CHAPTER 5

PROBLEM OF CUTTING THE SHEETS WITH UNCERTAIN DEMAND

5.1. INTRODUCTION

Consider m sheets (e.g. metallic, textile or paper) of equal width and of lengths $a_1, a_2, \ldots, a_m$. These sheets are required to be cut into the pieces of length $s_1 > s_2 > \ldots > s_m$. The demand $d_j$ for the number of pieces of length $s_j$ is uncertain; but its probability distribution $P_j(d_j)$ is known. The gain per piece from the pieces of length $s_j$ is $c_j$. We seek to find a cutting pattern that will maximize the gain and the demands are met. The problem of cutting the sheets or rolls into standard length is formulated and solved as the Knapsack problem in Gilmore and Gomory, (1961), (1963). Here we treat the problem where on each standard length there is an uncertain demand with known discrete probability distribution. This is a linear programming problem with uncertain demands. For its solution we give it a two stage formulation (Madansky (1962)).

5.2. A TWO STAGE MODEL

Let us denote by $x_{ij}$ the number of pieces of size $s_j$ taken from the $i$th sheet. The total supply of the pieces of size $s_j$ will be $\sum_{i=1}^{m} x_{ij} = x_j$, say.
Let an under supply of a piece of size \( s_j \) incure a loss of \( f_j \) (e.g. it is purchased from the local market and supplied to the customer on loss.)

Define

\[
L_j(d_j-x_j) = \begin{cases} 
0 & \text{for } d_j \leq x_j \\
 f_j(d_j-x_j) & \text{for } d_j > x_j 
\end{cases} \quad \ldots \ (1)
\]

The expected penalty cost from the discrepancies in the pieces of size \( s_j \) is then,

\[
L_j(x_j) = \mathbb{E} \left\{ L_j(d_j-x_j) \right\} = \mathbb{E} \left\{ f_j(d_j-x_j) \right\} \quad \ldots \ (2)
\]

The problem may now be stated as

Maximise

\[
\sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} L_j(x_j) \quad \ldots \ (3)
\]

subject to,

\[
\sum_{j=1}^{n} x_{ij} s_j \leq a_i, \quad i=1, \ldots, n \quad \ldots \ (4)
\]

\[
\sum_{i=1}^{n} x_{ij} = d_j, \quad j=1, \ldots, n. \quad \ldots \ (5)
\]
Here \( x_{ij} \) are the first stage variables. These are to be determined before the demand \( d_j \) is actually known.

At the second stage we do not have to solve an optimization problem, since the second stage variables

\[ y_j = d_j - x_j \]

are uniquely determined for each \( x_j \), when \( d_j \) are observed. This fact is exploited in the solution procedure of the problem.

5.3. SOLUTION PROCEDURE

Note that for each \( j \) the only possible values of \( d_j \) are integers. Let these values be \( o, 1, \ldots, \bar{d}_j \). The expected penalty cost from the under supply is then

\[ L_j(x_j) = \sum_{d_j=x_j}^{\bar{d}_j} f_j(d_j-x_j) P(d_j) \quad (7) \]

Let us determine \( L_j(x_j) \) for \( j = 1, \ldots, n \). We also find

\[ L_j(x_j) - L_j(x_j+1) \]

for each \( j \) where \( x_j \) ranges from \( o \) to \( \bar{d}_j \).

\( L_j(x_j) - L_j(x_j+1) \) are the incremental values. Observe that

the sum of the incremental values starting from \( x_j = \bar{d}_j - 1 \)
and going up to \( x_j^* \) equals \( L_j(x_j^*). \)

It is clear from (7) that \( L_j(x_j) \) is a decreasing function of \( x_j \). We may also prove that \( L_j(x_j) \) is convex. Since \( x_j \) are integers, it is sufficient to show (Wagner (1969), p 672) that

\[
L_j(d_j - x_j + 1) - L_j(d_j - x_j) \geq L_j(d_j - x_j) - L_j(d_j - (x_j - 1))
\]

i.e.

\[
f_j(d_j - (x_j + 1)) - 2f_j(d_j - x_j) \geq -f_j(d_j - (x_j - 1))
\]

or

\[
f_j \geq 0
\]

which is true. Thus

\[
L_j(x_j) = E \{L_j(d_j - x_j)\} = \sum_{D \mid \mathbf{x_j}} f_j(D - x_j) \cdot p(d_j = D).
\]

being a convex linear combination of \( L_j(d_j - x_j) \) is also convex.

From the fact that \( L_j(x_j) \) is decreasing and convex, it follows that the incremental values \( L_j(x_j) - L_j(x_j + 1) \) are decreasing.

The maximum possible demand that may occur is \( \sum_{j=1}^{n} d_j s_j \).

We may assume that \( \sum_{j=1}^{n} d_j s_j \) is greater than the total availability \( \sum_{i=1}^{m} a_i \). Let
For each piece taken from the slack there is a penalty associated with it. Let us choose the pieces from the slack such that the penalty is minimised. This is the same thing as to minimise

\[ \sum_{j=1}^{n} L_j(y_j) \]  \quad \ldots (9)

subject to, \[ \sum_{j=1}^{n} y_j s_j = a \]

where \( y_j \) (non-negative integer) is the number of pieces of size \( s_j \) taken from the slack.

The fact that \( \sum_{j=1}^{n} L_j(y_j) \) is convex and decreasing may be exploited to apply a technique for the Knapsack problem (Saaty (1970)), for solving the problem in (9).

Let us put the problem in the form

\[ \text{Max} \quad \frac{1}{\sum_{j=1}^{n} \frac{d_j}{L_j - \sum_{k=1}^{m} y_j^k}} \]  \quad \ldots (a)

subject to, \[ \sum_{j=1}^{n} y_j^k s_j = a \]  \quad \ldots (b)  \quad \ldots (10)

and \( y_j^k = 0 \) or 1 \quad \ldots (c)
where \[ \sum_{k=0}^{\infty} Y_{j}^{k} = Y_{j}. \]

The total number of variables \( Y_{j}^{k} \) is \( n \times \sum_{j=1}^{m} (d_{j}+1) = N \), say. We arrange the variables \( Y_{j}^{k} \) to the descending values of

\[ - \left\{ L_{j} \left( \bar{d}_{j} - \sum_{k=0}^{s} Y_{j}^{k} \right) - L_{j} \left( \bar{d}_{j} - \sum_{k=0}^{s+1} Y_{j}^{k} \right) \right\}. \]

The problem (10) may now be solved by enumerating the solution vectors in lexicographic ordering of the Knapsack problem with constraints 10(b), 10(c) and the linear objective function in which the cost coefficient of \( Y_{j}^{k} \) is

\[ - \left\{ L_{j} \left( \bar{d}_{j} - s \right) - L_{j} \left( \bar{d}_{j} - (s+1) \right) \right\}. \]

Note that the optimal solution, the sum of the costs associated with allocating \( a_{j} \) (unfilled demand) to the pieces of size \( s_{j} \) will equal the value of \( L_{j} \left( \bar{d}_{j} - \sum_{k=0}^{s} Y_{j}^{k} \right) \).

Let the optimal distribution of slack amongst various sizes be \( \bar{y}_{j} \), \( j = 1, \ldots, n \). After distributing the slack in this way we subtract \( \bar{y}_{j} \) from \( \bar{d}_{j} \) and then solve the following problem:

\[
\text{Maximise } \sum_{i=1}^{m} \sum_{j=0}^{\infty} c_{ij} x_{ij}
\]
subject to, \[ \sum_{j=1}^{n} a_j x_{ij} = a_i, \quad i=1, \ldots, n \]
\[ \sum_{i=1}^{m} x_{ij} = d_j - \bar{y}_j, \quad j=1, \ldots, n \]
\[ x_{ij}, \text{ integers}, \geq 0 \]

This is the deterministic problem of cutting the pieces and can be solved by the method given in Chapter 4.

5.4. NUMERICAL EXAMPLE

Consider the problem with the following data:

\[ m = 3 \]
\[ n = 4 \]
\[ q = (23, 12, 6, 2) \]
\[ s = (20, 12, 7, 3) \]
\[ s = (40, 30, 39) \]

where \( q, s \) and \( s \) are vectors (i.e., \( s = (a_1, a_2, a_3) \)). The demands \( d_1, d_2, d_3 \) and \( d_4 \) are uncertain with their probability distributions as

\[ P(d_1 = 1) = \ldots = P(d_1 = 5) = \frac{1}{5} \]
\[ P(d_2=2) = P(d_2=3) = \frac{1}{4}, \quad P(d_2=4) = \frac{1}{2} \]

\[ P(d_3=4) = P(d_3=5) = P(d_3=6) = \frac{1}{3} \]

and

\[ P(d_4=1) = P(d_4=2) = \frac{1}{2} \]

We have

\[ \frac{1}{j} \sum_{j=1}^{d_j} d_j - \frac{1}{2} n = 47 \]

Let the losses from under supply be \( f_1 = 2, f_2 = f_3 = f_4 = 1 \). By using (7) we find \( L_j(x_j) \) and then \( L_j(x_j+1) \) for all \( j = 1, \ldots, n \). We write the values in the following tabular form:

<table>
<thead>
<tr>
<th>( j )</th>
<th>( x_j )</th>
<th>( L_1(x_1) )</th>
<th>( L_1(x_1) - L_1(x_1+1) )</th>
<th>( L_2(x_2) )</th>
<th>( L_2(x_2) - L_2(x_2+1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>( 3f_1 )</td>
<td>( f_1 )</td>
<td>( 13/4 f_2 )</td>
<td>( f_2 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( 2f_1 )</td>
<td>( 4/5 f_1 )</td>
<td>( 9/4 f_2 )</td>
<td>( f_2 )</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>( 6/5 f_1 )</td>
<td>( 3/5 f_1 )</td>
<td>( 5/4 f_2 )</td>
<td>( f_2 )</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>( 3/5 f_1 )</td>
<td>( 2/5 f_1 )</td>
<td>( 1/2 f_2 )</td>
<td>( f_2 )</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>( 1/5 )</td>
<td>( 1/5 )</td>
<td>( f_2 )</td>
<td>( f_2 )</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
</tr>
</tbody>
</table>
How we solve the problem:

Minimize $\sum L_j(\bar{d}_j - \sum_{k=0}^{\infty} y_j^k)$

subject to, $\sum_{j=1}^{n} \sum_{k=0}^{\infty} y_j^k a_j = a_0$

and $y_j^k = 0$ or $1$.

The penalty functions are arranged in ascending order of magnitude from left to right. The allocations in the following table for solving the problem are done by using the rules for ordinary Knapsack problem discussed in Chapter 4.
The solution is \( y_5^5 = y_1^4 = y_3^4 = y_4^o = y_3^3 = 1. \)

Now the solution to the full problem can be obtained as shown in the following table:

<table>
<thead>
<tr>
<th>( a_j )</th>
<th>23</th>
<th>12</th>
<th>6</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_j )</td>
<td>20</td>
<td>12</td>
<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>

Supply

<table>
<thead>
<tr>
<th>slack</th>
<th>1</th>
<th>3</th>
<th>2</th>
<th>47</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>59</td>
</tr>
</tbody>
</table>

Maximum demand

| 5 | 4 | 6 | 2 |