CHAPTER 2

FLOW SHOP SCHEDULING WITH TRANSPORTATION TIME BETWEEN MACHINES

2.1 INTRODUCTION: The idea of two production stages in tandem was given by Jackson (1954) while studying a queueing system concerned with an industry in which the production of an item takes place in two distinct but successive stages. Such stages are called by Jackson in tandem (or in series). Johnson (1954) and Bellman (1956) studied the problem of scheduling of n jobs on two machines arranged in tandem where time required to transport jobs from the first machine to the second was assumed to be negligible. Maggu and Das (1980) introduced the concept of transportation time in going from one stage to the other. They studied a system in which an infinite number of transport agents were available and no transport agent was required to return to stage 1 from stage 2. Here we study the case where only a single transport agent is available who, after delivering the items at Machine 2 has to come back to machine 1 for transporting the next item. We assume that Machine A starts processing the next item immediately after finishing with the preceding one. Typical examples of varying transportation times between the two machines (stations) are found in the situations where the transportation from Machine A to Machine B has to pass through traffic lights sometimes during rush hours and sometimes during normal hours. The other situations considered in this chapter are: (1) machine A does not start processing the next item unless the preceding one has already been taken away by the transport
agent from machine A; (2)-machine A does not start processing the next item unless the preceding one has already been processed on machine B. These restrictions may be required due to the non-availability of a waiting room for the items processed by machine A or due to high inventory costs for holding these items within the passage between machine 1 and 2.

2.2 Two Machines in Tandem with a Single Transport Facility in Between-an Heuristic Approach:

Let us consider n items \( I_1, I_2, ..., I_n \) being processed through two machines (A and B) in the order AB with an agent who transports an item processed at machine A to the machine B and then returns back empty to A to transport the second item to B and so on until all the items were taken to B. Let \( t_i \) be the transportation time for item i to carry it from machine A to machine B; \( A_i, B_i \) are the service times on machines A and B respectively, and \( r_i \) is the returning time from machine B to A after delivering item i. This is described by the following diagram:

![Diagram](image)

Fig. (1) :- Two machines in tandem with a single transport agent

Note that by the time the transport agent finishes with item \( i-1 \), the job of \( i^{th} \) item on machine A may or may not get finished. As the machine A after processing item \( i-1 \) immediately takes up
item i for processing, the item i will wait for transport agent if it is not returned back by that time. The problem is to find an optimal schedule of items so as to minimize the total production time for completing all the items.

2.3 Development of the Procedure:

The following theorem provides a procedure for an optimal schedule:

**Theorem (1):** An optimal schedule is obtained by sequencing the items i-1, i, i+1 such that:

\[
\min (A_i + t_i + R_{i-1} + B_{i+1} + t_{i+1} + R_i) < \min (A_{i+1} + t_{i+1} + R_i, B_i + t_i + R_{i-1})
\]

Where \( R_{i-1} \) is defined as:

\[
R_{i-1} = \begin{cases} 
(t_{i-1} + r_{i-1} - A_i) & \text{if it is positive}, \\
0, & \text{otherwise}.
\end{cases}
\]

**Proof:** Let \( S \) and \( S' \) denote the sequences of items given by:

\[
S = (I_1, I_2, \ldots, I_{i-1}, I_i, I_{i+1}, I_{i+2}, \ldots, I_n).
\]

\[
S' = (I'_1, I'_2, \ldots, I'_{i-1}, I'_i, I'_{i+1}, I'_{i+2}, \ldots, I'_n).
\]

Let \((X_p, X'_p)\) and \((C_X, C'_X)\) be respectively the processing time and completion time of any item \( p \) on machine \( X \) (=A or B) for the sequences \((S, S')\). Let \((t_p, t'_p)\) denote respectively the transportation times of item \( p \) from machine A to machine B for the sequences \((S, S')\). \( r_p \) is the returning time of the transportation agent from machine B to A after delivering the \( p \)th item at machine B.

Note that we have defined

\[
R_{p-1} = t_{p-1} + r_{p-1} - A_p \geq 0.
\]

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The completion time of $p^{th}$ item on machine B is given by

$$C_{B_p} = \max \left( C_{A_p} + t_p + R_{p-1}, C_{B_{p-1}} \right) + B_p. \quad (1)$$

We will choose the sequence $S$ if:

$$C_B < C'B \quad (2)$$

i.e. if

$$\max \left( C_{A_n} + t_n + R_{n-1}, C_{B_{n-1}} \right) + B_n < \max \left( C'_{A_n} + t'_n + R'_{n-1}, C'B_{n-1} \right) + B'_n.$$

As $C_{A_n} + t_n + R_{n-1} = \sum A_i + t_n + R_{n-1} = C'_{A_n} + t'_n + R'_{n-1}$ and $B = B'$,

result (2) will be true if:

$$C_{B_{n-1}} < C'B_{n-1} \quad (3)$$

Proceeding in this way we get that inequality (2) is true if:

$$C_{B_p} < C'B_p \quad (p=i+1, i+2, \ldots, n, \text{ and } i = 1, \ldots, n-1) \quad (4)$$

We now calculate the values of $C_{B_{i+1}}$ and $C'B_{i+1}$:

$$C_{B_{i+1}} = \max \left( C_{A_{i+1}} + t_{i+1} + R_i, C_{B_{i+1}} \right) + B_{i+1}$$

$$= \max \left( C_{A_{i+1}} + t_{i+1} + R_i + B_i + B_{i+1}, \max \left( C_{A_{i+1}} + t_i + R_{i-1}, C_{B_i-1} \right) + B_i + B_{i+1} \right)$$

$$= \max \left( C_{A_{i+1}} + t_{i+1} + R_i + B_i + B_{i+1}, C_{A_i} + t_i + R_{i-1} + B_i + B_{i+1} \right).$$

$$C_{B_i-1} + B_i + B_{i+1} \right)$$

$$= \max \left( C_{A_i-1} + A_i + A_{i+1} + t_{i+1} + R_i + B_{i+1} \right)$$

$$= \max \left( C_{A_i-1} + A_i + t_i + R_{i-1} + B_i + B_{i+1} \right) + B_{i+1}$$

and

$$C_{B_i-1} + B_i + B_{i+1} \right). \quad (5)$$
Similarly

\[ C'B_{i+1} = \max (C'A_{i-1} + A_i + A_{i+1} + t'_{i+1} + R'_i + B'_{i+1}) \]

\[ C'A_{i-1} + A_i + t'_i + R'_{i-1} + B'_i + B'_{i+1} \]

\[ C'B_{i-1} + B'_i + B'_{i+1} \]  \hspace{1cm} (6)

For the sequences \( S \) and \( S' \) it is obvious that

\[ CA_{i-1} = C'A_{i-1} \]

\[ CB_{i-1} = C'B_{i-1} \]

\[ X_i = X'_{i+1} \ ; \ (X=A \text{ or } B) ; \ t_i = t'_{i+1} \]

\[ X_{i+1} = X'_i \ ; \ (X=A \text{ or } B) ; \ t_{i+1} = t'_i \]  \hspace{1cm} (7)

\[ R_{i-1} = R'_i \ ; \ R_i = R'_{i-1} \]

Writing (4) for \( p = i+1 \) and using (7) we get

\[ \max (CA_{i-1} + A_i + A_{i+1} + t_{i+1} + R_i + B_{i+1}, CA_{i-1} + A_i + t_i + R_{i-1} + B'_i + B_{i+1}) \]

\[ \max (CA_{i-1} + A_i + t'_i + R'_i + B'_i + B_{i+1}) \]

\[ \max (CA_{i-1} + A_i + t_i + R_{i-1} + B_i + B_{i+1}) \]  \hspace{1cm} (8)

Subtracting \( (C'B_{i-1} + B'_i + B_{i+1}) \) from both sides, the inequality (8) reduces to:

\[ \max (CA_{i-1} + A_i + A_{i+1} + t_{i+1} + R_i + B_{i+1}, CA_{i-1} + A_i + t_i + R_{i-1} + B'_i + B_{i+1}) \]

\[ < \max (CA_{i-1} + A_i + t_i + R_{i-1} + B_i, CA_{i-1} + A_{i+1} + t_{i+1} + R'_i + B'_i + B_{i+1}). \]
Further Subtracting \((CA + A + t + R + R + t + B + B)\) from each side we have:

\[
\max (-B_i - t_i - R_i - 1 - A_{i+1} - t_{i+1} - R_i) < \max(-B_{i+1} - t_{i+1} - R_i, -A_i - t_i - R_i - 1)
\]

or

\[
\min (A_i + t_i + R_i - 1, B_{i+1} + t_{i+1} + R_i) < \min (A_{i+1} + t_{i+1} + R_i, B_i + t_i + R_i - 1)
\]

\[\rightarrow (9)\]

**Remark:** Note that when we set \(R = 0\) in (9), our model will be the same as that of Maggu and Das (1980).

**Algorithm (1):** Our Problem can be represented in tableau form as follows:

<table>
<thead>
<tr>
<th>Item</th>
<th>Machine A</th>
<th>(t_i)</th>
<th>Machine B</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_i)</td>
<td>(r_i)</td>
<td>(B_i)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>A_1</td>
<td>(t_1)</td>
<td>(r_1)</td>
</tr>
<tr>
<td>2</td>
<td>A_2</td>
<td>(t_2)</td>
<td>(r_2)</td>
</tr>
<tr>
<td>3</td>
<td>A_3</td>
<td>(t_3)</td>
<td>(r_3)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>n</td>
<td>A_n</td>
<td>(t_n)</td>
<td>(r_n)</td>
</tr>
</tbody>
</table>

Where \(A_i, B_i\) are the service times on machines A and B respectively. \(t_i\) is the transportation time of item \(i\) from machine A to machine B and \(r_i\) is the returning time of the transport agent from machine B to machine A after delivering the \(i^{th}\) item.

The result of theorem (1) gives the following procedure for an optimal sequence:

**Step 1:** Assume there are two fictitious machines (G and H) in
place of A and B respectively. Assume that the service times for these fictitious machines are given by \( G_i \) and \( H_i \) where

\[
G_i = R_{i-1} + t_i + A_i
\]

\[
H_i = R_{i-1} + t_i + B_i
\]

**Step 2** Applying Johnson's (1954) rule to the fictitious machine times \( G \) and \( H \) constructed in step 1, we obtain the optimal sequence

**Example(1):** Let a machine tandem queueing problem be given in the following tableau form:

<table>
<thead>
<tr>
<th>Item (i)</th>
<th>Machine A ( (A_i) )</th>
<th>( t_i )</th>
<th>( r_i )</th>
<th>( R_i-1 )</th>
<th>Machine B ( (B_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>--</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>5(Times in Hours)</td>
</tr>
</tbody>
</table>

Where \( r_i = 3 \) (constant) for all \( i \) and \( A_i \), \( B_i \) and \( t_i \) are as defined before.

**Solution:** Let \( G \) and \( H \) be two fictitious machines representing \( A \) and \( B \) respectively. Let \( G_i \) and \( H_i \) be the service times of \( G \) and \( H \) respectively. Then our reduced problem is:
Now we apply Johnson's (1954) rule to the above reduced times. The optimal sequence obtained is: 1, 3, 5, 6, 2, 4. The minimum total production time (waiting time in the system for all the items) is calculated as follows:

<table>
<thead>
<tr>
<th>Item</th>
<th>Machine A</th>
<th>$t_i$</th>
<th>$r_i$</th>
<th>Machine B</th>
<th>$Y=C_iA+t_i+r_i$</th>
<th>Idle time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>in</td>
<td>out</td>
<td></td>
<td>in</td>
<td>out</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>9</td>
<td>3</td>
<td>3</td>
<td>15</td>
<td>23</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>17</td>
<td>6</td>
<td>3</td>
<td>23</td>
<td>32</td>
</tr>
<tr>
<td>6</td>
<td>17</td>
<td>23</td>
<td>3</td>
<td>3</td>
<td>32</td>
<td>37</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>30</td>
<td>4</td>
<td>3</td>
<td>37</td>
<td>41</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>37</td>
<td>5</td>
<td>3</td>
<td>44</td>
<td>47</td>
</tr>
</tbody>
</table>

$Y$ represent the time at which the transport agent returns to machine A to take the next item.

The total processing time of all the items through the system (i.e. total production time) is 47 hours. Idle time for machine A is 10 hours, for machine B it is 11 hours and for the rent is 5 hours. So
machine A is busy 78.7% of the time, machine B is busy 76.6% of the time, and the transportation agent is busy 89.4% of the time.

2.4 CASE OF HIGH INVENTORY COST BETWEEN THE TWO MACHINES:

Let us now consider the situation where machine A should not start processing the next item unless the preceding one has already been taken by the transport agent from machine A. This restriction is required when there is no space in front of machine A for the finished items or the holding cost for an item in front of machine A is very high.

Theorem (2): Under this assumption an optimal schedule can be obtained by sequencing the items i-1, i, i+1 such that:

\[ \min (A_i + t_i, B_{i+1} + t_{i+1}) < \min (A_{i+1} + t_{i+1}, B_i + t_i) \]

Proof: Let \( S, S', X_p, X'_p, C_X, C'_X, t_p, t'_p \) and \( r_p \) be as defined earlier. Under the assumption stated above we have that the completion time of the \( p \)th item at machine A is equal to

\[ \max (C_{A_p}, C_{A_{p-1}} + t_{p-1} + r_{p-1}) \]

Therefore, the completion time of the \( p \)th item on machine B is given by:

\[ C_{B_p} = \max (C_{A_p} + t_p, C_{A_{p-1}} + t_{p-1} + r_{p-1} + t_p) + B_p \]

The sequence \( S \) will be preferable to \( S' \) if (4) holds. Writing (4) for \( p = i+1 \) it follows that the sequence \( S \) will be preferable to \( S' \) if

\[ C_{B_{i+1}} < C'_{B_{i+1}} \]

(10)

Now \( C_{B_{i+1}} = \max (C_{A_{i+1}} + t_{i+1}, C_A + t_i + r_i + t_{i+1}) + B_{i+1} \)
\[ \text{max}(CA_{i+1} + t_{i+1}, CA_i + t_i + r_i + t_i + r_i + t_i + t_{i+1} + B_i + B_{i+1}) = \text{max}(CA_i + t_i + B_i + B_{i+1}, CA_{i-1} + t_{i-1} + r_{i-1} + r_i + t_i + t_{i+1} + B_i + B_{i+1}) \]

and similarly for the sequence \( S' \) we have:

\[ C'B_{i+1} = \text{max}(C'A_{i-1} + A'_{i-1} + t'_{i-1} + B'_{i-1}, \ C'A_i + t_i + B_i + B_{i+1}, \ C'A_{i-1} + A'_{i-1} + t'_{i-1} + B'_{i-1} + B_{i-1} + B_{i+1}) \]

Inequality (10) is equivalent to

\[ \text{max}(CA_{i-1} + A_i + A_{i+1} + t_i + B_i + B_{i+1}, CA_i + t_i + B_i + B_{i+1}, CA_{i-1} + A_i + t_i + B_i + B_{i+1}) \]

By subtracting the third term from both sides of the inequality (11), and further subtracting \( CA_{i-1} + A_i + A_{i+1} + t_i + t_{i+1} + B_i + B_{i+1} \) from the remaining terms of the inequality (11), we have:

\[ \text{max}(-t_i - B_i, -A_i - t_i) < \text{max}(-t_{i+1} - B_{i+1}, -A_i - t_i) \]

or

\[ \text{min}(A_i + t_i, B_{i+1} + t_{i+1}) < \text{min}(A_i + t_i + 1, B_{i+1} + t_i) \]

**Example (2):** We will solve the example 1 under the assumption that machine A does not start processing the next item unless it is already cleared from machine A by the transport agent.
Solution:-- The new processing times on machine A are as shown below. The processing time of the $i$th item = $\max(A_i, t_{i-1} + r_{i-1})$

<table>
<thead>
<tr>
<th>Items (i)</th>
<th>Machine A $(A_i)$</th>
<th>$t_i$</th>
<th>$r_i$</th>
<th>Machine B $(B_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>4</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>3</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>3</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

The problem can be reduced to that of two fictitious machines as follows:

<table>
<thead>
<tr>
<th>Item (i)</th>
<th>Machine G $G_i = A_i + t_i$</th>
<th>Machine H $H_i = B_i + t_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>8</td>
</tr>
</tbody>
</table>

The optimal sequence is: 1,3,5,4,6,2

The total production time can be calculated as follows:
Consider now the situation that machine A should not start processing the next item unless the preceding one has already been processed on machine B.

**Theorem (3):** Under this assumption the sequence $S$ will be preferable to the sequence $S'$ if:

$$\min(A_i + t_i + R_i - 1, B_i + 1) < \min (A_{i+1} + t_{i+1} + R_i, B_i)$$

**Proof:** Let $S$ and $S'$ be the sequences of items given by:

$$S = (I_1, I_2, \ldots, I_{i-1}, I_i, I_{i+1}, I_{i+2}, \ldots, I_n).$$

$$S' = (I'_1, I'_2, \ldots, I'_{i-1}, I'_i, I'_{i+1}, I'_{i+2}, \ldots, I'_n).$$

Let $X_p, X'_p, C_p, C'_p, t_p, t'_p, r_p$ and $R_p$ as defined in Case (1). The completion time of $p^{th}$ item on machine $B$ is given by:

$$C_B = \max(CA_{p-1} + t_{p-1} + R_{p-1} + A + t_p, C_B + t_p) + B_p$$

The sequence $S$ will be preferred to $S'$ if

$$C_{B_n} < C'B_n$$

(13)
As \( CA_n + t_n + R_{n-1} = \sum_{i=1}^{n} A_i + t_i + R_{i-1} = C'A_n + t' + R'_{n-1} \) and \( B'_n = B'_{n+1} \)

result (13) is true if: \( CB_{n-1} < C'B_{n-1} \) \hspace{1cm} (14)

Proceeding in this way we get that inequality (11) is true if:

\[
CB_p < C'B_p \hspace{1cm} (p=i+1, i+2, \ldots, n; \quad \text{and } i=1, 2, \ldots, n-1).
\]

Now we calculate values of \( CB_{i+1} \) and \( C'B_{i+1} \) as follows:

\[
CB_{i+1} = \max(CA_i + t_i + R_i + A_i + t_i + R_i + B_i) + B_{i+1}.
\]

\[
= \max[CA_i + \max(A_{i+1} + t_i + R_{i+1} + A_i + t_i + R_i + B_i + 1, CA_i + t_i + R_i + B_i + B_{i+1})]\)
\]

Similarly for the sequence \( S' \) we have:

\[
C'B_{i+1} = \max (C'A_i + t_i + R_i + A_{i+1} + t_i + R_i + B_i + B_i + B_{i+1})
\]

For the sequences \( S \) and \( S' \) it is obvious that:

\[
CA_{i-1} = C'A_{i-1}; \quad CB_{i-1} = C'B_{i-1}; \quad t_{i-1} = t'_{i-1}
\]

\[
X_i = X'_{i+1}; \quad X_i + 1 = X'_i, \quad (X = A \text{ or } B)
\]

\[
t_i = t'_{i+1}; \quad t_{i+1} = t'_i; \quad R_i = R'_{i+1}; \quad R_i = R'_{i-1}
\]

Writing (15) for \( p = i+1 \) and using (18) we get:
The third term on each side of the inequality (19), being equal may be deleted from both the sides. Further subtract:

\[
\text{max}(CA_{i-1} + t_{i-1} + R_{i-1} + A_i + A_{i+1} + t_i + t_{i+1} + R_i + B_{i+1}), \; \text{CB}_{i-1} + t_i + t_{i+1} + B_i + B_{i+1}) < \text{max}(CA_{i-1} + t_{i-1} + R_{i-1} + A_i + A_{i+1} + t_i + t_{i+1} + R_i + B_{i+1}), \; \text{CB}_{i-1} + t_i + t_{i+1} + B_i + B_{i+1})
\]

(19)

Example (3):-

We consider the example 1 under the assumption that machine A should not start processing the next item unless the preceding one has already been processed on machine B.

We calculate the new processing times on machine A as follows:

\[
A'_{i-1} = A_{i-1}
\]

\[
A'_i = t_{i-1} + R_{i-1} + B_{i-1} + A_i
\]

Therefore our problem becomes:
where \( R_i = t_i + r_i - A_{i+1} \)

Let \( G \) and \( H \) be two fictitious machines representing \( A \) and \( B \) respectively. Then our reduced problem is:

<table>
<thead>
<tr>
<th>Item</th>
<th>Machine ( G )</th>
<th>Machine ( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((i))</td>
<td>( A_i ) + ( t_i + R_{i-1} )</td>
<td>( B_i ) + ( t_i + R_{i-1} )</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>21</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>23</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>22</td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td>27</td>
<td>8</td>
</tr>
</tbody>
</table>

The optimal sequence is: 1, 5, 3, 6, 4, 2. The total production time can be calculated as follows:

<table>
<thead>
<tr>
<th>Item</th>
<th>Machine ( A )</th>
<th>Machine ( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((i))</td>
<td>( t_i )</td>
<td>( r_i )</td>
</tr>
<tr>
<td>1</td>
<td>0 - 5</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>15 - 22</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>37 - 41</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>52 - 58</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>66 - 73</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>81 - 88</td>
<td>4</td>
</tr>
</tbody>
</table>

60 60
Y* represents the time at which the transport agent returns to machine A to take the next item.

In this section we considered the minimization of total production time in a 2 machines flow-shop scheduling with a single transport agent. The general case where machine A starts processing the next item immediately after finishing with the preceding is considered first. In many practical situations this may not happen, so two more restrictions were introduced, namely, when machine A should not start processing the next item unless the preceding one has already been carried away by the transport agent and, secondly, machine A should not start processing the next item unless the preceding one has already been processed on machine B. Computational algorithms are proposed. Typical examples of varying transportation times between the two machines (stations) are found in the situations where the transportation from Machine A to Machine B has to pass through traffic lights sometimes during rush hours and sometimes during normal hours. The Computer Programs for the calculations in numerical examples are given in Appendix A.

2.6 The Problem of n Itmes and Three-Machine in Tandem Involving Transportation times and Break-Down of Machines

In many industrial and production processes, items are processed by a given number of machines in series. Sisson [1961] has pointed out that the researcher must be concerned not only with obtaining an optimal solution but also with the practical and economic application of the solution technique. It is this aspect of
the problem which has led us to look for the realistic scheduling situations. Recently Maggu etc. introduced the idea of break-down time intervals for machines in scheduling theory. This part extends the study due to Khan, Maggu, & Mudawi[1994] minimizing the total production time in 3-machine tandem problem where the new concept of a single transport agent has been introduced along with break-down time interval for machines.

Let us consider a flow-shop with n items (i=1,2,...,n) and three machines in series (U, V and W). Associated with each item are the processing times X_i on machines X(U,V,W).

Let t_i and q_i be the transportation times of item i from machines U to V and V to W, respectively. There is a single transport agent which carries produced items from machine U to V and returns back empty to U to take the next item and so on until all items were taken to V. Also there is another single transport agent which carries finished items from V to W and follows the same process as the first single transport agent. Each transport agent cannot carry more than one item at a time. Let r_{i1} and r_{i2} be their returning times from machine V to U and from W to V, respectively.

Define

\[ R_{i1} = \begin{cases} 
  t_i + r_{i1} - u_{i+1}, & \text{if } t_i + r_{i1} > u_{i+1} \\
  0, & \text{otherwise.}
\end{cases} \]

\[ R_{i2} = \begin{cases} 
  q_i + r_{i2} - v_{i+1}, & \text{if } q_i + r_{i2} > v_{i+1} \\
  0, & \text{otherwise.}
\end{cases} \]

The problem is to find an operation schedule for each machine so as to minimize the total production time necessary to process all the items.
Theorem (4) :- An optimal schedule is obtained by sequencing the items i-1, i, i+1, such that:

\[
\min \left( u_i + t_i + R_i + R_{(i-1)} + V_{i+1} + g_{i+1} + R_i + V_i \right)
\]

Let us first prove the following Lemma.

**Lemma :** Let \( \min u_i + t_i + R_i \geq \max v_i + t_i + R_i \)

Then the following holds: \( \left( u_{p+1} + t_{p+1} + R_{(p+1)} \right) \geq v_{(p+1)} \)

\((p=2, 3, 4, \ldots, n)\)

**Proof:** Let a statement \( p(q) \) for an arbitrary number \( q \) be

\[
p(q) : u_{q+1} + t_{q+1} + R_{q+1} > CV_{q} \quad (q=1, 2, 3, \ldots)
\]

Now for any arbitrary natural number \( q \) we have:

\[
CU_{q+1} = U_{q+1}
\]
\[
CV_{q+1} = u_{q+1} + t_{q+1} + v_{q+1} ; \quad CU_{q+1} = u_{q+1} + u_{q+2}
\]

or \( cu_{q+1} + t_{q+1} + R_{q+1} = u_{q+1} u_{q+2} + t_{q+1} + R_{q+1} \)

Processing times on machines \( u \) and \( v \) and \( v \) and \( w \) must satisfy either one or both of the following constraint relationships:

\[
\min \left( u_i + t_i + R_i + R_{(i-1)} \right) \geq \max \left( V_i + t_i + R_{(i-1)} \right)
\]

or

\[
\min \left( V_i + t_i + R_i + R_{(i-1)} \right) \geq \max \left( V_i + t_i + R_{(i-1)} \right)
\]

Now \( u_{q+1} + t_{q+1} + R_{q+1} > v_{q+1} + t_{q+1} \);
Therefore \( C_u + t_2 + R_{11} > CV_1 \)

Hence, \( p(q) \) is true for \( q = 1 \).

Let statement \( p(q) \) be true for \( q = m \), i.e.,

\[
C_{u_{m+1}} + t_{m+1} + R_{m} > CV_{m}
\]

Then,

\[
CV_{m+1} = \max \left( C_{u_{m+1}} + t_{m+1} + R_{m}, CV_{m} \right) + V_{m+1}
\]

\[
= C_{u_{m+1}} + \left( t_{m+1} + R_{m} + V_{m+1} \right)
\]

\[
C_{u_{m+2}} + t_{m+2} + R_{(m+1)} = C_{u_{m+1}} + \left( U_{m+2} + T_{m+2} + R_{(m+1)} \right)
\]

Now \( U_{m+2} + t_{m+2} + R_{(m+1)} \geq t_{m+1} + V_{m+1} + R_{m} \)

Hence \( C_{u_{m+2}} + t_{m+2} + R_{(m+1)} \geq CV_{m+1} \)

Therefore, \( p(q) \) is true for \( q = m + 1 \). Hence by induction hypothesis

Statement \( p(q) \) is true for every value of \( q \).

Remark 1: If \( \min \left( V_i + t_i + R_{(i-1)} \right) \leq \max \left( U_i + t_i + R_{(i-1)} \right) \) we can easily

proof following the same procedure that:

\[
CV_i \geq C_{u_{i+1}} + t_{i+1} + R_{i+1}
\]

Now we can proceed to the proof of our theorem:

Let \( S \) and \( S' \) be the sequence of items given by:

\[
S = \left( I_1, I_2, \ldots, I_{i-1}, I_i, I_{i+1}, I_{i+2}, \ldots, I_n \right)
\]

\[
S' = \left( I'_1, I'_2, \ldots, I'_{i-1}, I'_{i+1}, I'_i, I'_{i+2}, \ldots, I'_n \right)
\]

Let \( (X_p, X'_p) \) and \( (C_p, C'_p) \) denote the processing and completion times of \( p \)th item on machine \( x(=u, V, W) \) in the schedule \( S \) and \( S' \) respectively. Let \( (t_i, g_i) \) and \( (t'_i, g'_i) \) denote the transportation times of \( p \)th item from machines \( u \) to \( V \) and \( V \) to \( W \) in the process of
sequences \((S, S')\) respectively. Let \((r_1, r_2)\) and \((r'_1, r'_2)\) be the returning times of the transport agent from machines \(V\) to \(u\) and \(W\) to \(V\) in the schedule \(S\) and \(S'\) respectively.

Then it is obvious that:

\[
CV_p = \max \left\{ Cu_p + t_p + R(p-1), CV_{p-1} \right\} + V_p
\]

\[
= Cu_p + t_p + R(p-1) + V_p
\]

The completion time of \(p\)th item on machine \(W\) is given by:

\[
CW_p = \max \left\{ CV_p + g_p + R(p-1), CW_{p-1} \right\} + W_p
\]

\[
= \max \left\{ Cu_p + t_p + R(p-1) + V_p + g_p + R(p-1), CW_{p-1} \right\} + W_p \quad (2.6.1)
\]

Schedule \(S\) is preferable to \(S'\), if:

\[
CW_n < C'W_n \quad (2.6.2)
\]

\[
\max \left\{ Cu_n + t_n + R(n-1), V_n + g_n + R(n-1), CW_{n-1} \right\} + W_n
\]

\[
< \max \left\{ C'u_n + t'_n + R'(n-1), V'_n + g'_n + R'(n-1), C'W_{n-1} \right\} + W'_n
\]

Now

\[
Cu_n + t_n + R(n-1) + V_n + g_n + R(n-1) = C'u_n + t'_n + R'(n-1) + V'_n + g'_n + R'(n-1)
\]

\[
CW_n = C'W_n
\]

Therefore, equation (2.6.2) is true, if:

\[
CW_{n-1} < C'W_{n-1} \quad (2.6.3)
\]

Continuing in the same manner, one can get:

\[
CW_p < C'W_p \quad (p=1, 2, \ldots, n,
\]

\[
\text{and } i = 1, 2, \ldots, n-1 \quad (2.6.4)
\]

Now we proceed to calculate the values of \(CW_{i+1}\) and \(C'W_{i+1}\)
\[
\begin{align*}
\text{CW}_{i+1} & = \max \left\{ CV_{i+1} + g_{i+1} + R_{i_2}, \text{CW}_i \right\} + W_{i+1} \\
& = \max \left\{ \max \left\{ Cu_{i+1} + t_{i+1} + R_{i_1}, CV_i \right\} + V_{i+1} + g_{i+1} + R_{i_2}, \text{CW}_i \right\} + W_{i+1} \\
& = \max \left\{ Cu_{i+1} + t_{i+1} + R_{i_1} + V_{i+1} + g_{i+1} + R_{i_2}, \text{CW}_i \right\} + W_{i+1} \\
& = \max \left\{ Cu_{i+1} + t_{i+1} + R_{i_1} + V_{i+1} + R_{i_2}, \text{CW}_i \right\} + W_{i+1} \\
& = \max \left\{ CV_i + g_i + R_{(i-1)_2}, \text{CW}_{i-1} \right\} + W_i \right\} + W_{i+1} \\
\text{or } \text{CW}_{i+1} & = \max \left\{ Cu_{i+1} + t_{i+1} + R_{i_1} + g_{i+1} + R_{i_2}, CV_i + g_i + R_{(i-1)_2}, CV_{i-1} + W_i \right\} \\
& = \max \left\{ Cu_{i+1} + t_{i+1} + R_{i_1} + g_{i+1} + R_{i_2}, CV_i + g_i + R_{(i-1)_2}, CV_{i-1} + W_i \right\} \\
& = \max \left\{ Cu_{i+1} + t_{i+1} + R_{i_1} + g_{i+1} + R_{i_2}, CV_{i-1} + W_i \right\} + W_{i+1} \\
& = \max \left\{ Cu_{i-1} + u_i + t_{i+1} + R_{i_1} + g_{i+1} + R_{i_2}, CV_{i-1} + W_i \right\} + W_{i+1} \\
& = \max \left\{ Cu_{i-1} + u_i + t_{i+1} + R_{i_1} + g_{i+1} + R_{i_2}, CV_{i-1} + W_i \right\} + W_{i+1} \\
\text{CW}_{i-1} + W_i + W_{i+1} \right\} \\
\end{align*}
\]}

(2.6.5)

Similarly,
\[
\text{C'W}_{i+1} = \max \left\{ C'u_{i-1} + u_i + u_{i+1} + t_{i+1} + R_{i_1} + g_{i+1} + R_{i_2} + V_{i+1} + W_{i+1} \right\}
\]
Comparing the sequences $S$ and $S'$ it is very clear that:

$$C_u_{i-1} = C'u_{i-1}; \quad CW_{i-1} = C'W_{i-1}$$

$$X_i = X'_{i+1}; \quad X_{i+1} = X'_i; \quad (X = u, V \text{ or } W)$$

$$ti = t'_{i+1}; \quad ti+1 = t'_i, \quad g_i = g'_{i-1}, \quad g_{i+1} = g'_1$$

Using (2.6.7) in (2.6.4) gives:

$$\max \left\{ C'u_{i-1} + u_i + ti + R_{(i-1)} + V_i + g_i + R_i + W_i + W_{i+1} \right\}$$

$$\min \left\{ C'u_{i+1} + u_i + ti + R_{(i-1)} + V_i + g_i + R_i + W_i + W_{i+1} \right\}$$

Subtracting $(CW_{i-1} + W_i + W_{i+1})$ from both sides of the inequality (2.6.8), and further subtracting

$$\left\{ C'u_{i-1} + u_i + u_{i+1} + ti + t_{i+1} + R_{(i-1)} + R_i + V_i + g_i + R_i + W_i + W_{i+1} \right\}$$

$$\left\{ CW_{i-1} + W_i + W_{i+1} \right\}$$


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from both sides of the remaining inequality (2.6.8), we have

\[
\begin{align*}
\max & \left\{ -R_{(i-1)1} - R_{(i-2)z} - t_i - g_i - V_i - W_i, \\
& -R_{i1} - R_{iz} - t_{i+1} - g_{i+1} - U_{i+1} - V_{i+1} \right\} \\
< & \max \left\{ -R_{i1} - R_{iz} - t_{i+1} - g_{i+1} - V_{i+1} - W_{i+1}, \\
& -R_{(i-1)1} - R_{(i-2)z} - t_i - g_i - U_i - V_i \right\} \\
\text{Or} & \\
\min \left\{ u_i + t_i + R_{(i-1)1} + V_i + g_i + R_{(i-1)z}, \\
& t_{i+1} + R_{i1} + V_{i+1} + g_{i+1} + R_{iz} + W_{i+1} \right\} \\
< & \min \left\{ u_{i+1} + t_{i+1} + R_{i1} + V_{i+1} + g_{i+1} + R_{iz}, \\
& t_i + R_{(i-1)1} + V_i + g_i + R_{(i-1)z} + W_i \right\}
\end{align*}
\]

\textbf{Remark 2:-} If \( \min (g_i + R_{(i-1)z} + W_i) \geq \max (V_i + g_i + R_{(i-1)z}) \) is taken we can easily prove following the same procedure that:

\[
CW_i \geq CV_{i+1} + g_{i+1} + R_{iz}
\]

In the following, an algorithm is described which will determine an optimal schedule for the problem,

Our problem can be represented in tableau form as follows:
<table>
<thead>
<tr>
<th>Item</th>
<th>Machine u</th>
<th>t</th>
<th>r</th>
<th>Machine V</th>
<th>g</th>
<th>r</th>
<th>Machine W</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>(u_i)</td>
<td>t_i</td>
<td>r_{ij}</td>
<td>(vi)</td>
<td>g_i</td>
<td>r_{iz}</td>
<td>(W_i)</td>
</tr>
<tr>
<td>1</td>
<td>u_1</td>
<td>t_1</td>
<td>r_{11}</td>
<td>V_1</td>
<td>g_1</td>
<td>r_{1z}</td>
<td>W_1</td>
</tr>
<tr>
<td>2</td>
<td>u_2</td>
<td>t_2</td>
<td>r_{21}</td>
<td>V_2</td>
<td>g_2</td>
<td>r_{2z}</td>
<td>W_2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>n</td>
<td>u_n</td>
<td>t_n</td>
<td>r_{ni}</td>
<td>V_n</td>
<td>g_n</td>
<td>r_{nz}</td>
<td>W_n</td>
</tr>
</tbody>
</table>

Where \( u_i, V_i \) and \( W_i \) are the processing times on machines \( u, V \) and \( W \) respectively. \( t_i \) and \( g_i \) are the transportation times of item \( i \) from machine \( u \) to \( V \) and \( V \) to \( W \) respectively. Whereas \( r_{ij} \) and \( r_{iz} \) are the returning times of the transport agent from machine \( V \) to machine \( u \) and machine \( W \) to machine \( V \) after delivering the \( i \)th item. All the information in the above table must satisfy at least one of the following structural relationship.

\[
\min \left(u_i + t_i + R_{(i-1)} \right) \geq \max \left(V_i + t_i + R_{(i-1)} \right)
\]

or

\[
\min \left(W_i + g_i + R_{(i-1)} \right) \geq \max \left(V_i + g_i + R_{(i-1)} \right)
\]

The result of theorem (1) gives the following procedure for an optimal sequence:

**Algorithm (2):**

**step 1:** Reduce the given problem to two machines problem. Assume \( G \) and \( H \) be the fictitious machines with processing times \( G_i \) and \( H_i \) defined by:

\[
G_i = \left[u_i + t_i + R_{(i-1)} + V_i + g_i + R_{(i-1)} \right]
\]

\[
H_i = t_i + R_{(i-1)} + V_i + g_i + R_{(i-1)} + W_i
\]
Step 2: Applying Johnson's [1954] rule to the fictitious machine times $G_i$ and $H_i$ constructed in step 1, we obtain the optimal sequence.

Example (4): Let a machine tandem queueing problem be given in the following tableau form:

<table>
<thead>
<tr>
<th>Item (i)</th>
<th>Machine $u_i$</th>
<th>$t_i$</th>
<th>$r_{i1}$</th>
<th>$R_{i1}$</th>
<th>Machine $v_i$</th>
<th>$q_i$</th>
<th>$r_{i2}$</th>
<th>$R_{i2}$</th>
<th>Machine $w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>7</td>
<td>3</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>9</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>12</td>
</tr>
</tbody>
</table>

(Times in hours)

Where $r_{i1} = 3$ & $r_{i2} = 4$ for all $i$ and $u_i, V_i, W_i, t_i, q_i, R_{i1}$ and $R_{i2}$ are as defined before.

Solution: Let $G$ and $H$ be two fictitious machines. Then our reduced problems is:

<table>
<thead>
<tr>
<th>Item (i)</th>
<th>Machine $G$</th>
<th>Machine $H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_i)</td>
<td>28</td>
<td>27</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>32</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>35</td>
<td>32</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>34</td>
</tr>
</tbody>
</table>

where $G_i$ and $H_i$ as defined in Algorithm 1 (Step 1).

Now we apply Johnson's [1954] rule to the above reduced times. The optimal sequence obtained is: 2, 5, 4, 3, 1. The minimum total production time will be calculated as follows:
\[ Y_1^* \text{ and } Y_2^* \text{ represent the time at which the transport agents return to machine } u \text{ and } V \text{ respectively. The total production time is 82 hours.} \]

For the case of Machine break-down times the algorithm 2 should be changed as follows:

**Algorithm (3):**

**Step 1:** Find the optimal sequence of the items by applying Algorithm (2).

**Step 2:** Read the effect of break-down intervals of machines on all the items. Also find a new problem with the following processing items:

\[ X'_i = X_i, \text{ if } (a,b) \text{ has no effect on item } i. \]

\[ = X_i + L, \text{ if } (a,b) \text{ has an effect on item } i; \]

Where \( X = G = u_i + t_i + R_{(i-1)} + V_i + g_i + R_{(i-1)} \) \\
\( \text{or } X = H = t_i + R_{(i-1)} + V_i + g_i + R_{(i-1)} + W_i \) and \( L \) is the length of the interval \((a,b)\).

**Step (3)** Find the optimal sequence for the new reduced problem in step (2) using Johnson's [1954] rule.
Example (5) : We will solve the example 1 under the assumption that the break-down of machines occur in the interval (34-40) i.e. \( L = (40-34) = 6 \).

Solution : By applying step (1) we will find the optimal sequence of the items which is the same as in example 1. As per step (2), the new processing times are given in the following tableau:

<table>
<thead>
<tr>
<th>Item ((i))</th>
<th>Machine (G')</th>
<th>Machine (H')</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>28</td>
<td>27</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>32</td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>29</td>
</tr>
<tr>
<td>4</td>
<td>41</td>
<td>38</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

As per step (3), the optimal sequence for the original problem is \((2, 5, 4, 3, 1)\) which is the same as when break-down intervals of the machines is not taken into consideration.

Scheduling \(n\) Items on \(m\)-Machines in Tandem Involving Transportation Times:

The problem of scheduling of \(n\) items on \(m\) machines in tandem, the presence of transportation time between successive machine has been considered by (Maggu et al.[1980,1982]). The other authors have assumed the transportation times to be negligible. In many production and industrial processes the plants are located in different places,
and the items are transported from one machine to another for processing. In this section we consider the case of scheduling \( n \) items on \( m \) machines in the presence of a single transport agent working between each pair of successive machines.

Consider \( n \) items \((I_1, I_2, ..., I_n)\) which are processed through \( m \)-machines \( M_j (j=1,2,\ldots,m) \) in this order with no passing allowed. Let \( X_{ij} \) denote the processing time of item \( I_i \) on machine \( M_j \). Let \( t_{ij+j+1} \) be the transportation time of item \( I_i \) from machine \( M_j \) to the next machine \( M_{j+1} \). There is a single transport agent between each pair of subsequent machines, i.e., a single transport agent carries produced items from machine \( M_j \) to \( M_{j+1} \) and returns back empty to machine \( M_j \) to take the next item and so on until all the items were taken to machine \( M_m \). Similarly, there is another transport agent between machines \( M_j \) and \( M_{j+1} \) and so on up to \((M_{m-1} \) and \( M_m \)). Let \( r_i \) be the returning time of the transport agent from machine \( M_{j+1} \) to \( M_j \) after delivering the \( i \)th item at \( M_j \) \((j=1,2,\ldots,m)\).

**Theorem (5):** The optimal schedule minimizing the total processing time is given by the following rule:

Item \( I_i \) precedes item \( I_{i+1} \), if

\[
\min \left( G_i \right) \prec \min \left( G_{i+1} \right), \quad i=1,2,\ldots,n;
\]

Where
\[
G_i = X_{i1} + t_{i1} + R_{(i-1)1} + X_{i2} + t_{i2} + R_{(i-1)2} + \ldots
\]
\[
X_{i(m-1)} + t_{i(m-1)} + R_{(i-1)(m-1)};
\]
\[
H_i = t_{i1} + R_{(i-1)1} + X_{i2} + t_{i2} + R_{(i-1)2} + X_{i3} + \ldots
\]
\[
+ t_{i(m-1)} + R_{(i-1)(m-1)} + X_{im}.
\]"
and $R_{ij} = \begin{cases} t_{ij+j+1} + r_{ij} - x_{(i+1)j} & \text{if it is positive} \\ 0 & \text{otherwise} \end{cases}$ for $i=1,2,\ldots,n-1$ and $j=1,2,\ldots,m$.

Let the following structural relationship hold:

$$\min [t_{is} \rightarrow s+1] \geq \max [t_{is} \rightarrow s+1 + x_{i(s+1)}]$$

(S=1,2,\ldots,j-1).

In order to prove the above theorem, we must first prove the following lemma:

**Lemma:** The completion time of item $I_i$ on machine $j$ is given by

$$C_{ij} = x_{i1} + t_{i1} + r_{i1} + x_{i2} + t_{i2} + r_{i2} + \ldots + t_{ij}$$

$$+ x_{ij}, \quad (j=2,3,\ldots,m-1).$$

**Proof:** Let a statement $P(i)$ be defined as:

$$P(i): C_{ij} = x_{i1} + t_{i1} + r_{i1} + x_{i2} + t_{i2} + r_{i2} + \ldots + t_{ij}$$

$$+ x_{ij}, \quad (j=2,3,\ldots,m-1).$$

for any arbitrary natural number $i$.

We have: $C_{ij} = x_{i1}$

It is obvious that:

$$C_{ij} = x_{i1} + t_{i1} + r_{i1} + x_{i2} + t_{i2} + r_{i2} + \ldots + t_{ij} + x_{ij} + r_{ij}$$

Hence statement $P(i)$ is true for $i=1$, i.e. $P(1)$ is true.

Let statement $P(i)$ be true for $i=k$, i.e., assume $P(k)$ is true, then we have:

$$C_{kj} = x_{k1} + t_{k1} + r_{k1} + x_{k2} + t_{k2} + r_{k2} + \ldots + t_{kj} + x_{kj}$$

--- (I)

Let a new statement $P'(q)$ be defined as:

$$P'(q): C_{(k+1)q} + t_{(k+1)q} + r_{kq} \geq C_{k(q+1)}$$

(q=1,2,\ldots,m-2; m is a natural number)
From (I), we obtain:

\[ CX_{k2} = CX_{k1} + t_{k1\rightarrow 2} + R_{(k-1)}i + X_{k2} \]  

----- (III)

From structural relationship it is clear that:

\[ X_{(k+1)i} + t_{(k+1)i\rightarrow 2} + R_{k1} \geq t_{k1\rightarrow 2} + R_{(k-1)i} + X_{k2} \]  

----- (IV)

Now from (II), and (IV), we have:

\[ CX_{(k+1)i} + t_{(k+1)i\rightarrow 2} + R_{k1} \geq CX_{k2} \]

Hence statement \( P'(q) \) is true for \( q=1 \), i.e., \( P'(1) \) is true.

Let \( P'(q) \) be true for \( q=x \). Then we have:

\[ CX_{(k+1)x} + t_{(k+1)x\rightarrow x+1} + R_{kx} \geq CX_{k(x+1)} \]  

(V)

\[ CX_{(k+1)(x+1)} = \max \left( \left[ CX_{(k+1)x} + t_{(k+1)x\rightarrow x+1} + R_{kx}, CX_{k(x+1)} \right] + t_{(k+1)(x+1)} \right) \]

\[ = CX_{(k+1)x} + t_{(k+1)x\rightarrow x+1} + R_{kx} + t_{(k+1)(x+1)} \]  

[from (V)]

Therefore, from (I), we can get:

\[ CX_{k(x+1)} = CX_{k1} + t_{k1\rightarrow 2} + R_{(k-1)i} + X_{k2} + \cdots + X_{k(x+1)} \]

\[ CX_{k(x+z)} = CX_{k1} + t_{k1\rightarrow 2} + R_{(k-1)i} + X_{k2} + \cdots + X_{k(x+1)} + t_{k(x+1)\rightarrow (x+z)} + R_{(k-1)(x+1)} + X_{k(x+z)} \]  

(VI)

Also, \( CX_{(x+1)(x+1)} + t_{(k+1)(x+1)\rightarrow (x+2)} + R_{k(x+1)} \) = \( CX_{(VII)} \)

\[ CX_{(x+1)(x+1)} + t_{(k+1)(x+1)\rightarrow (x+2)} + R_{k(x+1)} \]

\[ + R_{kx} + X_{(k+1)(x+1)} + t_{(k+1)(x+1)\rightarrow (x+2)} + R_{k(x+1)} \]

From the structural relationship it is clear that:

\[ X_{(k+1)(x+1)} + t_{(k+1)(x+1)\rightarrow (x+2)} + R_{k(x+1)} \geq t_{k(x+1)\rightarrow (x+z)} + R_{(k-1)(x+1)} \]

\[ + X_{k(x+z)} \]  

(VIII)

From (V), (VI), (VII) and (VIII), we have:
Hence statement $P'(q)$ is true for $q = x + i$. By induction hypothesis $P'(q)$ is true. And

$$CX_{k+1}(q+1) \geq CX_{k}(q+1) \quad (IX)$$

(q = 1, 2, ..., m-2)

Let a new statement $P''(l)$ be defined as:

$$P''(l): CX_{k+l}(b+1) = CX_{k+l}(1) + t(k+1)_{1} + R_{k+1} + X_{k+l}(1+1) \quad (l = 1, 2, ..., m-2)$$

$$CX_{k+1} = \max \left( CX_{k+1}(l+1) + t(k+1)_{1} + R_{k+1}, CX_{k+2} \right) + X_{k+1}$$

$$= CX_{k+1}(1) + t(k+1)_{1} + R_{k+1} + X_{k+1} \quad \left[ \text{from (IX)} \right] .$$

Therefore, statement $P''(l)$ is true for $l = 1$.

Let statement $P''(l)$ be true for any arbitrary number $x$. Then we have:

$$CX_{k+l}(x+1) = CX_{k+l}(1) + t(k+1)_{1} + R_{k+1} + X_{k+l}(1+1) + t(k+1)_{2} + R_{k+2} + ...$$

$$+ X_{k+l}(x+1) \quad (X)$$

Now $CX_{k+l}(x+2) = \max \left( CX_{k+l}(k+1) + t(k+1)_{1} + R_{k+1} + X_{k+l}(1+1), \right.$

$$\left. CX_{k+2}(x+2) \right) + X_{k+l}(x+2) \quad \left[ \text{from (IX)} \right]$$

or, $CX_{k+l}(x+2) = CX_{k+l}(1) + t(k+1)_{1} + R_{k+1} + X_{k+l}(1+1) + t(k+1)_{2} + R_{k+2} + ... + X_{k+l}(x+2)$

$$+ t(k+1)(x+1) \rightarrow (x+2) + R_{k}(x+1) + X_{k+l}(x+2) \quad \left[ \text{from (X)} \right]$$

Also statement $P''(l)$ is true for $l = x+1$. By induction hypothesis statement $P''(l)$ is true for all values of $l$. Hence

$$CX_{k+l}(l+1) = CX_{k+l}(1) + t(k+1)_{1} + R_{k+1} + X_{k+l}(1+1) + t(k+1)_{2} + R_{k+2} + ... + X_{k+l}(l+1)$$

Put $l = j-1$. then we have:

$$CX_{k+l}(j) = CX_{k+l}(1) + t(k+1)_{1} + R_{k+1} + X_{k+l}(1+1) + t(k+1)_{2} + R_{k+2} + ... + X_{k+l}(j)$$

Hence statement $P(i)$ is true for $i = k+1$. Now we have $P(1), P(k)$ and
P(k+i) are true.

By induction hypothesis P(i) is true for all values of i.

Therefore, we have:

\[ CX_{i+j} = CX_{i} + t_{i} + R_{i-1} + X_{i+1} + t_{i+1} + R_{i-2} + \ldots + X_{i+j} \]  \hspace{1cm} (2.7.1)

Therefore, the lemma is proved.

Now we can proceed to the proof of our theorem.

**Proof of the theorem:** Let S and S' be the sequences of items given by:

\[ S = \{ I_1, I_2, \ldots, I_{i-1}, I_i, I_{i+1}, \ldots, I_n \} \]

\[ S' = \{ I_1', I_2', \ldots, I_{i-1}', I_i', I_{i+1}', \ldots, I_n' \} \]

Let \( (X_p, X'_p) \) and \( (C_X, C'_X) \) denote the processing and completion times of \( p \)th item on machine \( X = \{ M_1, M_2, \ldots, M_m \} \) in the schedule S and S' respectively. Let \( t_{p(m-1)} \rightarrow m \) and \( t'_{p(m-1)} \rightarrow m \) \((m=2, 3, \ldots)\) be the transportation times of \( p \)th item from machines \( M_i \) to \( M_j \) to \( M_k \) to \( M_l \) to \( M_m \) in the process of sequences \((S, S')\) respectively. Let \( r_1, r_2, \ldots, r_{m-1} \) and \( r'_1, r'_2, \ldots, r'_{m-1} \) be the returning times of the transport agent from machines \( M_m \) to \( M_{m-1} \) \((m=2, 3, \ldots)\) in the schedule S and S' respectively.

The completion time of \( p \)th item on machine \( m \) is given by:

\[ CX_{pm} = \max \left\{ CX_{p(m-1)} + t_{p(m-1)} \rightarrow m + R(p-1)(m-1), CX_{(p-1)m} + X_{pm} \right\} \]

\[ = \max \left\{ CX_{p1} + t_{p1} \rightarrow z + R(p-1) + X_{p2} + t_{p2} \rightarrow z + R(p-1)z + \ldots + X_{p(m-1)} + t_{p(m-1)} \rightarrow m + R(p-1)(m-1), CX_{(p-1)m} + X_{pm} \right\} \]

or \[ CX_{pm} = \max \left\{ CX_{(p-1)m} + X_{p1} + t_{p1} \rightarrow z + R(p-1) + X_{p2} + \ldots + X_{p(m-1)} + t_{p(m-1)} \rightarrow m + R(p-1)(m-1), CX_{(p-1)m} + X_{pm} \right\} \] \hspace{1cm} (2.7.2)

Now it is clear that:

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\[ G_p = \sum_{p=1}^{P} p \rightarrow z^p p^p + \cdots + X^p p^{(m-1)} + t^p (m-1) \rightarrow m^R (p-1)(m-1) \]

\[ (p=1,2,\ldots,n) \]

Hence \( CX_{pm} = \max \left[ CX_{(p-1)^m} + G_p, CX_{(p-1)^m} + X_{pm} \right] \rightarrow (2.7.3) \)

Schedule \( S \) is preferable to \( S' \), if:

\[ CX_{nm} < C'X_{nm} \rightarrow (2.7.4) \]

From (2.7.3), we have:

\[ \max \left[ \left( (n-1)^i + G_n \right) + \left( (n-1)^m + X_{nm} \right) \right] < \max \left[ C'X_{(n-1)^i} + G'_n, C'X_{(n-1)^m} + X'_{nm} \right]. \]

It is obvious that: \( CX_{(n-1)^i} + G_n = C'X_{(n-1)^i} + G'_n \) and \( X_{nm} = X'_{nm} \).

Therefore, equation (2.3.4) is true if:

\[ CX_{(n-1)^m} < C'X_{(n-1)^m} \rightarrow (2.7.5) \]

Continuing in this way, one can get:

\[ CX_{(i+1)^m} < C'X_{(i+1)^m} \rightarrow (2.7.6) \]

From (2.7.3), we have for \( p=i+1 \):

\[ CX_{(i+1)^m} = \max \left( \left( (i+1)^i + G_{i+1} \right) + \left( (i+1)^m + X_{(i+1)^m} \right) \right) \rightarrow (2.7.7) \]

Putting \( p=i \) in (2.7.3), we have:

\[ CX_{i^m} = \max \left[ CX_{(i-1)^i} + G_i, CX_{(i-1)^m} + X_{i^m} \right] \rightarrow (2.7.8) \]

Using equations (2.7.7) and (2.7.8), we get:

\[ CX_{(i+1)^m} = \max \left[ CX_{(i+1)^i} + G_{i+1}, CX_{(i-1)^i} + G_i + X_{i^m}, CX_{(i-1)^m} + X_{i^m} \right] \]

\[ + X_{(i+1)^m} \rightarrow (2.7.9) \]

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Similarly,

$$C'X_{(i+1)m} = \max \left\{ CX_{(i-1)l} + X_{(i+1)l} + G + X_{im}, CX_{(i-1)l} + G_{i+1} + X_{(i+1)m}, CX_{(i-1)l} + X_{(i+1)m} + X_{im} \right\}$$ (2.7.10)

Where $CX_{(i-1)l} = C'X_{(i-1)l}$.

$$CX_{(i-1)l} + X_{im} + X_{(i+1)m} = C'X_{(i-1)l} + X_{(i+1)m} + X_{im}$$ (2.7.11)

Putting (2.7.10) in (2.7.6) we get:

$$\max \left\{ CX_{(i-1)l} + X_{il} + G_{i+1} + X_{(i+1)m}, CX_{(i-1)l} + G_{i+1} + X_{im}, CX_{(i-1)l} + X_{(i+1)m} \right\} < \max \left\{ CX_{(i-1)l} + X_{im} + X_{(i+1)m} \right\}$$ (2.7.12)

Subtracting the third term and $CX_{(i-1)l}$ from each side of the inequality (2.7.12), we get:

$$\max \left\{ X_{il} + G_{i+1} + X_{(i+1)m}, G_{i+1} + X_{im} + X_{(i+1)m} \right\} < \max \left\{ X_{(i+1)l} + G_{i+1} + X_{im}, G_{i+1} + X_{(i+1)m} + X_{im} \right\}$$

or

$$\max \left\{ X_{il} + X_{(i+1)l} + t_{(i+1)l} + R_{i+1} X_{(i+1)m} + S_{i+1}, X_{il} + t_{il} + R_{i-1} X_{i2} + R_{i+1} X_{(i+1)m} + X_{(i+1)m} \right\}$$
Subtract \( \sum_{j=1}^{m} \left( X_{ij} + X_{(i+1)j} \right) \) from both sides of the inequality (2.7.13), we get:

\[
\max \left[ -t_{i1} \rightarrow z -R_{(i-1)} -X_{i2} -t_{i2} \rightarrow a -R_{(i-1)} z \ldots -X_{im},
\right.
\]

\[
-\left( X_{(i+1)1} -t_{(i+1)1} \rightarrow z -R_{(i+1)} \ldots -t_{(i+1)(m-1)} \rightarrow m -R_{(i+1)(m-1)} \right)
\]

\[
< \max \left[ -t_{(i+1)1} \rightarrow z -R_{(i+1)} -X_{(i+1)2} -t_{(i+1)2} \rightarrow a -R_{(i+1)} z \ldots -X_{(i+1)m},
\right.
\]

\[
-\left( X_{i1} -t_{i1} \rightarrow z -R_{(i-1)} \ldots -t_{i(m-1)} \rightarrow m -R_{(i-1)(m-1)} \right)
\]

\[
i.e. \max \left[ -H_i, -G_{i+1} \right] < \max \left[ -H_{i+1}, -G_i \right]
\]

Or

\[
\min \left[ G_i, H_{i+1} \right] < \min \left[ G_{i+1}, H_i \right]
\]

**Optimal Scheduling:** In the following, we give the procedure for solving the stated problem.

Assume that there are two fictitious machines (G and H) with processing times \((G_i \text{ and } H_i)\) defined by:

\[
G_i = X_{i1} + t_{i1} \rightarrow z + R_{(i-1)} + X_{i2} + t_{i2} \rightarrow a + R_{(i-1)} z + \ldots + X_{i(m-1)} + t_{i(m-1)} \rightarrow m + R_{(i-1)(m-1)}.
\]

\[
H_i = t_{i1} \rightarrow z + R_{(i-1)} i + X_{i2} + t_{i2} \rightarrow a + R_{(i-1)} z + i_{(m-1)} + t_{im} > m
\]

Applying Johnson's [1954] rule to the fictitious machine times
(G_i and H_i) constructed above, we obtain the optimal sequence.

Example (6):- Let a machine tandem queueing problem be given in the following tableau form:

<table>
<thead>
<tr>
<th>Item</th>
<th>M_1</th>
<th>t_1</th>
<th>r_1</th>
<th>M_2</th>
<th>t_2</th>
<th>r_2</th>
<th>M_3</th>
<th>t_3</th>
<th>r_3</th>
<th>M_4</th>
<th>t_4</th>
<th>r_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19</td>
<td>6</td>
<td>4</td>
<td>0</td>
<td>9</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>8</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>7</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>10</td>
<td>5</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>9</td>
<td>6</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
<td>7</td>
<td>4</td>
<td>-</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>-</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>-</td>
</tr>
</tbody>
</table>

(Time in hours) | 3 | 3 | 11 | 6 | 4 | 1 | 26 |

Apply step 1, the new reduced problem is:

<table>
<thead>
<tr>
<th>Item</th>
<th>Machine G</th>
<th>Machine H</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_i)</td>
<td>(H_i)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>74</td>
<td>81</td>
</tr>
<tr>
<td>2</td>
<td>82</td>
<td>84</td>
</tr>
<tr>
<td>3</td>
<td>76</td>
<td>80</td>
</tr>
<tr>
<td>4</td>
<td>72</td>
<td>64</td>
</tr>
</tbody>
</table>

As per step 2, the optimal sequence of the problem is: 1,3,2,4.

The minimum total processing time to complete all the items will be calculated as follows:

<table>
<thead>
<tr>
<th>Item</th>
<th>M_1</th>
<th>M_2</th>
<th>M_3</th>
<th>M_4</th>
<th>M_5</th>
<th>Y_1</th>
<th>Y_2</th>
<th>Y_3</th>
<th>Y_4</th>
<th>Y_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_i)</td>
<td>(H_i)</td>
<td>in-out</td>
<td>in-out</td>
<td>in-out</td>
<td>in-out</td>
<td>in-out</td>
<td>in-out</td>
<td>in-out</td>
<td>in-out</td>
<td>in-out</td>
</tr>
<tr>
<td>1</td>
<td>0-19</td>
<td>25-34</td>
<td>39-47</td>
<td>50-53</td>
<td>57-68</td>
<td>74-100</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>19-31</td>
<td>35-41</td>
<td>45-64</td>
<td>70-76</td>
<td>83-91</td>
<td>96-112</td>
<td>29</td>
<td>42</td>
<td>53</td>
<td>70</td>
</tr>
<tr>
<td>2</td>
<td>31-49</td>
<td>54-61</td>
<td>67-77</td>
<td>82-89</td>
<td>95-104</td>
<td>111-131</td>
<td>39</td>
<td>48</td>
<td>73</td>
<td>86</td>
</tr>
<tr>
<td>4</td>
<td>49-70</td>
<td>77-82</td>
<td>85-89</td>
<td>93-98</td>
<td>101-110</td>
<td>131-144</td>
<td>58</td>
<td>70</td>
<td>85</td>
<td>98</td>
</tr>
</tbody>
</table>

The total production time is 144 hours.