5.1 Introduction: In this chapter we consider the problem of interchangeability of servers in tandem queues consisting of two service stations each having a single server. We assume that the system is initially empty. The service times are assumed to be mutually independent and also independent of the arrival process.

Let us denote by \((D_n)\), the departure process from station 2 for the order \((A \rightarrow B)\), and by \((D^n)\), the departure process when the two servers are interchanged (i.e. for the order \((B \rightarrow A)\)). We will study the effect on the departure process of customers from the last station by the interchange of the servers.

Friedman [1965] showed that the departure process of customers from the last station is not affected by the interchange of servers when the service at each station is deterministic. Weber [1979] and Lehtonen [1986] etc., have shown the same results for exponential service at each station. Friedman [1965] considered tandem systems with multiple servers at each station and infinite buffers between stations. He showed for an arbitrary arrival process, that when the service time at each station is deterministic, the epoch at which the customer departs from the system is independent of the order of the stations. His result does not hold in general, if the service time at each station is
random. However, for a single server at each station and the exponential service times at all the stations the result holds.

Weber [1979] showed for given finite number of empty $0/M/1$ queues in tandem, where customers arrive according to an arbitrary arrival process and receive service at each queue exactly once in some fixed order, that $(D_n)$ and $(\bar{D}_n)$ are statistically indistinguishable for any $t$. His proof is based on a Laplace-transform method. Pinedo [1982] also considered deterministic and nonoverlapping distributions for systems that are initially empty and have an infinite number of customers waiting for service and suggested some general rules. Whitt [1985] applied approximation methods for networks of queues to obtain heuristic design principles for queues in tandem. The same result of Weber [1979] was proved by Lehtonen [1986] who presented a different proof for this interchangeability result using a sophisticated coupling technique.

Kijima and Makimoto [1990] gave a simple and direct proof of interchangeability with a slight extension of the result. Chao and Pinedo [1990] considered a system consisting of two stations in tandem with an infinite buffer in front of the first station and no buffer between the stations. They assumed that customers arrive in batches according to a poisson process and arbitrary service time distributions at the two stations. They showed that if the service times are either both exponentially distributed with different means or both deterministically distributed with different means, an
interchange of the two stations does not affect the expected time of
a customer in system and the expected number of customers in system.
They also studied in [1992] the effect of the order of service
stations on the departure process in a tandem system with finite
buffers and blocking. They showed that a reversibility result holds
for a three station tandem system with no buffers between stations
and with communication blocking. They also stated a general
conjecture regarding the reversibility of tandem systems with finite
buffers and blocking.

5.2 Model When no Queue is Allowed Infront of Any Server:

Ding and Greenberg [1991] considered a queueing system with no
queue allowed to be formed for any of the two servers. Each customer
arrives according to a Poisson process at rate \( \lambda \). If the second
server is busy and the first server has completed his service on a
customer, then that customer is blocked at the first server, and he
stops working until the second server is free and the customer leave
him to go to the second server. Any arriving customer is lost, if he
finds that the first server is busy or blocked. Denote the first
server by \( S \) and the second server by \( Y \). The service times of the \( i^{th} \)
customer at the first server is denoted by \( S_i \) and at the second
server by \( Y_i \). Let \( X_i \) be time between the starting of service of
customer \( i-1 \) at the second server and the arrival of customer \( i \) at
the first server.

Consider first the arrangement of the two servers in the order
\( S --> Y \) and call this as A: and assume that at \( t=0 \) the system is empty.
Since the arrival process is Poisson, the epochs at which customers leave the first server and go to the second one constitute a renewal process and the $n^{th}$ customer departure from arrangement $A$ occurs at time:

$$D_n^A = X_1 + S_1 + \sum_{i=1}^{n-1} \max \left( X_{i+1} + S_{i+1}, Y_i \right) + Y_n,$$

where $X_i$ are independent and exponentially distributed with mean $\frac{1}{\lambda}$.

Now denote the arrangement of the two servers in the order $Y \rightarrow S$ by $B$, then for $B$ the $n^{th}$ departure occurs at time

$$D_n^B = X_1 + Y_1 + \sum_{i=1}^{n-1} \max \left( X_{i+1} + Y_{i+1}, S_i \right) + S_n.$$

Let $W_i^A$ equal $\left( X_i + S_i + Y_i \right)$, and $W_i^B$ equal $\max \left( X_i + S_i, Y_{i-1} \right)$ for $i=1,2,\ldots,n$. Similarly, define $W_i^B$ for $i = 1,2,\ldots,n$. Since $\left( X_i + S_i + Y_n \right)$ has the same distribution as $\left( X_1 + Y_1 + S_n \right)$, the conditions can be determined on $S$ and $Y$ that yield:

$$W^A = \max \left( X+S, Y \right) \geq \max \left( X+Y, S \right) = W^B.$$

Since $S \geq Y$, we can construct the independent random vectors $(S_i, Y_i)$, such that $S_i \geq Y_i$, for $i=1,2$. For fixed $x \geq 0$, define $h_i = \max(x+S_i, Y_2)$, $h_2 = \max(x+S_2, Y_1)$, $g_1 = \max(x+Y_1, S_2)$ and $g_2 = \max(x+Y_2, S_1)$. It can easily be checked that $h_1 + h_2 \geq g_1 + g_2$. This implies that $E \left[ W^A \right] \geq E \left[ W^B \right]$. Since the mean time between renewals is larger in the first order than in the second, the mean time between departures is also larger in the first order than the second. It is, thus, concluded that it is better to have the faster server first.

In all the above studies, it is assumed that the transportation time of moving the item from one server to the next is negligible. In many realistic situations (cited in chapter 2) this is not true.
We consider here a model in which the transportation time from one station to the next is positive. For such models it is shown that the interchanging of servers does not effect the departure of customers from the system.

5.3 Model With Positive Transportation Time:

We assume that the service times are mutually independent and also independent of the arrival process. Let $A_n$ and $B_n$ be the service times of the $n^{th}$ customer at stations 1 and 2, respectively; and $U_n$ be his transportation time from station 1 to station 2. Let $t_n(0 \leq t_1 \leq t_2 \leq ...)$ be the arrival epoch of the $n^{th}$ customer. Let $S_n^{(1)}$ and $S_n^{(2)}$ be the departure times of the $n^{th}$ customer from the system for the orderings $(A\rightarrow B)$ and $(B\rightarrow A)$ respectively.

5.4 Interchangeability of the Servers:

The departure time of the $n^{th}$ customer from the system for both orderings (i.e. $A\rightarrow B$ and $B\rightarrow A$) is

$$S_n = \max\{t_n', S_{n-1}'\} + U_n + X_n; \quad (n=1,2,...) \text{ and } X = (A \text{ or } B) \quad \Rightarrow (5.4.1)$$

Proof: By induction on $n$, it follows that:

$$S_n = \max_{1 \leq i \leq n} \left[t_i + U_i + \sum_{j=i}^{n} X_j\right], \quad \text{where } X = (A \text{ or } B) \quad \text{and} \quad (n=1,2,...) \quad \Rightarrow (5.4.2)$$

For the ordering $(A\rightarrow B)$, we have from (5.4.2):
The departure time of nth customer from station 1 is:

\[ L_n = \max_{1 \leq i \leq n} \left\{ t_i + \sum_{k=i}^{n} A_k \right\} \]  \hspace{1cm} \text{------>(5.4.6)}

His waiting time up to his entrance service in the second station is:

\[ L'_n = \max_{1 \leq i \leq n} \left\{ t_i + \sum_{k=i}^{n} A_k \right\} + U_i \]

It then follows that:

\[ L'_n = \max \left\{ L_{n-1}, t_n \right\} + A_n + U_n \]  \hspace{1cm} \text{------>(5.4.7)}

and

\[ S^{(1)}_n = \max \left\{ S^{(1)}_{n-1}, L'_n \right\} + B_n \]

\[ = \max \left\{ S^{(1)}_{n-1} + B_n, L_{n-1} + A_n + U_n + B_n, t_n + A_n + U_n + B_n \right\} \]  \hspace{1cm} (5.4.8)

Fix \( S^{(1)}_i = s^{(1)}_i, \ldots, S^{(1)}_{n-1} = s^{(1)}_{n-1} \) and \( A_n + U_n + B_n = G \). Then from (5.4.8):

\[
S^{(1)}_n = \begin{cases} 
S_{n-1} + G - \min \left[ A_n, S_{n-1} - L_{n-1}, S_{n-1} - t_n \right], & S_{n-1} \geq t_n \\
\left( t_n + G \right), & S_{n-1} < t_n 
\end{cases} \]  \hspace{1cm} (5.4.9)
For $S_{n-1} \geq t$, we note that:

$$\mathbb{P}\left[ \min\left\{ X_n, S_{n-1} - L_{n-1} \right\} > x \left| S^{(1)}_1 = s_1, \ldots, S^{(1)}_{n-1} = s_{n-1}, A_n + U_n + B_n = G \right. \right]$$

$$= \mathbb{P}\left[ L_{n-1} < S_{n-1} - x \left| S^{(1)}_1 = s_1, \ldots, S^{(1)}_{n-1} = s_{n-1} \right. \right] \cdot \mathbb{P}\left[ A_n + U_n > x \left| A_n + U_n + B_n = G \right. \right]$$

(5.4.10)

$L_{n-1}$ and $A_n$ are conditionally independent.

Now consider the system in which $A_n$ and $B_n$ are interchanged. For this case (i.e. $B \rightarrow A$) the departure times $R_n$ and $S^{(2)}_n$ from stations 1 and 2, respectively obtained in a similar way are

$$S^{(2)}_n = \max_{i \leq i \leq n} \left( t_i + U_i + \max_{i \leq j \leq n} \left\{ \sum_{k=i}^{j} B_k + \sum_{k=j}^{n} A_k \right\} \right)$$

(5.4.11)

$$R_n = \max_{i \leq i \leq n} \left\{ t_i + \sum_{k=i}^{n} B_k \right\}$$

(5.4.12)

The waiting time of $n$th customer up to his entrance for service in the second station is:

$$R'_n = \max_{i \leq i \leq n} \left\{ t_i + \sum_{k=i}^{n} B_k \right\} + U_i$$

$$= \max \left\{ R_{n-1}', t_n \right\} + U_n + B_n$$

(5.4.13)

and $S^{(2)}_n = \max\left\{ S^{(2)}_{n-1}, R'_n \right\} + B_n$

In this case i.e. $B \rightarrow A$, we have for $S_{n-1} \geq t$:

$$\mathbb{P}\left[ \min\left\{ B_n, S_{n-1} - R_{n-1} \right\} > x \left| S^{(2)}_1 = s_1, \ldots, S^{(2)}_{n-1} = s_{n-1}, A_n + U_n + B_n = G \right. \right]$$

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Suppose:

\[ P \left[ R_{n-1} < X_{n-1} - X \left| S_1^{(1)} = S_1, \ldots, S_{n-1}^{(1)} = S_{n-1} \right. \right] \cdot P \left[ A_n + U_n > X \left| A_n + U_n + B_n = G \right. \right] \] (5.4.14)

\[ = P \left[ R_{n-1} < X_{n-1} - X \left| S_1^{(2)} = S_1, \ldots, S_{n-1}^{(2)} = S_{n-1} \right. \right] \cdot P \left[ B_n + U_n > X \left| A_n + U_n + B_n = G \right. \right] \] (5.4.15)

for any \( 0 \leq S_1 \leq \ldots \leq S_{n-1} \), \( G \geq 0 \) and \( 0 \leq X \leq S_{n-1} \), for which the conditional distribution are well defined. Then:

\[
\left[ S_n \left| S_1^{(1)} = S_1, \ldots, S_{n-1}^{(1)} = S_{n-1} \right. \right] = \left[ S_n \left| S_1^{(2)} = S_1, \ldots, S_{n-1}^{(2)} = S_{n-1} \right. \right];
\]

where = stands for equality in law. Thus, if \( S_1^{(1)}, \ldots, S_{n-1}^{(1)} = S_1^{(2)}, \ldots, S_{n-1}^{(2)} \) then \( S_1^{(1)}, \ldots, S_{n-1}^{(1)} = S_1^{(2)}, \ldots, S_{n-1}^{(2)} \). We have \( S_1^{(1)} = S_1^{(2)} \) for any \( t \). Therefore, if equation (5.4.15) holds for all \( n \geq 2 \), the departure processes \( (D_n) \) and \( (\overline{D}_n) \) are statistically equivalent.