1. Introduction

A random variable $X$ is said to have power function distribution if the probability density function (pdf) of $X$ is of the form

$$f(x) = p\nu^{-p}x^{p-1}; \quad 0 < x \leq \nu, \nu > 0$$

$$= 0, \text{ otherwise}$$

and the corresponding distribution function (df) is

$$F(x) = \nu^{-p}x^p; \quad 0 < x \leq \nu, \nu > 0$$

Therefore, for the power function distribution, we have

$$F(x) = \frac{x}{p} f(x)$$

Malik (1967) obtained the explicit expression for moment of power function distribution whereas Khan et al. (1983 a, b) established the recurrence relations for single and product moments of order statistics for truncated power function distribution. For record values, Ahsanullah (2004) has given the expression for moments of power function distribution. Also he established the recurrence relations for the moments of record values for power function distribution. In this chapter we have obtained simple expressions for the exact moments of dual generalized order statistics from power function distribution.

---

Part of the results of this chapter is contained in Athar et al. (2007 b)
2. Single moments

**Lemma 2.1:** For the power function distribution as given in (1.2) and any non-negative finite integers $a$ and $b$.

\[
J_\alpha(a,b) = \frac{1}{(m+1)^b} \sum_{i=0}^{b} (-1)^i \binom{b}{i} J_\alpha[a + (m+1)i, 0]
\]  

(2.1)

\[
= \frac{\nu^\alpha}{(m+1)^b} \sum_{i=0}^{b} (-1)^i \binom{b}{i} \frac{1}{t_\alpha[a + (m+1)i]}, m \neq -1
\]  

(2.2)

\[
= \frac{b! p^b \nu^\alpha}{[t_\alpha(a)]^{b+1}}, m = -1
\]  

(2.3)

where \( J_\alpha(a,b) = \int_0^\nu x^{\alpha-1} [F(x)]^a g_m[F(x)] dx \)  

(2.4)

\[
J_\alpha(a,0) = \frac{\nu^\alpha}{t_\alpha(a)}
\]  

(2.5)

and \( t_\alpha(a) = \alpha + a \eta \)

**Proof:** When \( m \neq -1 \), we have

\[
g_m^b[F(x)] = \left[ \frac{1}{m+1} \left( 1 - (F(x))^{m+1} \right) \right]^b
\]

\[
= \frac{1}{(m+1)^b} \sum_{i=0}^{b} (-1)^i \binom{b}{i} [(F(x))^{m+1}]^i
\]

Thus

\[
J_\alpha(a,b) = \frac{1}{(m+1)^b} \int_0^\nu x^{\alpha-1} [F(x)]^a \sum_{i=0}^{b} (-1)^i \binom{b}{i} [(F(x))^{m+1}]^i dx
\]

hence the result (2.1).
Using (2.5) in (2.1), we get (2.2).

At \( m = -1 \) in (2.2), we have

\[
J_\alpha(a,b) = \frac{0}{0} \quad \text{as} \quad \sum_{i=0}^{b} (-1)^i \binom{b}{i} = 0
\]

Since (2.2) is of the form \( 0/0 \) at \( m = -1 \), therefore we have

\[
J_\alpha(a,b) = \frac{\nu^\alpha}{(m+1)^b} \sum_{i=0}^{b} (-1)^i \binom{b}{i} \frac{1}{t_\alpha[a+(m+1)i]}
\]

\[
= \frac{\nu^\alpha}{(m+1)^b} \sum_{i=0}^{b} (-1)^i \binom{b}{i} [\alpha + \{a + (m+1)i\} p]^{-1}
\]

Differentiating numerator and denominator \( b \) times, we have

\[
J_\alpha(a,b) = \frac{\nu^\alpha}{b!} \sum_{i=0}^{b} (-1)^i \binom{b}{i} (-1)^b b! p^b i^b [\alpha + \{a + (m+1)i\} p]^{-b-1}
\]

Thus applying L’Hospital rule we have

\[
\lim_{m \to -1} J_\alpha(a,b) = \frac{p^b \nu^\alpha}{(\alpha + a p)^{b+1}} \sum_{i=0}^{b} (-1)^i+b \binom{b}{i} i^b, \; b > 0
\]

But for all integers \( n \geq 0 \) and for all real numbers \( x \), we have Ruiz (1996)

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x - i)^n = n!
\]

Therefore,

\[
\sum_{i=0}^{b} (-1)^i+b \binom{b}{i} i^b = b!.
\]
Hence \( \lim_{m \to -1} J_\alpha(a,b) = \frac{b! \, p^b \, v^\alpha}{(\alpha + a \, p)^{b+1}}. \)

**Theorem 2.1:** For the power function distribution as given in (1.2) and \( \gamma_r \geq 1, k \geq 1, 1 \leq r \leq n, m \neq -1. \)

\[
E\left(X^{r \alpha} (r, n, m, k)\right) = \frac{p \, C_{r-1}}{(r-1)!} J_\alpha(\gamma_r, r-1)
\]

\[
= \frac{p \, v^\alpha}{(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\Gamma(\gamma_{r-i})}
\]

**Proof:** We have

\[
E\left(X^{r \alpha} (r, n, m, k)\right) = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^\alpha \left[F(x)\right]^\gamma_r \frac{f(x) \, g_m^{r-1} [F(x)] \, dx}{(m+1)^{r-1} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\Gamma(\gamma_{r-i})}}
\]

Now applying (1.3), we get

\[
E\left(X^{r \alpha} (r, n, m, k)\right) = \frac{p \, C_{r-1}}{(r-1)!} \int_0^\infty x^\alpha \left[F(x)\right]^\gamma_r \frac{f(x) \, g_m^{r-1} [F(x)] \, dx}{\sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\Gamma(\gamma_{r-i})}}
\]

and hence the theorem, in view of (2.4) and (2.5).

**Identity 2.1:** For \( \gamma_r \geq 1, k \geq 1, 1 \leq r \leq n \) and \( m \neq -1. \)

\[
\sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\Gamma(\gamma_{r-i})} = \frac{(m+1)^{r-1} (r-1)!}{\prod_{j=1}^{r} \gamma_j}
\]

**Proof:** (2.10) can be proved by putting \( \alpha = 0 \) in (2.7).

**Remark 2.1:** If we put \( m = 0, k = 1 \) in (2.7), we get the result for order statistics.
\[ E(X'^\alpha (r, n, 0, 1)) = E(X'^\alpha_{n-r+1:n}) = p v^\alpha C_{n-r+1:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{t_\alpha (n+i-r+1)} \]  
\[ \quad \text{where } C_{n-r+1:n} = \frac{n!}{(n-r)!(r-1)!} \]

For \( m = 0 \) and \( k = 1 \), Identity 2.1 becomes

\[ \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{(n-r+i+1)} = \frac{(r-1)!}{\prod_{j=1}^{r} (n-j+1)} \]

Thus, we have

\[ E(X'^\alpha_{n-r+1:n}) = p v^\alpha C_{n-r+1:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{t_\alpha (n+i-r+1)} \]

\[ = p v^\alpha C_{n-r+1:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{[\alpha + (n+i-r+1)p]} \]

\[ = v^\alpha C_{n-r+1:n} \frac{(r-1)!}{\prod_{j=1}^{r} (n+\frac{\alpha}{p} - j+1)} \]

And hence we have

\[ E(X'^\alpha_{n-r+1:n}) = \frac{\Gamma(n+1)\Gamma[(\alpha/p) + n-r+1]v^\alpha}{\Gamma(n-r+1)\Gamma[n+(\alpha/p)+1]} \]  
\[ \text{as obtained by Malik (1967).} \]

**Remark 2.2:** Moments of \( k-th \) lower record values from the power function distribution may be obtained in view of \( (2.3) \) and \( (2.6) \) at \( m = -1 \).
Exact moments of dual generalized order statistics from power function distribution

\[ E(X^{\alpha}(r,n,-1,k)) = \frac{(pk)^n \nu^\alpha}{[t\alpha(k)]^n} \]  

(2.13)

by noting \( \gamma_i = k \) and \( C_{r-1} = k^r \).

**Remark 2.3:** For \( m = 0 \) and \( \gamma_r = \alpha - r + 1, \alpha \in \mathbb{R}_+ \), we get the moment of order statistics with non-integral sample size

\[ E\left(X^{\alpha}_{\alpha' - r + 1: \alpha'}\right) = p \nu^\alpha C_{\alpha' - r + 1: \alpha'} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{t\alpha(\alpha' + i - r + 1)} \]  

(2.14)

### 3. Product moments

**Lemma 3.1:** For the power function distribution as given in (1.2) and non-negative integers \( a,b,c \) with \( m \neq -1 \)

\[ J_{\alpha, \beta}(a,0,c) = \frac{\nu^{\alpha+\beta}}{t\beta(c)t\alpha+\beta(a+c)} \]  

(3.1)

where

\[ J_{\alpha, \beta}(a,b,c) \]

\[ = \int_0^\alpha \int_0^\alpha x^{\alpha-1} y^{\beta-1} [F(x)]^a [h_m(F(y)) - h_m(F(x))]^b [F(y)]^c dy dx \]  

(3.2)

**Proof:** From (3.2), we have

\[ J_{\alpha, \beta}(a,b,c) = \int_0^\alpha \int_0^\alpha x^{\alpha-1} y^{\beta-1} [F(x)]^a [F(y)]^c dy dx \]

and hence using (1.2), we get

\[ J_{\alpha, \beta}(a,0,c) = \nu^{-p(a+c)} \int_0^\alpha \int_0^\alpha x^{\alpha+ap-1} y^{\beta+cp-1} dy dx \]

\[ = \frac{\nu^{\alpha+\beta}}{(\beta + cp)[\alpha + \beta + p(a+c)]} \]
and hence the lemma.

**Lemma 3.2:** For the power function distribution as given in (1.2) and any non-negative integers \(a, b, c\).

\[
J_{\alpha, \beta}(a, b, c) = \frac{\nu^{\alpha+\beta}}{(m+1)^{b}} \sum_{j=0}^{b} (-1)^{j} \binom{b}{j}
\]

\[
\times \frac{1}{t_{\beta}[c+(m+1)j]t_{\alpha+\beta}[(a+c)+(m+1)b]} , \quad m \neq -1
\]

(3.3)

\[
J_{\alpha, \beta}(a, b, c) = \frac{b! p^b \nu^{\alpha+\beta}}{[t_{\beta}(c)]^{b+1}[t_{\alpha+\beta}(a+c)]}; \quad m = -1
\]

(3.4)

**Proof:** When \(m \neq -1\):

We have

\[
[h_{m}(F(y)) - h_{m}(F(x))]^{b} = \frac{1}{(m+1)^{b}} [(F(x))^{m+1} - (F(y))^{m+1}]^{b}
\]

\[
= \frac{1}{(m+1)^{b}} \sum_{j=0}^{b} (-1)^{j} \binom{b}{j} [(F(y))^{m+1}]^{j}[(F(x))^{m+1}]^{b-j}
\]

Thus, we have

\[
J_{\alpha, \beta}(a, b, c) = \frac{1}{(m+1)^{b}} \sum_{j=0}^{b} (-1)^{j} \binom{b}{j}
\]

\[
\times \int_{0}^{\infty} \int_{0}^{\infty} x^{\alpha-1} y^{\beta-1} [F(x)]^{a} [(F(y))^{m+1}]^{j}[(F(x))^{m+1}]^{b-j} [F(y)]^{c} dy dx
\]

and hence, we get (3.3) on the application of (3.1).

When \(m = -1\):

Since at \(m = -1\) (3.3) is of the 0/0, so after applying L-Hospital’s rule (3.4) can be proved on the lines of (2.3).
Theorem 3.1: For power function distribution as given in (1.2) and \( \gamma_r, \gamma_s \geq 1, \ k \geq 1, \ 1 \leq r < s \leq n, \ m \neq -1 \).

\[
E(X'^\alpha (r, n, m, k)X'^\beta (s, n, m, k)) = \frac{p^2}{(m + 1)^{r-1}} \frac{C_{s-1}}{(r-1)!(s-r-1)!} \\
	imes \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} J_{\alpha, \beta}[(m+1)(j+1), (s-r-1, \gamma_s)] \tag{3.5}
\]

\[
eq \frac{p^2 v^{\alpha+\beta}}{(m+1)^{s-2}} \frac{C_{s-1}}{(r-1)!(s-r-1)!} \\
	imes \sum_{j=0}^{r-1} \sum_{l=0}^{s-r-1} (-1)^{j+l} \binom{r-1}{j} \binom{s-r-1}{l} \frac{1}{t_\beta(\gamma_{s-l}) t_{\alpha+\beta}(\gamma_{r-j})} \tag{3.6}
\]

and subsequently for \( s = r + 1 \)

\[
E(X'^\alpha (r, n, m, k)X'^\beta (r+1, n, m, k))
\]

\[
= \frac{p^2 v^{\alpha+\beta}}{(m+1)^{r-1}} \frac{C_r}{(r-1)!} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{1}{t_\beta(\gamma_{r+1}) t_{\alpha+\beta}(\gamma_{r-j})} \tag{3.7}
\]

Proof: We have

\[
E(X'^\alpha (r, n, m, k)X'^\beta (s, n, m, k))
\]

\[
= \frac{C_{s-1}}{(r-1)!(s-r-1)!} \left[ v^\alpha y^\beta [F(x)]^m f(x) \right. \\
	imes \left. g_{m-1}^r (F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_{s-1}} f(y) dy \right] dx \tag{3.8}
\]

Since
\[ g_m(F(x)) = \left\{ \frac{1}{m+1} \right\}^{r-1} \left[1 - (F(x))^{m+1}\right] \]

\[ = \frac{1}{(m+1)^{r-1}} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} (F(x))^{(m+1)j} \]

Therefore in view of (1.3), we get

\[ E(X'^\alpha (r,n,m,k) X'^\beta (s,n,m,k)) \]

\[ = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \]

\[ \times \int_0^r \int_0^r x^{\alpha-1} y^{\beta-1} [F(x)]^{(m+1)(j+1)} \]

\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma s} dy \, dx \]

Thus the Theorem is proved in view of lemma 3.1 and lemma 3.2.

**Identity 3.1:** For \( \gamma_r, \gamma_s \geq 1, k \geq 1, 1 \leq r < s \leq n \) and \( m \neq -1 \).

\[ \sum_{l=0}^{s-r-1} (-1)^l \binom{s-r-1}{l} \frac{1}{\gamma_{s-l}} = \frac{(m+1)^{s-r-1}(s-r-1)!}{\prod_{i=r+1}^{s} \gamma_i} \] (3.9)

**Proof:** At \( \alpha = \beta = 0 \) in (3.6), we have

\[ 1 = \frac{C_{s-1}}{(m+1)^{s-2} (r-1)!(s-r-1)!} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{1}{\gamma_{r-j}} \]

\[ \times \left\{ \sum_{l=0}^{s-r-1} (-1)^l \binom{s-r-1}{l} \frac{1}{\gamma_{s-l}} \right\} \]

Now on application of (2.10), we get the required result.
At $r = 0$, (3.9) reduces to (2.10).

**Remark 3.1:** At $m = 0$ and $k = 1$, the product moment of order statistics is

$$E(X'^\alpha (r,n,0,1)X'^\beta (s,n,0,1)) = E(X_{n-r+1:n}^\alpha \cdot X_{n-s+1:n}^\beta )$$

$$= C_{n-s+1,n-r+1:n} \frac{\beta^{\alpha+\beta}}{\Gamma(n-s+1 + \alpha/p) \cdot \Gamma(n-r+1 + \beta/p) ^{\alpha+\beta}}$$

$$\times \sum_{j=0}^{r-1} \sum_{l=0}^{s-r-1} (-1)^{j+l} \binom{r-1}{j} \binom{s-r-1}{l} \frac{1}{t_\beta(n-s+l+1) \cdot t_\alpha+\beta(n-r+j+1)}$$

(3.10)

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} = C_{n-s+1,n-r+1:n}$

Using (2.10) and (3.9), (3.10) may be re-written as

$$E(X_{n-r+1:n}^\alpha \cdot X_{n-s+1:n}^\beta )$$

$$= \frac{\Gamma(n+1) \cdot \Gamma[(\alpha/p) + n-s+1] \cdot \Gamma[(\alpha + \beta)/p + n-r+1]}{\Gamma(n-s+1) \cdot \Gamma[n-r+1 + (\alpha/p)] \cdot \Gamma[n + (\alpha + \beta)/p + 1]} \cdot \frac{\beta^{\alpha+\beta}}{\Gamma(n-s+1 + \alpha/p) \cdot \Gamma(n-r+1 + \beta/p) ^{\alpha+\beta}}$$

(3.11)

At $\alpha = \beta = 1$, (3.11) reduces to

$$E(X_{n-r+1:n} \cdot X_{n-s+1:n})$$

$$= \frac{\Gamma(n+1) \cdot \Gamma[(1/p) + n-s+1] \cdot \Gamma[(2/p) + n-r+1]}{\Gamma(n-s+1) \cdot \Gamma[n-r+1 + (1/p)] \cdot \Gamma[n + (2/p) + 1]} \cdot \frac{\beta^2}{\Gamma(n-s+1 + \alpha/p) \cdot \Gamma(n-r+1 + \beta/p) ^{\alpha+\beta}}$$

(3.12)

as obtained by Malik (1967).

**Remark 3.2:** At $m \to -1$ in (3.7), the moments of $k-th$ record value is given by

$$E(X'^\alpha (r,n,-1,k)X'^\beta (r+1,n,-1,k)) = \frac{(pk)^{n+1} \cdot \beta^{\alpha+\beta}}{t_\beta(k) \cdot t_\alpha+\beta(k)^n}$$

(3.13)
**Remark 3.3:** At $\beta = 0$ in (3.6), we have

$$E \left(X^{\alpha} (r, n, m, k) \right) = \frac{p \nu^\alpha}{(m + 1)^{s-2}} \frac{C_{s-1}}{(r - 1)!(s - r - 1)!} \times \sum_{j=0}^{r-1} \sum_{l=0}^{s-r-1} (-1)^{j+l} \binom{r-1}{j} \binom{s-r-1}{l} \frac{1}{\gamma_{s-l}t_\alpha(\gamma_{r-j})}$$

In view of (3.9), (3.14) becomes

$$E \left(X^{\alpha} (r, n, m, k) \right) = \frac{p \nu^\alpha}{(m + 1)^{r-1}} \frac{C_{r-1}}{(r - 1)!} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{1}{t_\alpha(\gamma_{r-j})}$$

as obtained in (2.7).