5.1. INTRODUCTION

Vajda (1970) has investigated the properties of entropy of order \( \alpha \) for two probability distributions \( P \) and \( Q \) in continuous case over the same measurable space and he has established the relationship between \( H_\alpha(P;Q) \) and Bayes' risk where

\[
(5.1.1) \quad H_\alpha(P;Q) = \int p^\alpha q^{1-\alpha} \, d\mu, \quad \alpha \in (0,1),
\]

\( p, q \) being the Radon-Nikodym densities of probability distributions \( P \) and \( Q \) on measurable space \((X, \mathcal{X})\) with respect to another (dominating) probability distribution \( \mu \) on \((X, \mathcal{X})\). \( H_\alpha(P;Q) \) was simply called \( \alpha \)-entropy.

For discrete probability distributions \( P = (p_1, p_2, \ldots, p_n) \in \delta_n \) and \( Q = (q_1, q_2, \ldots, q_n) \in \delta_n \), we define \((\alpha, \beta)\)-information as

\[
(5.1.2) \quad D_n^{(\alpha, \beta)}(P;Q) = \sum_{i=1}^{n} p_i^\alpha q_i^{\beta-\alpha},
\]

where \( \alpha \neq \beta \), \( \alpha, \beta > 0 \).
In this chapter, we will give a characterization of $D_n^{(\alpha, \beta)}(P; Q)$ through certain axioms and discuss some of its special cases such as Matusita (1967) distance and Bhattacharya (1945-46) distance.

5.2. CHARACTERIZATION OF $D_n^{(\alpha, \beta)}(P; Q)$

We prove the following theorem.

**Theorem**: Let $K_n : S_n = \delta_n \times \delta_n \rightarrow R$ (reals) (n=2,3,...) be a sequence of functions of $p_1$'s and $q_1$'s satisfying the following posulates:

(i) **Symmetry**: $K_3$ is symmetric in pairs \{p_1, q_1\}, $i = 1, 2, 3$.

(ii) **Normalization**: $K_2(1,0; \frac{1}{2}, \frac{1}{2}) = 2^{\alpha-\beta}$

(iii) **Branching Property**: 

$$K_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = K_{n-1}(p_1+p_2, p_3, \ldots, p_n; q_1+q_2, q_3, \ldots, q_n) + (p_1+p_2)^\alpha (q_1+q_2)^{\beta-\alpha} (K_2(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}; \frac{q_1}{q_1+q_2}, \frac{q_2}{q_1+q_2}) - 1),$$

for all $n > 2$, $p_1+p_2 > 0$, $q_1+q_2 > 0$.

(Postulate (iii) explains the desired nature of combinations of the measures to be taken when the union of
two mutually exclusive events are considered.)

The measure of \((\alpha, \beta)\)-information defined in (5.1.2) is uniquely determined by the above postulates.

First we prove the following lemmas:

**Lemma 1.** Let

\[ h(r, s) = k_2(r, 1-r, s, 1-s) - 1, \text{ for } r, s \in [0, 1]. \]

Then

\[ h(r, s) = h(1-r, 1-s); r, s \in I. \]

**Proof:** From postulate (i) for \( n = 3 \), we get

\[ k_3(p_1, p_2, p_3; q_1, q_2, q_3) = k_3(p_2, p_1, p_3; q_2, q_1, q_3), \]

where \( p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1 \).

Applying postulate (iii), (5.2.3) becomes

\[
\begin{align*}
& k_2(p_1 + p_2, p_3; q_1 + q_2, q_3) + (p_1 + p_2)^\beta (q_1 + q_2) \\
& (k_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) - 1) \\
& = k_2(p_2 + p_1, p_3; q_2 + q_1, q_3) \\
& + (p_2 + p_1)^\alpha (q_2 + q_1) \left( k_2\left(\frac{p_2}{p_2 + p_1}, \frac{p_1}{p_2 + p_1}; \frac{q_2}{q_2 + q_1}, \frac{q_1}{q_2 + q_1}\right) - 1 \right) ,
\end{align*}
\]
Substituting
\[
 r = \frac{p_1}{p_1+p_2} \quad \text{and} \quad s = \frac{q_1}{q_1+q_2}
\]
in (5.2.4), we get
\[
 K_2 (r, l-r; s, l-s) = K_2 (1-r, r; 1-s, s).
\]

i.e.
\[
 h(r,s) = h(l-r, l-s), \text{ using (5.2.1)}.
\]

This proves the lemma.

**Lemma 2.** \( h(r,s) \) satisfies the following functional equation

\[
 (5.2.5) \quad h(r,s) + (1-r)(1-s) \alpha \beta-\alpha \frac{u}{l-r} \frac{v}{l-s} \quad h(-\frac{u}{l-r}, -\frac{v}{l-s})
\]

\[
 = h(u,v) + (1-u)(1-v) \alpha \beta-\alpha \frac{r}{l-u} \frac{s}{l-v} \quad h(-\frac{r}{l-u}, -\frac{s}{l-v}),
\]

for \( r,s,u,v \in [0,1] \) and \( r+u \leq 1, s+v \leq 1 \).

**Proof:** From postulate (i) for \( n = 3 \)
\[ K_3(p_1, p_2, p_3; q_1, q_2, q_3) = K_3(p_3, p_2, p_1; q_3, q_2, q_1). \]

Now using branching property for \( n = 3 \), we get

\begin{align*}
(5.2.6) \quad & K_2(p_1+p_2, p_3; q_1+q_2, q_3) + (p_1+p_2) \alpha (q_1+q_2)^{\beta-\alpha} \\
& = K_2(p_3+p_2, p_1; q_3+q_2, q_1) + (p_3+p_2) \alpha (q_3+q_2)^{\beta-\alpha} \\
& \quad (K_2(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}; \frac{q_1}{q_1+q_2}, \frac{q_2}{q_1+q_2}) - 1)
\end{align*}

or

\begin{align*}
(5.2.7) \quad & K_2(1-p_3, p_3; 1-q_3, q_3) + (1-p_3) \alpha (1-q_3)^{\beta-\alpha} \\
& = K_2(1-p_1, p_1; 1-q_1, q_1) + (1-p_1) \alpha (1-q_1)^{\beta-\alpha} \\
& \quad (K_2(\frac{p_1}{1-p_3}, \frac{p_2}{1-p_3}; \frac{q_1}{1-q_3}, \frac{q_2}{1-q_3}) - 1).
\end{align*}

as \( \sum_{i=1}^{3} p_i = \sum_{i=1}^{3} q_i = 1 \).

Now by putting \( p_3 = r, q_3 = s, p_1 = u \) and \( q_1 = v \) in (5.2.7), we get
Using (5.2.1), we have

\[ K_2(l-r; 1-s, 1-s) + (1-r)(1-s)^{\beta-\alpha} \]
\[ = K_2(\frac{l-r}{1-r}, \frac{1-u}{1-u}; \frac{1-\nu}{1-\nu}, \frac{1-\nu}{1-\nu}) \]
\[ + (1-u)^{\beta-\alpha}(1-\nu)^{\beta-\alpha} K_2(\frac{r-u}{1-u}; \frac{r-s}{1-s}; \frac{s}{1-\nu}, \frac{s}{1-\nu}) \]

Further using (5.2.2) we get (5.2.5).

This proves the Lemma 2.

**Lemma 3.** \( h(r,s) \) satisfying the functional equation (5.2.5), is uniquely determined as

\[ (5.2.8) \quad h(r,s) = C \left( r s + (1-r)(1-s)^{\beta-\alpha} \right) \]

for \( r, s \in I \),

where \( C \) is a constant to be determined by normalizing condition (ii).

**Proof:** Putting
\[ u = x_1, \quad \frac{v}{1-r} = x_2, \quad l-r = y_1 \text{ and } l-s = y_2 \text{ in } \]

(5.2.5), we get

\[ \alpha \beta - \alpha \]
\[ h \left( 1-x_1, 1-y_2 \right) + y_1 y_2 \quad h(x_1, x_2) = h(x_1 y_1, x_2 y_2) \]
\[ + \left( \frac{1-x_1}{1-x_2} \right) \left( \frac{1-y_1}{1-y_2} \right) \quad h \left( \frac{1-x_1 y_1}{1-x_2 y_2} \right), \]

for \( y_1 \in (0,1] \), \( x_1 \in [0,1] \), \( x_1 y_1 \neq 1( i=1, 2) \).

Using (5.2.2) in (5.2.9), we get

\[ \alpha \beta - \alpha \]
\[ h(y_1, y_2) + y_1 y_2 \quad h(x_1, x_2) = h(x_1 y_1, x_2 y_2) \]
\[ + \left( \frac{1-x_1}{1-x_2} \right) \left( \frac{1-y_1}{1-y_2} \right) \quad h \left( \frac{1-x_1 y_1}{1-x_2 y_2} \right). \]

Now consider the function

\[ \alpha \beta - \alpha \]
\[ g(x_1, x_2, y_1, y_2) = h(y_1, y_2) + y_1 y_2 \]
\[ + \left( \frac{1-y_1}{1-y_2} \right) \quad h(x_1, x_2) \]

for \( x_1, x_2, y_1, y_2 \in (0,1) \).

We will show that \( g(x_1, x_2, y_1, y_2) \) is symmetric in pairs \((x_1, y_1)\) and \((x_2, y_2)\). Interchanging the \( y \)'s and \( x \)'s in
(5.2.10), we get

\[(5.2.12) \quad h(x_1, x_2) + x_1 x_2 h(y_1 y_2) = h(x_1 y_1, x_2 y_2)\]

\[\alpha \beta - \alpha + (1-x_1 y_1) (1-x_2 y_2) h(y_1 y_2) = h(x_1 y_1, x_2 y_2).\]

Now subtract the respective sides of (5.2.12) from (5.2.10), to obtain

\[h(y_1, y_2) + y_1 y_2 h(x_1, x_2) - h(x_1, x_2) + x_1 x_2 - \alpha h(y_1, y_2)\]

\[= (1-x_1 y_1) (1-x_2 y_2) h(y_1 y_2) = h(x_1 y_1, x_2 y_2) - h(y_1, y_2)\]

That is

\[(5.2.13) \quad h(y_1, y_2) + y_1 y_2 h(x_1, x_2) = h(x_1, x_2)\]

\[\alpha \beta - \alpha + x_1 x_2 h(y_1, y_2) + (1-x_1 y_1) (1-x_2 y_2)\]

\[= h(y_1, y_2) - h(x_1 y_1, x_2 y_2) = h(x_1 y_1, x_2 y_2) - h(y_1, y_2)\]

Setting \[y_i = \frac{1-y_i}{1-x_i y_i}\] for \(i=1, 2\) in (5.2.10), we get
\[(5.2.14) \quad h\left(\frac{1-y_1}{1-x_1y_1}, \frac{1-y_2}{1-x_2y_2}\right) + h\left(\frac{1-y_1}{1-x_1y_1}, \frac{1-y_2}{1-x_2y_2}\right) = h\left(\frac{1-x_1y_1}{1-x_2y_2}, \frac{1-x_2y_2}{1-x_1y_1}\right) \]

\[(5.2.15) \quad (1-x_1y_1)(1-x_2y_2) \left[ h\left(\frac{1-y_1}{1-x_1y_1}, \frac{1-y_2}{1-x_2y_2}\right) \right] = h\left(\frac{1-x_1y_1}{1-x_2y_2}, \frac{1-x_2y_2}{1-x_1y_1}\right) \]

Using (5.2.2) in (5.2.14), we get

\[ h\left(\frac{1-y_1}{1-x_1y_1}, \frac{1-y_2}{1-x_2y_2}\right) = h\left(\frac{1-x_1y_1}{1-x_2y_2}, \frac{1-x_2y_2}{1-x_1y_1}\right) \]

or

\[ h\left(\frac{1-y_1}{1-x_1y_1}, \frac{1-y_2}{1-x_2y_2}\right) - h\left(\frac{1-x_1y_1}{1-x_2y_2}, \frac{1-x_2y_2}{1-x_1y_1}\right) \]

or

\[ h\left(\frac{1-y_1}{1-x_1y_1}, \frac{1-y_2}{1-x_2y_2}\right) = h\left(\frac{1-x_1y_1}{1-x_2y_2}, \frac{1-x_2y_2}{1-x_1y_1}\right) \]

or

\[ (5.2.15) \quad (1-x_1y_1)(1-x_2y_2) \left[ h\left(\frac{1-y_1}{1-x_1y_1}, \frac{1-y_2}{1-x_2y_2}\right) \right] = h\left(\frac{1-x_1y_1}{1-x_2y_2}, \frac{1-x_2y_2}{1-x_1y_1}\right) \]
Thus equation (5.2.13) and (5.2.15) give

\[
\begin{align*}
&= (1-x_1)^\alpha (1-x_2)^\beta h(y_1, y_2) - (1-y_1)^\alpha (1-y_2)^\beta h(x_1, x_2). \\
&= (1-x_1)^\alpha (1-x_2)^\beta h(y_1, y_2) - (1-y_1)^\alpha (1-y_2)^\beta h(x_1, x_2), \\
i.e.\ &\frac{\alpha y_1^\alpha y_2^\beta - h(x_1, x_2) + (1-y_1)^\alpha (1-y_2)^\beta h(x_1, x_2)}{(1-x_1)^\alpha x_1^\alpha x_2^\beta + (1-y_1)^\alpha (1-y_2)^\beta h(x_1, x_2),} \\
i.e.\ &h(x_1, x_2)(y_1^\alpha y_2^\beta + (1-y_1)^\alpha (1-y_2)^\beta - 1) \\
&= h(y_1, y_2) (x_1^\alpha x_2^\beta + (1-x_1)^\alpha (1-x_2)^\beta - 1), \\
i.e.\ &h(y_1, y_2) = \frac{\alpha y_1^\alpha y_2^\beta + (1-y_1)^\alpha (1-y_2)^\beta - 1}{(1-x_1)^\alpha x_1^\alpha x_2^\beta + (1-y_1)^\alpha (1-y_2)^\beta h(x_1, x_2)} = \text{C (say)} \\
i.e.\ &(5.2.16) h(y_1, y_2) = \text{C(} y_1^\alpha y_2^\beta + (1-y_1)^\alpha (1-y_2)^\beta - 1) ,
\end{align*}
\]

where C is constant depending on the parameters \(\alpha\) and \(\beta\).
Now on using the normalizing condition given in postulate (ii), \( C \) must be equal to 1 and (5.2.16) reduces to

\[
(5.2.17) \quad h(y_1, y_2) = y_1^\alpha y_2^{\beta-\alpha} + (1-y_1)^\alpha (1-y_2)^{\beta-\alpha} - 1,
\]

for \( y_1, y_2 \in (0,1) \),

i.e.

\[
(5.2.18) \quad K_2(y_1, 1-y_1; y_2, 1-y_2) = y_1^\alpha y_2^{\beta-\alpha} + (1-y_1)^\alpha (1-y_2)^{\beta-\alpha}.
\]

We observe that \( K_2(y_1, 1-y_1; y_2, 1-y_2) = 1 \), whenever \( y_1 = y_2 \) which implies that the measure of \((\alpha, \beta)\)-information is maximum when the probability distributions \( P \) and \( Q \) coincide.

Further we have to show that (5.2.17) can be extended to \( y_1, y_2 \in [0,1] \). In (5.2.5) setting \( r = 0 \) and \( v = 0 \), we get

\[
\beta-\alpha\quad h(o,s)+(1-s)h(u,o) = h(u,o)+(1-u)h(o,s),
\]

i.e.

\[
(5.2.19) \quad ((1-u)^\alpha - 1)h(o,s) = ((1-s)^{\beta-\alpha} - 1)h(u,o).
\]

Since \( u \)'s are arbitrary in \([0,1]\), (5.2.19) gives

\[
(5.2.20) \quad h(o,s) = C_1((1-s)^{\beta-\alpha} - 1),
\]
where \( C_1 = \frac{h(u, o)}{(1-u)^\alpha - 1} \) is constant not involving \( s \).

But \( h(o, s) = h(1, 1-s) \), (according to 5.2.2). Therefore replacing \( s \) by \( (1-s) \) in (5.2.20), we get

\[
(5.2.21) \quad h(1, s) = C_1(s^{\beta - \alpha} - 1), \text{ for } s \in (0, 1].
\]

Also, it is evident from (5.2.20) and (5.2.21) that \( s \) may be taken as \( o \) in (5.2.20) and as unity in (5.2.20). Now \( s + \nu \leq 1 \) (as in 5.2.5). Take \( s + \nu = 1 \) and since \( s \in (0, 1] \) and \( \nu = 1 \), we have \( s = o \) and (5.2.21) reduces to

\[
(5.2.22) \quad h(1, o) = C_1(-1).
\]

Similar result is easily obtained for unity and zero in \( h(\ ) \). Hence, in general using (5.2.2), (5.2.1), and normalizing condition given in postulate (ii), we have

\[
(5.2.23) \quad h(x_1, x_2) = x_1^\alpha x_2^{\beta - \alpha} + (1-x_1)^\alpha (1-x_2)^{\beta - \alpha} - 1,
\]

for \( x_1, x_2 \in I \).

Thus we have

\[
(5.2.24) \quad k_2(x_1, 1-x_1; x_2, 1-x_2) = x_1x_2^{\beta - \alpha} + (1-x_1)^\alpha (1-x_2)^{\beta - \alpha},
\]

for \( x_1, x_2 \in I \).
This proves the Lemma 3.

**Proof of the Theorem.** By repeated application of the postulate (iii), we get

\[(5.2.25) \quad K_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) - 1 = \sum_{i=2}^{n} p_i^{\alpha} Q_i^{\beta-\alpha} \left( K_2 \left( \frac{p_i}{p_1}, \frac{1 - p_i}{p_1}; \frac{q_i}{Q_i}, 1 - \frac{q_i}{Q_i} \right) - 1 \right), \]

where

\[
P_i = p_1 + p_2 + \ldots + p_i \]
\[
Q_i = q_1 + q_2 + \ldots + q_i \quad ; \quad i = 1, 2, \ldots, n.
\]

Since \( K_2(y_1, 1-y_1; y_2, 1-y_2) = y_1^{\alpha} y_2^{\beta-\alpha} (1-y_1)^{\alpha} (1-y_2)^{\beta-\alpha} \),

we get

\[
K_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) - 1 = \sum_{i=2}^{n} p_i^{\alpha} q_i^{\beta-\alpha} \left( \frac{p_i}{p_1} \frac{1 - p_i}{p_1} \left( \frac{q_i}{Q_i} \frac{1 - q_i}{Q_i} \right) - 1 \right)
\]

\[
= \sum_{i=2}^{n} p_i^{\alpha} q_i^{\beta-\alpha} + \sum_{i=2}^{n} p_i^{\alpha} q_i^{\beta-\alpha} - \sum_{i=2}^{n} p_i^{\alpha} q_i^{\beta-\alpha}
\]

\[
= n \sum_{i=2}^{n} p_i^{\alpha} q_i^{\beta-\alpha} - \sum_{i=2}^{n} p_i^{\alpha} q_i^{\beta-\alpha} = \sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta-\alpha} - 1 \quad (\text{since } p_n = Q_n = 1)
\]
i.e. \[ K_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta - \alpha}. \]

This completes the proof of the theorem.

**Particular cases of** \( D_n^{(\alpha, \beta)}(P; Q) \)

(a_1) If we take \( \alpha = \frac{1}{2}, \beta = 1 \) then \( D_n^{(\alpha, \beta)}(P; Q) \) reduces to

\[ (5.2.26) \quad D_n^{(1/2, 1)}(P; Q) = \sum_{i=1}^{n} p_i^{1/2} q_i^{1/2}, \]

which is a measure of affinity between the distributions \( P \) and \( Q \) and the Matusita (1967) distance \( M_n(P; Q) \) takes the form

\[ (5.2.27) \quad M_n(P; Q) = 2(1 - D_n^{(1/2, 1)}(P; Q)). \]

(a_2) The quantity \( -\log_2(D_n^{(1/2, 1)}(P; Q)) \) is a measure of distance proposed by Bhattacharya (1945-46).

**5.3. INFORMATION THEORETIC MEASURE**

In this section, we consider the generalized functional equation in three variables defined as

\[ (5.3.1) \quad \sum_{i=1}^{m} \sum_{j=1}^{n} F(x_i y_j, u_i v_j, s_i t_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i u_i s_i F(y_j, v_j, t_j) \]

\[ + \sum_{i=1}^{m} \sum_{j=1}^{n} y_i v_i t_j F(x_i, u_i, s_i), \]
for $x_i, y_j, u_i, v_j, s_i, t_j, \alpha, \Sigma x_i = \sum_{i=1}^{n} y_j = 1, \Sigma u_i \leq 1, \Sigma v_j \leq 1, \Sigma s_i \leq 1, \text{ and } \Sigma t_j \leq 1$, where all the $x, y, u, v, s, t > 0$.

If $F$ is the continuous function satisfying functional equation (5.3.1) under some suitable conditions, then we define information measure associated with discrete probability distributions $P, Q$ and $R$ as

$$I(P;Q;R) = \sum_{i=1}^{n} F(p_i, q_i, r_i).$$

**Theorem 1.** The only continuous solution of the functional equation (5.3.1) is

$$F(p, q, r) = \mu(\beta - \alpha \quad \alpha - \beta \quad \lambda - \delta \quad \delta - \lambda),$$

where $\mu$ is a constant depending on the parameters $\alpha, \beta, \delta$ and $\lambda$ such that $\alpha \neq \beta$ and $\delta \neq \lambda$.

**Proof:** Let $m, n, a, b, c$ and $d$ be the positive integers such that $1 \leq m \leq a, c$ and $1 \leq n \leq b, d$. Now setting

$$x_i = \frac{1}{m}, \quad u_i = \frac{1}{a}, \quad s_i = \frac{1}{c} \quad (i=1, \ldots, m)$$

$$y_j = \frac{1}{n}, \quad v_j = \frac{1}{b}, \quad t_j = \frac{1}{d} \quad (j=1, 2, \ldots, n)$$
in equation (5.3.1), we obtain

\[
\begin{align*}
\text{mn} \ F\left(\frac{1}{m}, \frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) &= \text{mn} \ F\left(\frac{1}{n}, \frac{1}{b}, \frac{1}{d}\right)(\frac{1}{a})^{\beta-\alpha} (\frac{1}{c})^{\alpha-\beta} \\
+ \text{mn} \ F\left(\frac{1}{m}, \frac{1}{a}, \frac{1}{b}\right)(\frac{1}{n})(\frac{1}{b})^{\lambda-\delta} (\frac{1}{d})^{\delta-\lambda} ,
\end{align*}
\]

i.e.

\[
(5.3.4) \quad (\frac{1}{m})(\frac{1}{a})(\frac{1}{b})(\frac{1}{c})^{\lambda-\delta} F\left(\frac{1}{m}, \frac{1}{a}, \frac{1}{c}\right).
\]

By putting

\[
\frac{1}{m} = p, \frac{1}{n} = f, \frac{1}{a} = q, \frac{1}{b} = r, \frac{1}{c} = g, \frac{1}{d} = h \quad \text{in} \quad (5.3.4)
\]

we get

\[
F(pf, qg, rh) = p^{\beta-\alpha} q^{\alpha-\gamma} r^{\lambda-\delta} h^{\delta-\lambda} F(pf, qg, rh) + f g^{\lambda-\delta} h^{\delta-\lambda} F(pq, rh).
\]

Since (due to symmetry property), we have

\[
F(pf, qg, rh) = F(fp, qg, hr),
\]

\[
\begin{align*}
\text{or} \quad f g h^{\lambda-\delta} + p q r^{\lambda-\delta} h^{\delta-\lambda} F(p, q, r) \\
&= f g h^{\lambda-\delta} + p q r^{\lambda-\delta} h^{\delta-\lambda} F(p, q, r) + p q r^{\lambda-\delta} h^{\delta-\lambda} F(p, q, r) + p q r^{\lambda-\delta} h^{\delta-\lambda} F(p, q, r)
\end{align*}
\]
or
\[
\frac{F(p,q,r)}{p(q - r - q r)} = \frac{F(f,g,h)}{f(g - h - g h)} = \mu \text{ (say)}
\]
i.e.
\[
(5.3.5) \quad F(p,q,r) = \mu(p(q - r - q r)).
\]

The solution (5.3.5) can also be extended to the case when \( p, q \) and \( r \) are rational numbers. For this, let
\[
x = \frac{m}{n} (m < n), \quad y = \frac{p}{q} (p < q), \quad u = \frac{p^*}{q} (p^* < q^*).
\]
Choose integers \( K \) and \( K^* \) sufficiently large such that
\[
m < KP, \quad K^* p^*, \quad n < Kq, \quad K^* q^*; \quad \frac{q(n-m)}{n(q-p)} \leq K
\]
and \( \frac{q^*(n-m)}{n(q^*-p^*)} \leq K^* \).

Set
\[
x_1 = \frac{m}{n}, \quad x_2 = x_3 = \ldots = x_{n-m+1} = \frac{1}{n}
\]
\[
y_1 = y_2 = \ldots = y_m = \frac{1}{m}
\]
\[
u_1 = \frac{p}{q}, \quad u_2 = u_3 = \ldots = u_{n-m+1} = \frac{1}{nk}
\]
\[
v_1 = v_2 = \ldots = v_m = \frac{1}{pk}
\]
\[
s_1 = \frac{p^*}{q}, \quad s_2 = s_3 = \ldots = s_{n-m+1} = \frac{1}{nk^*}
\]
\[
t_1 = t_2 = \ldots = t_m = \frac{1}{p^*k^*}
\]
(5.3.6)

Taking \( m \) as \( n-m+1 \) and \( n \) as \( m \) in equation (5.3.1), we get
Now (5.3.7) alongwith (5.3.3) gives

\[ F\left(\frac{m}{n}, \frac{p}{q}, \frac{p^*}{q^*}\right) = \mu\left(\left(\frac{m}{n}\right)(\frac{p}{q})^{\beta-\alpha} \left(\frac{p^*}{q^*}\right)^{\alpha-\beta} \right) \]

i.e.

(5.3.8) \[ F(x, u, s) = \mu(x \{u \leq s - u \leq s\}), \]

for all rationals \( x, u, s \in [0, 1] \).

From the continuity of \( F \) it follows that (5.3.8) is valid for all real numbers \( x, u, s \in [0, 1] \).

**Theorem 2.** Corresponding to the continuous solution (5.3.5), the information theoretic measure associated with the distribution \( P, Q \) and \( R \) is

(5.3.9) \[ I_{n}^{\alpha, \delta}(P; Q; R) = (2^{-2}) \sum_{i=1}^{n} q_i^{\beta-\alpha} r_i^{\alpha-\beta} \]

\[ - \sum_{i=1}^{n} p_i q_i^{\lambda-\delta} r_i^{\delta-\lambda}. \]
Remark: If the original and final predictions are the same then the information improvement is zero. In mathematical terms it means

\[ I_n(P;Q|R) = \sum_{i=1}^{n} F(p_i, q_i, r_i) = 0, \]

whenever \( q_i = r_i \) for all \( i \). Therefore from (5.3.9),

\[ I_{\alpha, \delta}^{(\beta, \lambda)} (P;Q;R) = 0. \]

The quantity (5.3.9) may be called type \((\alpha, \beta, \delta, \lambda)\)-information improvement.

We add an additional condition which in a way determines the unit of the measure, i.e.

\[ (5.3.10) \quad I(1, \frac{1}{2}, 1) = F(1, \frac{1}{2}, 1) = 1. \]

Proof of the theorem 2: Equation (5.3.5) for \( F(1, \frac{1}{2}, 1) = 1 \) yields

\[ (5.3.11) \quad \mu = \frac{1}{2^{\alpha-\beta - \delta - \lambda}}. \]

Therefore (5.3.9) follows from (5.3.2), (5.3.5) and (5.3.11).