CHAPTER 6

INTEGER QUADRATIC PROGRAMMING

6.1. INTRODUCTION

The general integer non-linear programming problem (INLP) is

\[ \text{Maximise} \quad z = f(x) \]

subject to \[ g_i(x) \leq b_i, \quad i = 1, \ldots, m \]

\[ x \geq 0 \] and integer.

Kurai and Oettli (1965) are the first to describe a cutting plane algorithm for solving problems in which \( f(x) \) is quadratic and the constraints \( g_i(x) \) are linear. Haurd (1970) extends the method of centers given in Haurd (1967) to develop a cutting plane algorithm for binary INLPs. A generalisation of the additive algorithm for INLPs of Balas (1967) to the case where both \( f(x) \) and the constraints are quadratic has been given by V. Ginsburgh and Van Peeterssean (1969). The Gomory type cuts for mixed integer quadratic programs are introduced by Agrawal (1974).

Some enumeration procedures for INLP are discussed in Garfinkel and Nemhauser (1972), Agrawal (1974) uses a branch
and bound technique for solving integer quadratic programs. In his approach the branching is done by using the Land and Doig's rule and the created subproblems are solved by Beal's method. Here we consider a variation of Agrawal's approach in which the branching is made by introducing the inequality constraints and the created subproblems are solved by using the Wolf's procedure for solving the quadratic programs (Hadley (1964)). The equivalent programs to be solved with some modification of the simplex technique are easily derived for the created subproblems.

6.2. ANALYSIS OF THE PROCEDURE

Let us consider the problem of finding $x' = (x_1, \ldots, x_n)$ that

\[
\text{Maximise } f(x) = c^T x + x' D x
\]

subject to, $Ax \leq b$  \hspace{1cm} (ii)

\[
x \geq 0
\]

where $A$ is $m \times n$, and $x_j$ integers, $j = 1, \ldots, n$  \hspace{1cm} (iv)

We assume that the matrix $D$ is negative semidefinite and symmetric. A problem equivalent to the quadratic program 2(i) to 2(iii) is to find
\((\bar{z} + x + \bar{y} + \bar{z}) \geq 0\) \hspace{1cm} (1)

that satisfy
\[2\bar{z} - \lambda'x + y = -\bar{g} \hspace{1cm} (ii)\]
\[\lambda + \bar{y} = \bar{b} \hspace{1cm} (iii)\]
\[\bar{y}' \bar{y} = 0, \quad \bar{y}' \bar{z} = 0 \hspace{1cm} (iv)\]

It is known that the solution of the linear program 3(i)-3(iii) (i.e. finding a basic feasible solution) by the simplex technique with the basis entry restricted by 3(iv) also solves the quadratic program 2(i) to 2(iii). If all the components of \(x\) are also integers, then the solution to the problem 2(i) to 2(iv) is obtained. Now consider the case where all the components of \(x\) are not integers. Let fractional component be \(x^j\) whose value is \(x^j_0\) and let \([x^j_0]\) \(= a\). The two new subproblems to be solved are created by adding in (2) respectively the constraints

\[x^j \leq a \hspace{1cm} \ldots (4)\]
\[\text{and } x^j \geq a + 1 = q^j \hspace{1cm} \ldots (5)\]

The condition \(x^j \leq a\) is equivalent to
\[\bar{x}^j - a = x^j \geq 0 \hspace{1cm} \ldots (6)\]

The equivalent Kuhn-Tucker conditions for the problem
2(1) - 2(111) together with (6) are

\[
\begin{pmatrix} x \cdot \bar{y} \cdot z \cdot \bar{z} \end{pmatrix} \geq 0 \tag{1}
\]

\[2\bar{D}x - A'y + Ig = - g \tag{11} \]

\[\bar{A}\bar{x} + Ly = b \tag{111} \]

\[\bar{u}' \bar{y} = 0, \quad \bar{x}' \bar{g} = 0 \tag{iv} \]

where the components of \( \bar{x} \) are the same as that of \( x \) except that \( \bar{x}_j = a - x_j \) and \( \bar{I} \) is a unit matrix with the jth diagonal element replaced by \(-1\). (since \( \frac{\partial f}{\partial x_j} = - \frac{\partial f}{\partial \bar{x}_j} \).)

Similarly the condition \( x_j > q \) is equivalent to

\[\bar{x}_j = x_j - q \geq 0 \tag{8} \]

The equivalent Kuhn-Tucker conditions for the problem
2(1) - 2(111) together with (8) are

\[
\begin{pmatrix} \bar{x} \cdot \bar{y} \cdot \bar{z} \cdot \bar{y} \end{pmatrix} \geq 0 \tag{1}
\]

\[2\bar{D}\bar{x} - A'y + I\bar{g} = - g \tag{11} \]

\[\bar{A}\bar{x} + Ly = b \tag{111} \]

\[\bar{u}' \bar{y} = 0, \quad \bar{x}' \bar{g} = 0 \tag{iv} \]
Now the two subproblems (7) and (9) are solved by the simplex method using the restricted entry into the basis according to (iv).

The upper bounds on the solutions are obtained using the Dakin's approach. The conditions for the convergence of the procedure are the same as those for the Wolfe's procedure.

6.3. NUMERICAL EXAMPLE

We consider the same example, as given by Agrawal (1974).

Maximise \( z = -2x_1^2 + 2x_1x_2 - 2x_2^2 + 6x_1 - 6 \)
subject to, \( x_1 + x_2 \leq 2 \)
\( x_1 \geq 0, x_2 \geq 0, \)
\( x_1 \text{ and } x_2 \text{ integers.} \)

The non-integral solution obtained by applying the artificial basis technique to the transformed problem in (5) is

\( x_1 = 3/2 , x_2 = 1/2 , u = 1 \text{ and } z = -1/2. \)

The two subproblems are created by taking \( x_1 \leq 1 \)
and \( x_1 \geq 2 \). The equivalent Kuhn-Tucker conditions as
given in (7) are

\[ \bar{x}_1, x_2, y, u, s_1, s_2 \geq 0 \quad (1) \]
\[ -4\bar{x}_1 - 2x_2 + u + s_1 = 2 \quad (11) \]
\[ -2\bar{x}_1 - 4x_2 - u + s_2 = -2 \]
\[ -\bar{x}_1 + x_2 + y = 1 \quad (111) \]
\[ uy = 0, \bar{x}_1 s_1 = 0, x_2 s_2 = 0 \quad (iv) \]

where \( \bar{x}_1 = 1 - x_1 \).

A solution to (10) is

\[ \bar{x}_1 = 0, x_2 = \frac{1}{2}, s_1 = 3, y = 2. \]

Similarly we enumerate the other solutions of the created subproblems. The various iterations can be shown in the following diagram:
The problem has three solutions obtained on the nodes (2), (3) and (4).