]

3.9.1

which may be written in the following form:

\[-\sum_\nabla (-1)^\nabla \langle \psi_A\psi_B\|\prod_{i=1}^{A}\prod_{j=1}^{B}(\Gamma_{ij})^{\nabla_{ij}}\|\psi_A\psi_B\rangle,\]

where

\[\nabla_{ij} = \begin{cases} 1 & \text{when } \Gamma_{ij} \text{ appears in the expansion term} \\ 0 & \text{is absent from the expansion term} \end{cases} \]

3.9.3

The constant

\[\nabla = \sum_\nabla \sum_{ij}^{AB} \nabla_{ij}\]

3.9.4

is called the rank of summation and \(\sum_\nabla\) runs through all \(2^{A\times B}\) possible \(\nabla(\nabla_{ij})\). Thus the complete expansion 3.9.1 is lengthy. Of course, not all terms are different because all these terms in eq. 3.9.1 whose corresponding \(\nabla\) matrices can be transformed into each other by doing some row or column permutations, give the same contribution to the Glauber theory. These terms can be grouped into orbits, and the rank of the term is called the length of the corresponding orbit.
Details of the calculations particularly the integrations of the general terms which appear in a recurrence relation is given in ref. 17.

They used the double Gaussian density function\textsuperscript{18} for α-particle and the usual Gaussian parametrisation of the NN scattering amplitude while calculating the $^4\text{He}-^4\text{He}$ elastic scattering at 1.98 GeV. The parameters in the density functions are determined by requiring that the corresponding charge distribution yield a good fit to the electromagnetic form factor measured by electron elastic scattering\textsuperscript{18}.

The Fig. 3.9 shows the convergence of the expansion terms. It is observed that the series converges up to 10 terms and other neglected terms are of little importance.

This completes our discussion on the various methods that have been used so far to calculate nucleus-nucleus elastic scattering within the framework of the Glauber model.
References

16 M. A. Alvi, Ph. D. Thesis, Aligarh Muslim University, Aligarh, UP, India.
Fig. 3.1 Elastic scattering of 1.37 GeV $\alpha$ particles on $^{40}\text{Ca}$. Dotted curve, optical limit calculation; solid curve, calculation with the two-body density term but without short range calculation; chain curve, same as the solid curve but including the short range correlation.
Fig. 3.2  Elastic scattering of 1.37 GeV $\alpha$ particles on $^{12}_C$. The description of the curves is the same as in Fig. 3.1.
Fig. 3.3  Elastic scattering of 1.37 GeV α particles on $^{42,44,48}$Ca. Solid curve shows the parameter free calculation using the effective $N\alpha$ amplitude corresponding to the $\sigma_\alpha=107$ mb, $\beta_\alpha=0.33$, $\beta'=31.02$ (GeV/c)^2, and $\lambda_R=102.3$ (GeV/c)^{-2}. 
Fig. 3.4 Elastic scattering of 1.37 GeV $\alpha$ particles on $^{12}\text{C}$. Solis curve, the parameter free calculation with the effective $N\alpha$ amplitude corresponding to the parameters $\sigma_\alpha = 107 \text{ mb}$, $f_\alpha = 0.3$, $\beta_\alpha = 27 \text{ (GeV/c)}$ and $\lambda_\alpha = 70.5 \text{ (GeV/c)}$ a. Dashed curve, the RPM calculation of Ahmad b.
Fig. 3.5 Convergence of the expansion. The solid curve correspond to the full series calculation. Crossed, chain, dotted and dashed curves correspond to six, eight, ten and twelve terms respectively.