CHAPTER - I

INTRODUCTION

1.1 OPTIMIZATION TECHNIQUES :

Optimization is the process of obtaining the best possible result under any given set of circumstances. Optimization covers a wide range of examples and application. Some typical examples of optimization that arise in practice are one Economic problem, Commercial problem, Aerodynamics problem and the Engineering science etc. Most of the problem in the analysis of industrial process and manufacturing plants can be reduced to an optimization problem. Thus the scope of the optimization problem can be entire business enterprise, a process, a single operation or any intermediate stage between these. It should be noted that optimization may not only modify the system variables, but in addition can alter the system itself.

Since the optimization may usually be written in some mathematical form, it follows that the actual physical situation need be taken into account only when constructing the model. The optimization problem is then a purely mathematical one. The first, step, therefore in a mathematical optimization problem is to determine the maximum or minimum value of a function of several variables subject possibly to one or more constraints. The constraints are equalities and inequalities which must be satisfied by the variables of the problem. But many other types of constraints are possible, e.g., a solution in integers may be required. The next step is to use a
mathematical method to solve the optimization problems, such methods are usually called optimization techniques or algorithm. The optimization method considered here are known collectively as mathematical programming methods.

Generally we can define a mathematical program as:

Minimize \( f(x) \) ...........(1.1.1)
subject to \( g_i(x) \leq 0, i = 1, \ldots, m \) ...........(1.1.2)
\( h_j(x) \leq 0, j = 1, \ldots, n \) ...........(1.1.3)

Where \( x \) is an \( n \)-dimensional vector called the column vector. A mathematical programming problem is to determine a vector \( x \) that satisfies (1.1.2) and (1.1.3) such that the value of objective function is optimum, i.e., minimum. If both objective function \( f(x) \) and all the constraints are linear function of the \( x_j \), we call it a linear programming problem. Linear programming is still one of the most widely used optimization techniques. If one or more of the function are nonlinear in \( x_j \), we call it a nonlinear program.

A linear programming problem can be solved by the simplex algorithm, which was devised by G.B. Dantzig(1947). It takes huge amount of literature exists on this topic, including S. I. Gass(1958), G. Hadley(1962), and G.R. Walsh(1971).

Unlike linear programming, nonlinear programming does not follow some or all of the properties of the linear programming problem. The early work in the theory of general nonlinear programming was directed towards characterizing its solution, the derivation of necessary and sufficient condition for a solution and the construction of closely
related problems "dual" to a given problem. The practical importance of some of the nonlinear programming problem has inspired the rapid development of many algorithms for solving them. Of these the gradient projection method by Rosen (1960), the cutting plane method by Kelley (1960), the method of feasible direction by Zontendizk (1960), penalty function method by Zangwill (1967), method of centers by Huard (1967), sequential unconstrained minimization techniques by Fiacco and McCormick (1968) and ellipsoid algorithm by Ecker and Kupferschmid (1983) are worth mentioning. An extensive study regarding the convergence of the procedure has been made by Zangwill (1969).

A nonlinear programming problem in which a convex objective function is to be minimized and the constraints which form a convex set is called a convex programming problem. Of special interest are the problem with nonlinear constraints and convex objective function. In Chapter V of this thesis we have considered the chance constrained programming problem with nonlinear constraints in which the idea of Ecker and Kupferschmid for ellipsoid algorithm has been used. The procedure is shown to be convergent for convex functions.

Linear programming problem is a decision making problem in which all the parameters of the problem are deterministic. But, if some or all of the parameters appearing in mathematical programming problem are stochastic or random variable, we call it a stochastic linear programming problem. A stochastic linear programming problem can be solved by means of chance constrained and two stage programming
method, which was discovered by A. Charnes and W.W. Cooper (1959) and G.B. Dantzig (1955).

When all the problem variables are integer in a mathematical programming problem, we call it an (all) integer programming problem. But, if some of the problem variables are integer we call it a mixed integer programming problems. All integer and mixed integer linear programming problems can be solved by the cutting plane algorithm of Gomory (1960) and the branch and bound algorithm of Land and Doig (1960).

1.2 OPTIMIZATION UNDER UNCERTAINTY:

1.3 STOCHASTIC PROGRAMMING PROBLEM

Many optimization problems arisen in practice are formulated as stochastic programming problems. However, in most of them, the parameters describing the problems are unknown or known with uncertainty. Stochastic programming deal with situations where some or all random or stochastic parameters appear in a formulation of mathematical program. Examples of random parameter are random demands, random inflows, random yields, etc.

Decision models of stochastic programming have been designed to treat the cases when a decision has to be chosen before a realization of random parameter can be observed. Two recent developments in the field of stochastic programming have raised new interest in obtaining robust solution in management decision problem under uncertainty. One is the development of the principle of scenario aggregation by Rockafellar and Wets (1987) and the second development is the approach of data envelopment analysis, originally developed by Charnes et al. (1978) and extended by Charnes et al. (1989) and Sengupta (1988).

Management decision making usually requires consideration of uncertainty, as well as multiple, often conflicting objectives. This uncertainty may be inconsequential, making assumption of certainty appropriate under certain conditions. However, many management decisions involve statistically determined measures of risk where the probability distribution of outcomes can be described. Consideration of more precise description of uncertainty allows more accurate prediction of decision outcomes.
A stochastic linear programming problem can be stated as follows:

Minimize \( \sum_{j=1}^{n} c_j x_j \) ......(1.3.1)

subject to \( \sum_{j=1}^{n} a_{ij} x_j \leq b_i \), \( i=1,2,\ldots,m \) ......(1.3.2)

\( x_j \geq 0 \), \( j=1,2,\ldots,n \) ......(1.3.3)

where \( x_j \) are decision variables, \( c_j \), \( a_{ij} \), and \( b_i \) are random variables with known probability distribution.

The basic idea of all stochastic programming problem is to convert the stochastic or probabilistic problem into an equivalent deterministic problem. The unknown values of the decision variable may be assumed deterministic.

An alternative procedure is the use of expected values in the objective function. Usually the objective function is either a profit maximization or a cost minimization. If we also take the expected value of the random variables occurring in the constraints and then solve the resulting deterministic problem, such a solution will be called as expected value solution Madansky(1962).

The other approaches have been developed to handle special case of the general problem, the idea of employing deterministic equivalence will be illustrated by introducing the technique of chance constrained programming and two stage programming which is described in the succeeding sections.
1.4 CHANCE CONSTRAINED PROGRAMMING

Chance constrained programming was developed as a means of describing constraints in the form of probability levels of attainment (Charnes and Cooper (1959, 1962, 1963)). Consideration of chance constraints allows the decision maker to consider objective in terms of probability of their attainment. If $\alpha$ is a predetermined confidence level desired by the decision maker, the implication is that a constraint will be violated at most $(1-\alpha)$ of the time.

Chance constrained programming permits the constraints to be violated by a small probability. It is not required that the constraint should always hold, but these must hold simultaneously or individually with given probabilities. The general chance constrained linear program is of the form,

$$\text{Minimize } \sum_{j=1}^{n} c_j x_j \quad \text{........(1.4.1)}$$

subject to $\quad \Pr\left[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \right] \geq \alpha_i, i=1,2,\ldots,m \quad \text{........(1.4.2)}$

and $\quad x_j \geq 0, \quad j=1,2,\ldots,n \quad \text{........(1.4.3)}$

Where $c_j$, $a_{ij}$ and $b_i$ are random variables and $\alpha_i$ are specified probabilities ($0 \leq \alpha_i \leq 1$). This means that the constraint $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ may be violated some of the time through at its most for $100(1-\alpha)\%$ of the time.

A general approach to solving the above type of problem is to reduced them to ordinary linear programming problem by finding the deterministic constraint (i.e., constraints not containing any probabilistic element), which are equivalent to chance constraint.

The deterministic equivalent of the chance constrained problem is
nonlinear. In Chapter V we have considered the above problem in which the idea of ellipsoid algorithm has been used.

Application of chance constrained model exist in many areas: energy planning, industrial production, capital budgeting, water system planning (see Hogan, Morris and Thompson 1981).

1.5 TWO STAGE PROGRAMMING

For solving a stochastic programming problem G.B. Dantzig suggested another programming problem called two stage programming problem. The two stage programming does not permit any constraint to be violated in contrast to chance constraint programming.

The solution of two stage stochastic programming problem consists of deterministic and random vectors. At the first stage in the solution of problem the deterministic plan is considered. It is done prior to the random conditions of the problem. Once the random vector becomes known, is called the second stage of the problem. Usually we minimize the mean (expected) value of summary costs, which includes not only the expenditure at the initial planning stage but also at the second stage when it is necessary to compensate for the divergencies in the system of constraints for the problem.

A general formulation of the above situation can be stated as follows (Beale(1955) and Dantzig(1955)):

\[
\text{Minimize } \sum_{\mathbf{y}} (c \mathbf{x} + f \mathbf{y}) \\
\text{subject to } A\mathbf{x} + B\mathbf{y} = \mathbf{b} \\
\mathbf{x}, \mathbf{y} \geq 0
\]
Where $A$ is a random $m \times n_1$ matrix with known distribution, $B$ is a known $m \times n_2$ matrix, $x$ and $y$ are $n_1$ and $n_2$ dimensional vectors and $b$ is random dimensional vector with known distribution and $c$ and $f$ are known $n_1$ and $n_2$ dimensional vectors.

The above program requires that a vector $x \geq 0$ must be found before the actual values of the random components in the problem become known, and when they become known a recourse $y$ must be found from the following second stage program:

Find $y$ which

$$\text{Minimizes } f'y$$

subject to $By = b - Ax$

$$y \geq 0$$

where $b$, $x$, $d$, $B$ and $A$ are all known. If we denote the minimum by $h(x,b)$ then the original two stage problem can then be written as the equivalent deterministic program:

$$\text{Minimize } c'x + E[h(x,b)], \ x \geq 0$$

where $h(x,b) = \text{Min } f'y$

The conditions for the existing a minimum to the above program under various situation has been studied by Charnes, Kirby and Raike(1967) and Wets(1972). The necessary and sufficient conditions for the minimum of the above program are also given in Walkup and Wets(1967,1970).

We have used this formulation in chapter IV for solving the problem of investment optimally when demand is random with known probability distribution.
1.6 KNAPSACK PROBLEM WITH UPPER BOUNDS

A knapsack problem is an integer linear programming problem with only a single constraint. A knapsack problem without the upper bounds is as follows:

\[
\begin{align*}
\text{Maximize} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b \\
& \quad x_j \geq 0 \text{ and integers}
\end{align*}
\]

where the variable are ordered so that \( c_1/a_1 \geq \ldots \geq c_n/a_n \).

A procedure for its solution is as follows, see Saaty (1960).

The first lexicographic solution is

\[
x^0 = \left[ \frac{b - \sum_{j=1}^{i-1} a_j x^0_j}{a_i} \right], \quad i=1, \ldots, n
\]

Where \([x]\) denote the integer part of \( x \).

An optimal solution can be obtained by enumerating all the feasible solutions in lexicographic orderings. (A lexicographic ordering of a set of solution is an ordering of the solutions according to the first components and if there a tie then according to the second components, and soon, the solution with the larger components being considered larger). A number of feasible solutions are eliminated from the enumeration process as follows.

Let the \( k \)th feasible solution enumerated in the procedure be \( x^k_1, \ldots, x^k_n \), such that \( x^k_s > 0 \) and \( x^k_{s+1} = \ldots = x^k_n = 0 \). Define a vector \( y^k \) which coincides with \( x^k \), except that \( y^k_s = x^k_{s-1} \geq 0 \).

Compute

\[
g^k = \sum_{j=1}^{n} c_j y^k_j + \frac{c_{s+1}}{a_{s+1}} \left( b - \sum_{j=1}^{s} a_j y^k_j \right)
\]
Now let \( \bar{x} \) yield the maximum value \( \sum_{j=1}^{n} c_j x_j \) among all the solutions which come on or before \( x^k \) in the ordering. Then if \( g^k \leq \bar{c} \bar{x} \), \( x^{k+1} = y_k \) is taken to be the next vector for enumeration.

If \( g^k > \bar{c} \bar{x} \), then the next vector enumeration \( x^{k+1} \) is taken to be the next lexicographic vector.

Now consider the problem in (1.6.1) together with the upper bounds on \( x_j \):

\[
x_j \leq u_j, \quad j = 1, \ldots, n
\]

.....(1.6.4)

The largest lexicographic solution is obtained as

\[
x_j = \min \left\{ \left[ \left\lfloor \frac{b - \sum_{i=1}^{j-1} a_{i_j} x_i} {a_j} \right\rfloor \right] , \left\lfloor (u_j) \right\rfloor \right\}, \quad j = 1, \ldots, n
\]

.....(1.6.5)

At \( k+1 \)th step if \( g^k < \bar{c} \bar{x} \), we take \( x^{k+1} = y_k \). In case \( g^k > \bar{c} \bar{x} \), the solution next to \( x^k \) is obtained by decreasing the last positive component of \( x^k \), say \( x^k_s \), by unity and then increasing \( x^k_{s+1} \) as much as possible such that it remains \( \leq u_{x+1} \).

In chapter II of this thesis we have developed a procedure for solving the cutting stock problem (knapsack type problem).

1.7 KNAPSACK PROBLEM WITH MINIMUM SLACK :

For obtaining a solution of a knapsack problem with minimum slack, we try to solve the following knapsack problems successively for \( k=0,1,2,3, \ldots \):
Maximize \[ \sum_{j=1}^{n} c_j x_j \]
subject to \[ \sum_{j=1}^{n} a_j x_j + k = b \]
\[ x_j \geq 0 \text{ and integers}. \]

If a feasible solution is obtained for \( k = 1 \), while no solution is found for \( k = 1, \ldots, l-1 \) then the first lexicographic solution with minimum slack to the problem (1.7.1.) will be the first lexicographic solution to

Maximize \[ \sum_{j=1}^{n} c_j x_j \]
subject to \[ \sum_{j=1}^{n} a_j x_j + 1 = b \]
\[ x_j \geq 0 \text{ and integers}. \]

1.8 ELLIPSOID ALGORITHM:

Ellipsoid algorithm has been applied for solving a variety of important optimization problems. Ellipsoid algorithm was first introduced by D.B. Iudin and A.S. Nemirovskii (1976) for linear program and then clarified by N.Z. Shor (1977). Later, L.G. Khachiyan (1979) modified the model to obtain a polynomial time algorithm for the feasibility problem for the system of the linear inequalities. Later, Robert G. Bland, Donald Goldfarb and M.J. Todd (1981) extended the Khachiyan results.

For solving nonlinear programming problems a general formulation to this method was first proposed by Shor (1977). A general description of the basic problem may also be found in M.

The basic idea of the ellipsoid algorithm is as follows:

We generate a sequence of successively smaller ellipsoids each containing the optimal point. This is done by cutting the previous ellipsoid in half and then enclosing the half that is known to contain the optimal point with the smallest possible new ellipsoid. By doing so, we observe that whether the current ellipsoid centre is feasible or not. If it is not feasible, i.e., a constraint is violated at the current ellipsoid centre, we compute the cutting hyperplane that supports the constraints contour at the current ellipsoid centre and constructing the next ellipsoid. The phase 1 iteration of ellipsoid algorithm is completed. In phase 2, if all the violated constraints are satisfied, the next ellipsoid centre will be feasible, so the objective function must be used for defining the cutting hyperplane. Continuing in this manner, we get sequence of iterates converging to the final values.

The average convergence of the ellipsoid algorithm is linear. For the surety of convergence of this algorithm, the functions must be convex, but often the algorithm converges to K.K.T point even, if the convexity requirement is not satisfied.