Chapter 2

EIKONAL DESCRIPTION OF HIGH ENERGY NUCLEAR SCATTERING
2.1 INTRODUCTION

Generally speaking, quantum mechanical equations of motion pertaining to complex systems are prohibitively difficult to solve exactly. Therefore, approximation methods have been developed to tackle such problems and obtain approximate solutions to a desired degree of accuracy depending upon the nature of the problem. The generally used approximate methods are: perturbation method, variational method, and semi-classical method. The perturbation method is more appropriate when the perturbing potential is small while the variational method works better in situations when there is enough insight of the problem to enable to choose a good trial wavefunction. The semi-classical method which forms the basis of the present study is used in situations involving large quantum numbers or when the de Broglie wavelength associated with the relative motion is much smaller than the characteristic dimensions of the system. More precisely semi-classical methods are used to obtain expressions for wavefunctions and other quantities of interest which are correct in the limiting case when the Planck's constant is small in comparison with the action functions occurring in the corresponding classical problems. It differs from the perturbation and variational methods in
that the perturbation method produce a series expansion for the quantities of interest in powers of a variable which specifies the departure of the given problem from an exactly solvable case, while the variational methods produce a best estimate out of a given class of trial solutions.

The semi-classical method to solve quantum scattering problems, assumes a rather simple form in the domain of medium and high energies where it may be safely assumed that the projectile follows a straight line trajectory within the interaction region. This straight line or the eikonal approach has been extensively studied and developed for medium and high energy nuclear scattering by Glauber\(^\text{1}\) and others\(^{2-15}\) and is known in the literature as the Glauber model.

In this chapter, we will mainly describe the Glauber model\(^\text{1}\). The emphasis will be on development of the model rather than on describing its extensive applications to interpret experimental data. A convenient formulation for describing the high energy potential scattering will be given first. The result so obtained will next be used to develop microscopic description of the elastic and inelastic hadron-nucleus scattering as well as nucleus-nucleus scattering. This will give a necessary background for the work presented in this thesis.
2.2 POTENTIAL SCATTERING IN EIKONAL APPROXIMATION

The problem of the scattering of a high energy particle of mass $m$ and momentum $p(=\hbar\kappa)$ from the interaction potential $V(r)$ which for simplicity is assumed to be spin independent, consists of solving the Schrödinger equation:

$$ (\nabla^2 + k^2)\psi_k(r) = \frac{2m}{\hbar^2} V(r) \psi_k(r) $$

subject to the boundary condition that at large distances from the interaction region the wavefunction $\psi_k(r)$ has the asymptotic form:

$$ \psi_k(r) \rightarrow e^{ik\cdot r} + f(\theta) \frac{e^{ikr}}{r} $$

where $f(\theta)$ is the scattering amplitude which gives the differential scattering cross section through the relation:

$$ \frac{d\sigma}{d\Omega} = |f(\theta)|^2 $$

To obtain an expression for the scattering state wavefunction $\psi_k(r)$ we introduce the free particle Green's function $G_0$ appropriate to the outgoing boundary condition:

$$ G_0(\xi, \xi') = \frac{-m}{2\pi\hbar^2} \frac{\exp[ik|\xi-\xi'|]}{15-\xi'-1} $$
which satisfies the equation

\[(\nabla^2 + k^2) G_0(r, r') = \frac{2m}{\hbar^2} \delta(r-r'). \]  \hspace{2cm} 2.2.3

Now it is easy to see that

\[\Psi_k(r) = e^{i\mathbf{k} \cdot \mathbf{r}} + \int G_0(r, r') V(r') \Psi_k(r') \, dr'. \]  \hspace{2cm} 2.2.6

which leads to the following expression for the elastic scattering amplitude for a particle of initial momentum \(\hbar \mathbf{k}\) and final momentum \(\hbar \mathbf{k}'\) [\(f(\mathbf{k})=f(\mathbf{k}, \mathbf{k}')\)]:

\[f(\mathbf{k}, \mathbf{k}') = \frac{-m}{2\pi \hbar^2} \int e^{-i\mathbf{k} \cdot \mathbf{r}} V(r) \Psi_k^*(r) \, dr. \]  \hspace{2cm} 2.2.7

Upto this point no approximation has been made and hence eq. 2.2.7 gives an exact expression for the scattering amplitude.

To evaluate the scattering amplitude from eq. 2.2.7, one needs \(\Psi_k^*\). This amounts to solving the Schrodinger equation with the appropriate boundary condition. Since, in general, it is a difficult proposition, physical considerations are invoked to obtain an approximate expression for \(\Psi_k^*(r)\) to be substituted in eq. 2.2.7 for evaluating the scattering amplitude.

In the domain of high energy scattering, if the kinetic energy of incident particle greatly exceeds the interaction
potential $V(r)$ and the associated wavelength $\lambda(=k^{-1})$ is much smaller than the typical variation length 'a' of the interaction:

$$|V|/E \ll 1 ; \quad ka \gg 1 \quad , \quad 2.2.8$$

it is reasonable to assume that the incident particle follows almost a straight line trajectory along the incident direction even in the interaction region (Fig. 2.1). In other words if the energy of the incident particle is sufficiently high, it is a good approximation to assume that the
wavefunction $\psi_k^*(r)$ is of the form:

$$\psi_k^*(r) = e^{ik \cdot r} \rho(r), \quad 2.2.9$$

where $\rho(r)$ is a slowly varying function which satisfies the boundary condition (This assumes that $k$ is along positive $z$-axis):

$$\rho(r) \rightarrow 1 \quad \text{as} \quad z \rightarrow -\infty \quad 2.2.10$$

Substituting eq. 2.2.9 in the wave equation 2.2.1 and neglecting $\nabla^2$ one obtains

$$\frac{\partial \rho}{\partial z} = \frac{-i \hbar \rho}{2m} V(r) \rho(r). \quad 2.2.11$$

This gives

$$\rho(x,y,z) = \exp(-i \frac{\hbar}{2m} \int_{-\infty}^{z} V(x,y,z') dz'). \quad 2.2.12$$

so that ($\nu = \hbar k/m$)

$$\psi_k^*(r) = e^{ik \cdot r - i \frac{\hbar}{2m} \int_{-\infty}^{z} V(x,y,z') dz'}. \quad 2.2.13$$

Next substituting the above wavefunction into the expression 2.2.7 for the scattering amplitude, one gets:

$$f(k,k') = \frac{-m}{2\hbar^2} e^{i k' \cdot r} V(r) e^{-i k \cdot \int_{-\infty}^{z} V(b + k^z') dz'} dzd^2b, \quad 2.2.14$$

where $r = b + \hat{k}$ and $dr = d^2bdz$ (see Fig. 2.1). The symbol $\hat{k}$ denotes
the unit vector along $k$. The above equation may be put in the form

$$f(k, k') = \frac{-m}{2 \pi \hbar^2} \int e^{i(k-k').(b+kz)} V(b+kz)$$

$$_x e^{-\frac{i}{\hbar \nu} \int_{-\infty}^{z} V(b+kz') dz'} dz d^2b . \quad 2.2.15$$

Now, energy conservation requires $|k'| = |k|$. Thus, the error in equating $\exp[i(k-k').kz]$ to unity is not large. Therefore, the scattering amplitude as given by eq. 2.2.15 may be put in the following simple form:

$$f(k, k') = \frac{i k}{2 \pi} e^{i(k-k').b} [1 - e^{-\frac{-i}{\hbar \nu} \int_{-\infty}^{z} V(b+kz) dz}] d^2b . \quad 2.2.15$$

For small scattering angles the vector $k-k'$ is approximately perpendicular to $k$ and hence, introducing the momentum transfer

$$q = k-k' \quad 2.2.17$$

Fig. 2.2
and writing
\[ \chi(b) = -\frac{1}{4\pi} \int_0^a d^2b \ V(b+\hat{k}z) \ dz, \]
we may write eq. 2.2.16 as
\[ f(q) = \frac{ik}{2\pi} \int d^2b \ e^{iq.b} [1 - e^{-ib}] \]
This is the basic result of high energy potential scattering. For potentials with azimuthal symmetry we may go one step further by noting that
\[ \frac{1}{2\pi} \int_{\phi}^{2\pi} e^{-i\lambda \cos(\phi)} d\phi = J_0(\lambda), \]
where \( J_0(\lambda) \) is the zeroth order Bessel function. This gives
\[ f(q) = ik \int_0^a db \ b \ J_0(qb) [1 - e^{-ib}], \]
where
\[ q = 2k \sin(\theta/2) \]
The straight line motion of the projectile within the potential range limits the validity of the theory for small angles. This introduces a kind of asymmetry between the incident momentum \( k \) and the final momentum \( k' \). A better description may be obtained by assuming that, in the interaction region, the projectile moves in the direction of the average momentum \( \bar{k} \) defined by (Fig. 2.3)
Now, the momentum operator $\hat{p}$ in the wave equation is expanded along $K$ and approximated as ($\hbar=1$):

$$(p)^2 = 2 \cdot K \cdot \hat{p} - k^2.$$  \hspace{1cm} \text{(2.2.24)}$$

Substituting the above approximation for $p^2$ in the wave equation, taking the z axis along $K$ and proceeding as before, one obtains the same expression for $f(q)$ as given by eq. 2.2.19 with the understanding that now the phase function $\chi(q)$ as given by eq. 2.2.18 is to be evaluated by performing integration along $K$. In this approach one need not invoke the approximation

$$e^{i(k-k') \cdot \hat{k} z} \approx 1.$$
2.3 GLAUBER MODEL FOR NUCLEON-NUCLEUS SCATTERING

We are now fairly equipped to consider the problem of the scattering of a high energy projectile on a system of bound particles. We particularize the discussion by considering the scattering of a nucleon described by the coordinate \( \mathbf{r} \) on a target nucleus of mass number \( A \) governed by the Hamiltonian \( H_0 \). We denote the target nucleon coordinates by \( \mathbf{r}_i (i=1,2,...,A) \) and the target states by \( \Phi_i \). For simplicity we neglect the spin and isospin coordinates. The target states \( \Phi_i \) satisfy the equation

\[
H_0 \Phi_i = \varepsilon_i \Phi_i ,
\]

where \( \varepsilon_i \) are the energy eigen values.

The time dependent equation for the present problem may be written as

\[
\left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_A) + H_0 \right) \Psi = i\hbar \frac{\partial \Psi}{\partial t} ,
\]

where

\[
V = \sum_{i=1}^{A} V_{\text{NN}}(\mathbf{r} - \mathbf{r}_i)
\]

with \( V_{\text{NN}} \) as the NN interactional potential.

Next, we introduce the state \( \Psi \) through the transformation:
\[ \psi = e^{-\frac{iH_0 t}{\hbar}} \psi \]  \hspace{1cm} 2.3.4

to obtain

\[ \left[ -\frac{\hbar^2}{2m} \nabla^2 + e \frac{\hbar}{\hbar} V\left(\xi_1, \xi_2, \ldots, \xi_n\right)e^{-\frac{iH_0 t}{\hbar}} \right] \psi = i\hbar \frac{\partial \psi}{\partial t} \]  \hspace{1cm} 2.3.5

or

\[ \left[ -\frac{\hbar^2}{2m} \nabla^2 + V\left(\xi_1, \hat{r}_i(t)\right) \right] \psi = i\hbar \frac{\partial \psi}{\partial t} \]  \hspace{1cm} 2.3.6

where \( \hat{r}_i(t) \) are the coordinate operators in the Heisenberg picture:

\[ \hat{r}_i(t) = e^{-\frac{iH_0 t}{\hbar}} \xi_i e^{\frac{-iH_0 t}{\hbar}}. \]  \hspace{1cm} 2.3.7

Now, we write eq. 2.3.6 as

\[ \left[ -\frac{\hbar^2}{2m} \nabla^2 + i\hbar \frac{\partial}{\partial t} \right] \psi = V\left(\xi_1, \hat{r}_i(t)\right) \psi \]  \hspace{1cm} 2.3.8

and assume that the desired solution is of the form

\[ \psi(\xi, t) = e^{\frac{i(k \cdot \xi - t)}{\hbar}} \rho(\xi, t) \Phi_0 \]  \hspace{1cm} 2.3.9

where \( \Phi_0 \) is the ground state wavefunction of the target, \( E = \hbar \omega \), and \( \hbar k \) is the momentum transfer corresponding to the relative motion. Here \( \rho \) is an operator implicitly dependent on \( \hat{r}_i(t) \). Substituting eq. 2.3.9 in eq. 2.3.8 and making the high energy approximation \( (\nabla^2 \rho = 0 \text{ for all } t) \), we get
\[ \left( \frac{\partial}{\partial z} + \frac{1}{\nu} \frac{\partial}{\partial t} \right) \rho(z,t) = -\frac{i}{\hbar \nu} \nabla [z; \hat{r}_1(t)] \rho(z,t) \] \hspace{1cm} 2.3.10

Using the transformation
\[ t' = t - \frac{z}{\nu}; \hspace{0.5cm} z' = z \] \hspace{1cm} 2.3.11

the above equation may be put in the following form
\[ \frac{\partial \rho}{\partial z}(b+\hat{k}z, t'+z/\nu) = -\frac{i}{\hbar \nu} \nabla [b+\hat{k}z; \hat{r}_1(t'+z/\nu)] \rho(b+\hat{k}z, t'+z/\nu). \] \hspace{1cm} 2.3.12

In general, the potential at different points along the z trajectory do not commute. As a consequence, the integration of eq. 2.3.12 is not so simple as was the case in a similar situation before.

Displaying only the z dependence the solution of eq. 2.3.12 with the boundary condition \( \rho(z) \rightarrow 1 \) as \( z \rightarrow -\infty \) may be expressed as
\[ \rho(z) = 1 - \frac{i}{\hbar \nu} \int_{-\infty}^{z} V(z') dz' + \left( \frac{1}{\hbar \nu} \right) \int_{-\infty}^{z} dz' \int_{-\infty}^{z'} dz'' V(z') V(z'') + \ldots \] \hspace{1cm} 2.3.13

Introducing the z ordering operator \( \mathcal{Z} \) which arranges the product of the z dependent operators in increasing order of z from right to left e.g.
\[ \mathcal{Z} V(z_1) V(z_2) = V(z_1) V(z_2); \hspace{0.5cm} z_1 > z_2 \] \hspace{1cm} 2.3.14.

\[ = V(z_2) V(z_1); \hspace{0.5cm} z_1 < z_2 . \]
to write eq. 2.3.13 in the form

$$\rho(z) = Z e^{\frac{-i}{\hbar} \int_{-\infty}^{z} V(z') \, dz'}.$$  \hspace{1cm} 2.3.15

When the potentials at different points along the z trajectory commute the $Z$ ordering is unnecessary and one gets the usual exponential form for $\rho$.

From eqs. 2.3.12 and 2.3.15 it follows that the modulating function $\rho(r,t)$ may be put in the form

$$\rho(r,t) = \frac{-i}{\hbar \nu} \int_{-\infty}^{z'} V[(b+\hbar kz')\hat{r}_1(t-z'\nu)] \, dz'.$$  \hspace{1cm} 2.3.16

Now, we are in a position to evaluate the probability amplitude for the target, making transition from the ground state $\hat{\phi}_0$ to some excited state $\hat{\phi}_f$ when the projectile is at $r$ at time $t$. Using eq. 2.3.9 the transition amplitude may be written as

$$(\hat{\phi}_f, \Psi(r,t)) = e^{\frac{i \nu}{\hbar} (\hat{r}_1(t-z'/\nu) - \hat{r}_1(t-z'/\nu))}.$$  \hspace{1cm} 2.3.17

Since

$$\hat{r}_1(t-z'/\nu) = e^{\frac{i \nu}{\hbar} (t-z'/\nu)} \hat{r}_1(z'/\nu) e^{-\frac{i \nu}{\hbar} (t-z'/\nu)},$$

we may write

$$(\hat{\phi}_f, \Psi(r,t)) = \exp\left(i\left[\left(k - \frac{\xi_f - \xi_0}{\hbar \nu}\right) z - \left(\xi_f - \xi_0\right) t\right]\right) \times \left(\hat{\phi}_f, Z e^{\frac{-i}{\hbar \nu} \int_{-\infty}^{z'} V[(b+\hbar kz')\hat{r}_1(z'/\nu)] \, dz'} \hat{\phi}_0\right).$$  \hspace{1cm} 2.3.18
It may be noted that $E_f - E_o$ is the energy transferred to the target nucleus. Hence the corresponding loss of energy of the projectile ($E_f - E_o$) and the loss in the magnitude of the momentum ($E_f - E_o)/v$ for $|E_f - E_o| \ll E$ is consistent with the energy and momentum conservation.

For low lying transitions and large projectile energy, the energy difference ($E_f - E_o$) and the time interval $z'/v$ may be neglected. Under this approximation we have

$$i(k \cdot r - t) \left( \phi_f, \psi(\xi, t) \right) = e^{-i \frac{z'}{v} \int V[(b + k^z') \cdot \hat{r}_1(o)] dz'} \left( \phi_f, \phi_o \right)$$

2.3.19

Further, we note that, in eq. 2.3.19, $\hat{r}_1(o) = r_1$, so that the interaction potential (which is assumed to be spin-isospin independent) at different points along the $z$ trajectory commute. Hence the $z$ ordering is unnecessary and one can omit the $\hat{z}$ operator. This, as will be seen below, simplifies the problem greatly. However, if the interaction potential, $V$, itself contains such spin dependent components as to violate the commutativity along the $z$ trajectory, then the $z$ ordering operator becomes important.

Considering situations when the $z$ ordering operator is unimportant eq. 2.3.19 implies that

$$\psi(\xi, t) = e^{-i \frac{z'}{v} \int V[(b + k^z') \cdot \hat{r}_1] dz'} \phi_o.$$  

2.3.20
Hence the time independent wavefunction under the eikonal approximation may be identified as

$$
\psi_{b+kz}(r) = e^{ik \cdot r - \frac{i}{\hbar \nu} \int_{-\infty}^{z} V[b+kz'; r'] dz'}
$$

2.3.21

Now, the scattering amplitude $F_{fo}(q)$ for transition from the ground state $\Phi_0$ to some excited state $\Phi_f$ is given by

$$
F_{fo}(k, k') = \frac{m}{2\hbar h^2} \int e^{-ik \cdot r} (\Phi_f, V\Phi_0) dr
$$

2.3.22

Substituting eq. 2.3.21 into eq. 2.3.22, we may write

$$
F_{fo}(q) = (\Phi_f, \hat{F}(q) \Phi_0)
$$

2.3.23

where

$$\hat{F}(q) = \frac{m}{2\hbar h^2} \int dz e^{iq \cdot r} V(r, r') e^{\frac{-i}{\hbar \nu} \int_{-\infty}^{z} V(b+kz'; r') dz'}
$$

The above equation may be treated in the same manner as for the potential scattering (sect. 2.2) to obtain

$$
\hat{F}(q) = \frac{ik}{2\pi} d^2b e^{iq \cdot b} \left( 1 - e^{-i\chi_T(b)} \right)
$$

2.3.25

where the total phase function $\chi_T(b)$ is given by

$$\chi_T(b) = \frac{-i}{\hbar \nu} \int_{-\infty}^{z} V(b+kz; r') dz'
$$

2.3.26

Next, substituting eq. 2.3.3 in eq. 2.3.26 and writing

$r = s_1 + k z_1$
where \( \vec{s}_i \) are the projections of \( \vec{r}_i \) in the plane perpendicular to \( \hat{k} \) (Fig. 2.4), we obtain the following expression for the total phase function:

\[
\chi_T(b) = \sum_{i=1}^{A} \chi_{NN}(b-\vec{s}_i) . \tag{2.3.27}
\]

In the above equation

\[
\chi_{NN}(b-\vec{s}_i) = \frac{1}{\hbar\nu} \int_{-\infty}^{\infty} \chi_{NN}(b-\vec{s}_i + \hat{k}z_i) \, dz . \tag{2.3.28}
\]

This is the phase function for the scattering of the projectile nucleon with the target nucleon located at \( \vec{r}_i \) with the impact parameter \( b-\vec{s}_i \) as shown in Fig. 2.4.

\[\text{Fig. 2.4}\]

Thus we find that the total phase function for nucleon-nucleus scattering is the sum of the phase functions for the scattering of the projectile nucleon on target nucleons. This
principle of additivity of phase functions is a direct consequence of the additivity of the two-body interaction and the approximation of neglecting the target dynamics during the passage of the projectile (viz. replacing the Heisenberg coordinate operator \( \hat{r}_1(\mathbf{r}/v) \) by \( \hat{r}_1(0) \) in eq. 2.3.18). The approximation is clearly justified for sufficiently fast moving projectiles.

Finally, substituting eq. 2.3.27 in eq. 2.3.25 and using eq. 2.3.23 we get the following expression for the nucleon-nucleus scattering amplitude:

\[
\mathbf{F}_{N}(q) = \frac{1}{2\pi} \int d^2 b \ e^{i \mathbf{q} \cdot \mathbf{b}} \left( \langle \Phi_f | [1 - e^{i \sum_{1=1}^{A} \chi_{NN}(b-s_1)}] | \Phi_0 \rangle \right). \tag{2.3.29}
\]

### 2.4 MULTIPLE SCATTERING EXPANSION

The essential result of the previous section is that the scattering amplitude for N-nucleus scattering may be obtained by taking the matrix element of the operator \( \hat{F}(q) \) between the target ground state \( \Phi_0 \) and the final state \( \Phi_f \), where (cf. eq. 2.3.25)

\[
\hat{F}(q) = \frac{k_N}{2\pi} \int d^2 b \ e^{i \mathbf{q} \cdot \mathbf{b}} [1 - e^{i \sum_{1=1}^{A} \chi_{NN}(b-s_1)}] . \tag{2.4.1}
\]

The operator \( \hat{F}(q) \) is in fact the scattering amplitude for a fixed configuration of the target nucleus with \( k_N = k \) is the
nucleon momentum.

We now use eq. 2.4.1 to demonstrate that the additivity of the phase function implies a finite multiple scattering series for the scattering amplitude. This series is called the Glauber multiple scattering series\(^\dagger\). For this, we use eq. 2.2.15 for NN scattering which will then give the relation between the NN scattering amplitude \(f_{\text{NN}}(q)\) and the NN phase function \(\varphi_{\text{NN}}(b)\)

\[
f_{\text{NN}}(q) = \frac{ik_N}{2\pi} \int d^2b e^{i\mathbf{q} \cdot \mathbf{b}} \varphi_{\text{NN}}(b),
\]

\[
\varphi_{\text{NN}}(b) = 1 - e^{-i\varphi_{\text{NN}}(b)}.
\]

The function \(\varphi_{\text{NN}}\) is the NN profile function.

It is seen that the \(f_{\text{NN}}(q)/ik_N\) may be interpreted as the two dimensional Fourier transform of the profile function \(\varphi_{\text{NN}}(b)\). Therefore, taking the inverse Fourier transform, we have

\[
\varphi_{\text{NN}}(b) = \frac{1}{2\pi i k_N} \int e^{-i\mathbf{q} \cdot \mathbf{b}} f_{\text{NN}}(q) d^2 q.
\]

Since the profile function \(\varphi_{\text{NN}}(b)\) through eq.2.4.4 is a directly measurable quantity (except for the overall phase), the Glauber theory seems very useful to analyze medium and high energy hadron scattering experiments. If the NN amplitude is known one can learn about the nuclear
wavefunctions. Alternatively if the target wavefunctions are known one can obtain information on the NN amplitude.

We now use eq. 2.4.3 to write eq. 2.4.1 in the form

\[ \hat{F}(q) = \frac{1}{2}\int d^2b \ e^{-i\frac{q}{2}b} [1 - \hat{S}(b)] , \]

2.4.5

where S-matrix operator, \( \hat{S}(b) \), is given by:

\[ \hat{S}(b) = \prod_{i=1}^{A} [1 - \Gamma_{NN}(b-s_i) \]  

2.4.6

Next, expanding the product in eq. 2.4.6 as

\[ \hat{S}(b) = 1 - \sum_{i=1}^{A} \Gamma_{NN}(b-s_i) + \sum_{i<j} \Gamma_{NN}(b-s_i) \Gamma_{NN}(b-s_j) \]

\[ - \sum_{i<j<k} \Gamma_{NN}(b-s_i) \Gamma_{NN}(b-s_j) \Gamma_{NN}(b-s_k) + \ldots \]  

2.4.7

and substituting the same in eq. 2.4.5 the following expansion may be easily obtained:

\[ \hat{F}(q) = \sum_{i=1}^{A} \hat{F}_{i}(q) , \]

2.4.8

\[ \hat{F}_{i}(q) = \sum_{i=1}^{A} f_{i}(q) e^{i\frac{q}{2} \cdot r_{i}} \]

Next,

\[ \hat{F}_{2}(q) = \frac{-1}{2\pi i k_{F}} \sum_{i<j} \int d^2q_1 d^2q_2 e^{i(q_1 \cdot r_i + q_2 \cdot r_j)} f_{NN}(q_1) f_{NN}(q_2) \]

\[ \times \delta^{2}(q - q_1 - q_2) \]

2.4.9

\[ \hat{F}_{2}(q) = \frac{-1}{2\pi i k_{F}} \sum_{i<j} \int d^2q_1 d^2q_2 e^{i(q_1 \cdot r_i + q_2 \cdot r_j)} f_{NN}(q_1) f_{NN}(q_2) \]

\[ \times \delta^{2}(q - q_1 - q_2) \]

2.4.10

\[ \hat{F}_{2}(q) = \frac{-1}{2\pi i k_{F}} \sum_{i<j} \int d^2q_1 d^2q_2 e^{i(q_1 \cdot r_i + q_2 \cdot r_j)} f_{NN}(q_1) f_{NN}(q_2) \]

\[ \times \delta^{2}(q - q_1 - q_2) \]
The first term $F_1(q)$ in the above expansion gives the contribution of single scattering in which the projectile nucleon scatters with any one target nucleon once and leaves the nucleus, the second term $F_2(q)$ gives the contribution of the double scattering in target before leaving the nucleus, and so on. Only up to A-tuple scatterings are allowed by the model. This is mainly because of the assumed forward propagation of the projectile which precludes rescattering from the target nucleus.

Finally the expansion 2.4.8 may be applied to calculate the elastic and inelastic scattering amplitudes using eq. 2.3.23:

$$F_{10}(q) = \sum_{i=1}^{A} (\hat{\Phi}_f, \hat{F}_i(q) \hat{\Phi}_0). \tag{2.4.9}$$

However, application of eq. 2.4.9 is convenient only when $A$ is small or when the scattering cross section is to be calculated in the cluster model of the target nucleus$^{3,16}$.

2.5 INCLUSION OF COULOMB SCATTERING

So far we have disregarded consideration of Coulomb scattering which is by no means unimportant for charged projectiles like protons. It may be included in the calculation by considering the nucleus as a spherically
symmetric charge distribution and adding the corresponding phase function to the nuclear phase. The result for the elastic scattering amplitude may be expressed as\textsuperscript{17}

\[
F_0(q) = F_{\text{cou}}(q) + i k N \int db b J_0(qb)e^{\frac{1}{2}X_p(b)} \left[ 1 - e^{-X_c(b)} \right] \langle 0 | S(b) | 0 \rangle
\]

with

\[
F_{\text{cou}}(q) = -2\eta k e^{\frac{i\Phi_c}{q^2}}
\]

\[
X_p(b) = 2\eta \ln(kb),
\]

\[
X_c(b) = \beta \Delta \int dt t^2 \rho_\text{ch}(t) \left( \ln \frac{1+(1-b^2/t^2)^{1/2}}{(b/t)} - (1-b^2/t^2)^{1/2} \right)
\]

where \(\eta\) is the fine structure constant, \(\rho_\text{ch}(t)\) is the charge density of the nucleus and

\[
\Phi_c = -2\eta \left( \ln(q/2k) + \gamma \right) + 2 \sum_{r=0}^{\infty} \left[ \frac{\eta}{r+1} - \tan^{-1} \left( \frac{\eta}{r+1} \right) \right].
\]

Here \(\gamma\) is the Euler's constant.

2.6 NUCLEUS-NUCLEUS SCATTERING

The formalism for high energy nucleus-nucleus scattering to be described below is a straightforward generalization of the Glauber multiple scattering theory for hadron-nucleus scattering as described in the previous
Consider the scattering of a projectile nucleus of mass number $B$ from a target nucleus of mass number $A$. Let $\Psi_B(r_1', ..., r_B')$ and $\Psi_A(r_1, ..., r_A)$ be their ground state wavefunctions; and $r_j'(j=1, 2, ..., B)$ and $r_i(i=1, 2, ..., A)$ their intrinsic coordinates as measured from the respective centres of mass. Further, let $\mathbf{p}_K$ be the momentum associated with the relative motion of the two nuclei, and $s_i(i=1, ..., A)$ and $s_j'(j=1, ..., B)$ be the projections of $r_i$ and $r_j'$ on a plane perpendicular to $\mathbf{K}$, respectively (Fig. 2.5).
Now if $b$ is the impact parameter for the collision of \( B \) on \( A \), the phase shift function for the collision of \( j \)-th projectile nucleon with the \( i \)-th target nucleon is: \( \chi_{NN}(b-\xi_i^{+}\xi_j^{-}) \), where \( \chi_{NN}(b) \) is the phase shift function for the NN scattering. Thus invoking the principle of additivity of phase shift functions the total phase function for the scattering of the projectile nucleus \( B \) on the target nucleus \( A \) may be written as

\[
\chi_T(b) = \sum_{i=1}^{A} \sum_{j=1}^{B} \chi_{NN}(b-\xi_i^{+}\xi_j^{-}) . \tag{2.6.1}
\]

Hence using eq. 2.3.29 and neglecting Coulomb scattering the elastic scattering amplitude may be expressed as

\[
F_{BA}(q) = \frac{iK}{2\pi} \int d^2b \, e^{iq\cdot b} \chi_T(b) \tag{2.6.2}
\]

where the S-matrix operator in this case reads as

\[
\hat{S}(b) = e^{i \sum_{i=1}^{A} \sum_{j=1}^{B} \chi_{NN}(b-\xi_i^{+}\xi_j^{-})} \hat{S}_{BA} \tag{2.6.3}
\]

or using eq. 2.4.3

\[
= \prod_{i=1}^{A} \prod_{j=1}^{B} [1 - \Gamma_{NN}(b-\xi_i^{+}\xi_j^{-})] . \tag{2.6.4}
\]

Thus, the basic problem in the study of nucleus-nucleus
scattering at medium and high energies is to evaluate the elastic S matrix element

\[ S_{BA}(b) = \langle \psi_A | \hat{S}(b) | \psi_B \rangle, \]

or

\[ S_{BA}(b) = \int |\psi_A(r_1,\ldots,r_A)|^2 |\psi_B(r'_1,\ldots,r'_B)|^2 \hat{S}(b,r_1,\ldots,r_A,r'_1,\ldots,r'_B) \]

2.6.5

\[ \int_A r_A^{r_A} \int_B r_B^{r_B} dr_1 \ldots dr_A dr'_1 \ldots dr'_B. \]

2.6.6

As it turns out that the problem posed by eq. 2.6.5 is not as simple and straightforward as it was thought to be initially. The main reason is that the multiple scattering series resulting from the double product in the expression 2.6.4 converges rather slowly. In consequence, fairly large number of higher order terms of the multiple scattering expansion are needed for realistic evaluation of \( S_{BA}(b) \). These higher order terms involve multidimensional integrals and higher order densities of the colliding nuclei which make their evaluation prohibitively tedious. In the next chapter we briefly discuss the various approximation methods that have been used to evaluate \( S_{BA}(b) \) approximately.

2.6.1 Coulomb scattering for nucleus-nucleus scattering

So far our discussion of the nucleus-nucleus scattering has disregarded the Coulomb scattering which is quite
important. This deficiency can be remedied rather easily as explained in the case of proton-nucleus scattering (sect. 2.5). The only nontrivial difference from that case being replacement of the finite charge distribution of the target by the folded nuclear charge distributions of the projectile and the target nuclei. Thus the formula we have used for calculating the differential cross section (eq. 2.3) reads as:

$$\frac{d\sigma}{d\Omega} = |F_{\text{cou}}(q) + F_{\text{NC}}(q)|^2,$$

2.6.7

where

$$F_{\text{NC}}(q) = \frac{iK}{2\pi} \int d^2q \ e^{-i\rho_{pt}(q)} \ e^{-i\rho_{t}(q)} \ [1-e^{-i\rho_{c}(q)} <00|S(q)|00>],$$

2.6.8

and $F_{\text{cou}}(q)$ is given by eq. 2.5.2.

The quantities $\chi_{pt}(b)$ and $\chi_{c}(b)$ are the same as given by eqs. 2.5.3 and 2.5.4, with the difference that the fine structure constant $\eta$ and $\rho_{ch}(t)$ should now be read as:

$$\eta = \frac{Z_A Z_B}{(137.036v_c)},$$

$$\rho_{ch}(t) = \int \rho_A^C(t) \rho_B^C(t-t) \ dt,$$

where $\rho_A^C$ and $\rho_B^C$ are the charge distributions of the target and the projectile nuclei, respectively and $v_c$ is the velocity of the projectile in the c.m. frame.
References


16 I. Ahmad, Phys. Letts. 36B, 301(1971).