1. Introduction

In this chapter a general form of continuous distributions is obtained by considering the conditional expectation of function of dual generalized order statistics (dgos) and then characterizing relationships are established for distributions through dgos. Further, various deductions for order statistics and lower records are discussed.

2. Characterization of distributions when \( m_i = m_j = m \)

Let \( X'(r,n,m,k) \), \( r = 1,2,\ldots,n \) be dgos, then the conditional pdf of \( X'(s,n,m,k) \) given \( X'(r,n,m,k) = x \), \( 1 \leq r < s \leq n \)

\[
f_{s|r}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \times \frac{[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1} [F(y)]^y_{s-1}}{(m+1)^{s-r-1} [F(x)]^{y}_{r+1}} f(y)
\]

and the conditional pdf of \( X'(r,n,m,k) \) given \( X'(s,n,m,k) = y \), \( 1 \leq r < s \leq n \) is

\[
\]

---

Part of the results of this chapter is contained in Khan et al. (2007 b).
Theorem 2.1: Let \( \xi(x) \) be a monotonic and continuous function of \( x \). If

\[
E[\xi\{X'(s,n,m,k)|X'(r,n,m,k) = x\}] = g_{s|r}(x), \quad 1 \leq r < s \leq n \quad (2.1)
\]

then for two consecutive values \( r \) and \( r + 1 \),

\[
F(x) = \exp \left[ -\frac{1}{\gamma_{r+1}} \int_{\gamma_{r+1}}^{\beta} \frac{g_{s|r}(t)}{g_{s|r+1}(t) - g_{s|r}(t)} \, dt \right] \quad (2.2)
\]

where \( g(.) \) is a finite and differentiable function of \( x \).

Proof: We have

\[
E[\xi\{X'(s,n,m,k)|X'(r,n,m,k) = x\}] = g_{s|r}(x)
\]

that is,

\[
\frac{C_{s-1}}{C_{r-1}(s - r - 1)!(m + 1)^{s-r-1}} \times \int_{\gamma_{s-r-2}}^{\gamma_{s-r-1}} \xi(y)((F(x))^{m+1} - (F(y))^{m+1})^{s-r-1}[F(y)]^{\gamma_y-1} f(y) \, dy
\]

\[
= g_{s|r}(x)[F(x)]^{\gamma_{r+1}} \quad (2.3)
\]

Differentiating (2.1) both sides with respect to \( x \), we get

\[
\frac{C_{s-1}[F(x)]^m f(x)}{C_{r-1}(s - r - 2)!(m + 1)^{s-r-2}} \times \int_{\gamma_{s-r-2}}^{\gamma_{s-r-1}} \xi(y)((F(x))^{m+1} - (F(y))^{m+1})^{s-r-2}[F(y)]^{\gamma_y-1} f(y) \, dy
\]

\[
= g'_{s|r}(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1} g_{s|r}(x)[F(x)]^{\gamma_{r+1} - 1} f(x)
\]
Characterization of continuous distributions Through conditional expectation of functions of...

\[ [F(x)]^{\gamma r+2} [F(x)]^m \gamma r+1 f(x) g_{s|r+1}(x) \]

\[ = g'_{s|r}(x) [F(x)]^{\gamma r+1} + \gamma r+1 g_{s|r}(x) [F(x)]^{\gamma r+1} f(x) \]

\[ \gamma r+1 f(x) g_{s|r+1}(x) = \frac{g'_{s|r}(x) [F(x)]^{\gamma r+1}}{[F(x)]^{\gamma r+2} [F(x)]^m} + \gamma r+1 \frac{g_{s|r}(x) [F(x)]^{\gamma r+1} f(x)}{[F(x)]^{\gamma r+2} [F(x)]^m} \]

\[ \gamma r+1 g_{s|r+1}(x) = \frac{g'_{s|r}(x) [F(x)]}{f(x)} + \gamma r+1 g_{s|r}(x) \]

\[ \frac{f(x)}{F(x)} = \frac{1}{\gamma r+1} \frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} \]

\[ \frac{d}{dx} \ln F(x) = \frac{1}{\gamma r+1} \frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} \]

and hence the theorem.

**Corollary 2.1:** \( E[X'(s,n,m,k) | X'(r,n,m,k) = x] = a_{s|r}^* x + b_{s|r}^* = g_{s|r}(x) \) (2.4)

if and only if

\[ F(x) = [ax + b]^c, \quad \alpha < x < \beta \] (2.5)

where \( a, b \) and \( c \) are such that \( F(\alpha) = 0 \) and \( F(\beta) = 1 \), and

\[ a_{s|r}^* = \prod_{i=r+1}^s \frac{c_{i/r}}{1 + c_{i/r}}, \quad \text{and} \quad b_{s|r}^* = -\frac{b}{a} (1 - a_{s|r}^*) \] (2.6)

**Proof:** We have

\[ E[X'(s,n,m,k) | X'(r,n,m,k) = x] \]

\[ = \frac{C_{s-1}}{C_{r-1} (s - r - 1)! (m + 1)^{s-r-1}} \]

\[ \times \int_a^x \left( \frac{F(y)}{F(x)} \right)^{\gamma s-1} \left[ 1 - \left( \frac{F(y)}{F(x)} \right)^{m+1} \right]^{s-r-1} \frac{f(y)}{F(x)} dy \] (2.7)
Let \( u = \frac{F(y)}{F(x)} = \frac{ay + b}{ax + b} \)

then \( y = \frac{(ax + b)u^c - b}{a} \)

Thus (2.7) becomes

\[
E[X'(s,n,m,k) \mid X'(r,n,m,k) = x] = \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \cdot \frac{1}{(m+1)^{s-r-1}} \int_0^{\frac{1}{a}} (ax + b)^{\frac{1}{c(m+1)}} - b \cdot t^{\frac{\gamma_{s-1}}{m+1} - \frac{m}{m+1} [1-t]^{s-r-1}} dt
\]

Set \( u^{m+1} = t \) to get

\[
E[X'(s,n,m,k) \mid X'(r,n,m,k) = x] = \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \cdot \frac{1}{(m+1)^{s-r-1}} \int_0^{\frac{1}{a}} \frac{(ax + b)u^c - b}{a} \cdot t^{\frac{\gamma_{s-1}}{m+1} - \frac{m}{m+1} [1-t]^{s-r-1}} dt
\]

\[= a_{s|r}^* x - \frac{b}{a} (1 - a_{s|r}^* )\]

where

\[
a_{s|r}^* = \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \cdot \frac{1}{(m+1)^{s-r-1}} \int_0^{\frac{1}{a}} \frac{1}{c(m+1)} \cdot t^{\frac{\gamma_{s-1}}{m+1} - \frac{m}{m+1} [1-t]^{s-r-1}} dt
\]

\[= \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \cdot \frac{1}{(m+1)^{s-r-1}} \cdot B \left( \frac{c\gamma_s + 1}{c(m+1)}, s-r \right)
\]

\[= \prod_{i=r+1}^{s} \frac{c\gamma_i}{1 + c\gamma_i}\]

To show that (2.4) implies (2.5), we have

\[g_{s|r+1}(x) - g_{s|r}(x) = a_{s|r+1}^* x + b_{s|r+1}^* - a_{s|r}^* x - b_{s|r}^* = (a_{s|r+1}^* - a_{s|r}^*) (x + \frac{b}{a})\]
as $\alpha_{s|r+1} = \prod_{i=r+1}^{s} \frac{c \gamma_i}{1 + c \gamma_i}$

Therefore,

$$\frac{1}{\gamma_{r+1} g_{s|r+1}(x) - g_{s|r}(x)} = \frac{ac}{ax + b}$$

Thus

$$\frac{f(x)}{F(x)} = \frac{ac}{ax + b}$$

implying

$$F(x) = [ax + b]^c$$

**Remark 2.1:** Let $\xi(x)$ be a monotonic and continuous function of $x$, then

$$E[\xi(X'(s,n,m,k)) \mid X'(r,n,m,k) = x] = \alpha_{s|r}^* \xi(x) + b_{s|r}^*$$  \hspace{1cm} (2.8)

if and only if

$$F(x) = [a \xi(x) + b]^c$$  \hspace{1cm} (2.9)

where $a, b$ and $c$ are such that $F(\alpha) = 0$ and $F(\beta) = 1$ and $\xi(x)$ is the monotonic and continuous function of $x$ in the interval $[\alpha, \beta]$.

This can be proved on the lines of Corollary 2.1 by considering

$$g_{s|r}(x) = \alpha_{s|r}^* \xi(x) + b_{s|r}^*$$
Table 2.1: $F(x) = [a \xi(x) + b]^c$

<table>
<thead>
<tr>
<th>Distribution function</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$\xi(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Power function</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F(x) = a^{-p}x^p$ ,</td>
<td>$a^{-q}$</td>
<td>0</td>
<td>$\frac{p}{q}$</td>
<td>$x^q$</td>
</tr>
<tr>
<td>$0 \leq x \leq a$</td>
<td>$a^{-1}$</td>
<td>0</td>
<td>$p$</td>
<td>$x$</td>
</tr>
<tr>
<td>2. Pareto</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F(x) = 1 - a^p x^{-p}$,</td>
<td>$-a^p$</td>
<td>1</td>
<td>1</td>
<td>$x^{-p}$</td>
</tr>
<tr>
<td>$a \leq x &lt; \infty$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Reflected exponential</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F(x) = e^{\lambda(x-\mu)}$ ,</td>
<td>$\lambda$</td>
<td>1</td>
<td>$1 - \frac{\lambda \mu}{c}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$-\infty &lt; x &lt; \mu$</td>
<td>$\frac{\lambda}{c}$</td>
<td>0</td>
<td>$\theta$</td>
<td>$e^{-x^{-p}}$</td>
</tr>
<tr>
<td>4. Inverse Weibull</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F(x) = e^{-\theta x^{-p}}$ ,</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$e^{-\theta x^{-p}}$</td>
</tr>
<tr>
<td>$0 \leq x &lt; \infty$</td>
<td>$\frac{-\theta}{c}$</td>
<td>0</td>
<td>$\theta$</td>
<td>$e^{-x^{-p}}$</td>
</tr>
<tr>
<td>5. Burr type III</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F(x) = (1 + \theta x^{-p})^{-\lambda}$ ,</td>
<td>$\theta$</td>
<td>1</td>
<td>$-\lambda$</td>
<td>$x^{-p}$</td>
</tr>
<tr>
<td>$0 \leq x &lt; \infty$</td>
<td>1</td>
<td>1</td>
<td>$-\lambda$</td>
<td>$\theta x^{-p}$</td>
</tr>
<tr>
<td>6. Cauchy</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$= \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x-\theta}{\lambda}\right)$ ,</td>
<td>$\frac{1}{\pi}$</td>
<td>$\frac{1}{2}$</td>
<td>$1 \tan^{-1}\left(\frac{x-\theta}{\lambda}\right)$</td>
<td></td>
</tr>
<tr>
<td>$-\infty &lt; x &lt; \infty$</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
Remark 2.2: If we denote \( X_{1:n} \geq X_{2:n} \cdots \geq X_{n:n} \) (lower order statistics) and \( X_{1:n} \leq X_{2:n} \cdots \leq X_{n:n} \) (order statistics), then for

\[
F(x) = [a \xi(x) + b]^c
\]

\[
E[\xi(X_{s:n}')] \bigg| X_{r:n}' = x \bigg] = a_s^* \xi(x) + b_s^* = E[\xi(X_{n-s+1:n}')] \bigg| X_{n-r+1:n} = x \bigg]
\]

(Burkschat et al., 2003)

where \( a_s^* = \prod_{j=r+1}^{s} \frac{c(n-j+1)}{c(n-j+1)+1} \) and \( b_s^* = -\frac{b}{a} (1 - a_s^*) \)

Therefore replacing \( n-s+1 \) by \( r \), we get

\[
E[\xi(X_{r:n})] \bigg| X_{s:n} = y \bigg] = a_2^* \xi(y) + b_2^*
\]

if and only if

\[
F(x) = [a \xi(x) + b]^c
\]

where \( a_2^* = \prod_{j=n-s+2}^{n-r+1} \frac{c(n-j+1)}{c(n-j+1)+1} = \prod_{l=1}^{s-r} \frac{c(s-l)}{c(s-l)+1} \) and \( b_2^* = -\frac{b}{a} (1 - a_2^*) \)

as obtained by Khan and Abouammoh (2000).

This can also be stated as [Khan and Alzaid, 2004]

\[
E[\xi(X_{n-r+1:n}')] \bigg| X_{n-s+1:n} = x \bigg] = a_2^* \xi(x) + b_2^*
\]

if and only if

\[
\bar{F}(x) = [a \xi(x) + b]^c
\]

And for lower record \( (m = -1) \) values

\[
E[\xi(X'(s,n,-1,k)) \bigg| X'(r,n,-1,k) = x \bigg] = a_s^* \xi(x) + b_s^* \bigg|_r
\]
where \( a_{s|r}^* = \left( \frac{c k}{1 + c k} \right)^{s-r} \) and \( b_{s|r}^* = \frac{b}{a} (1 - a_{s|r}^*) \).

**Theorem 2.2:** If for \( 1 \leq r < s \leq n \)

\[
E[\xi\{X'(r, n, m, k)\}| X'(s, n, m, k) = y] = g_{r|s}(y)
\]

then for two consecutive values \( s - 1 \) and \( s \),

\[
\frac{(m+1)f(y)[F(y)]^m}{1-[F(y)]^{m+1}} = \frac{g'_{r|s}(y)}{(s-1)[g_{r|s}(y) - g_{r|s-1}(y)]} = A(y), \quad m \neq -1
\]  

(2.11)

\[
-\frac{f(y)}{F(y) \log F(y)} = A(y) \quad m = -1
\]  

(2.12)

and

\[
F(x) = \left[ 1 - \exp \left( -\int_{x}^{y} A(y) dy \right) \right]^{m+1}, \quad m \neq -1
\]  

(2.13)

\[
F(x) = \exp \left[ e^{-\int_{p}^{x} A(y) dy} \right], \quad m = -1, x \geq p
\]  

(2.14)

where \( g(.) \) is a finite and differentiable function of \( y \) and

\[
\log F(p) = 1
\]

**Proof:** We have

\[
E[\xi\{X'(r, n, m, k)\}| X'(s, n, m, k) = y] = g_{r|s}(y)
\]

\[
\frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!} \int_{y}^{\beta} \xi(x)[F(x)]^m[1-(F(x))^{m+1}]^{r-1} \]

\[
\times [(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-1} f(x) dx
\]

\[
= g_{r|s}[1-(F(y))^{m+1}]^{s-1}
\]

Differentiating both sides with respect to \( y \), we have
\[ \frac{(s-1)! (m+1)^2 [F(y)]^m f(y)}{(r-1)! (s-r-2)!} \times \int_y^{\beta} \xi(x)[F(x)]^m[1-(F(x))^{m+1}]^{r-1}[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-2} f(x)dx \]

\[ = g_{r|s}'(y)[1-(F(y))^{m+1}]^{s-1} \]

\[ = (s-1)(m+1)[1-(F(y))^{m+1}]^{s-2} [F(y)]^m f(y) g_{r|s}(y) \]

implying

\[ -(s-1)! (m+1) [F(y)]^m f(y) g_{r|s}(y) \]

\[ = \frac{g_{r|s}'(y)[1-(F(y))^{m+1}]^{s-1}}{[1-(F(y))^{m+1}]^{s-2}} \]

\[ = (s-1)(m+1)[1-(F(y))^{m+1}]^{s-2} [F(y)]^m f(y) g_{r|s}(y) \]

After simplification we get (2.11). And (2.12) is obtained by taking limit \( m \to -1 \) in the LHS of (2.11). (2.13) is obtained by integrating (2.11) with respect to \( y \).

To prove (2.14), we note that

\[ \log F(x), \alpha < x < \beta \] is a non-decreasing function in \((-\infty, 0)\), therefore there exists a \( p, \alpha < p < \beta \), such that

\[ \log F(p) = 1 \]

Now at \( m = -1 \),

\[ -\frac{f(y)}{F(y) \log F(y)} = A(y) \]

and

\[ \int_p^\alpha -\frac{f(y)}{F(y) \log F(y)} dy = -\log[\log F(x)] = \int_p^\alpha A(y) dy \]
and hence the result.

**Corollary 2.2:**
\[
E[\{X'(r,n,m,k)\} \mid X'(s,n,m,k) = y] = a^*_{r|s} y + b^*_{r|s} = g_{r|s}(y) \tag{2.15}
\]
if and only if
\[
1 - [F(x)]^{m+1} = [ax + b]^c, \quad \alpha < x < \beta, \quad m \neq -1 \tag{2.16}
\]
where \(a, b\) and \(c\) are such that \(F(\alpha) = 0\) and \(F(\beta) = 1\) and
\[
a^*_{r|s} = \prod_{j=1}^{s-r} \frac{c(s-j)}{1 + c(s-j)}, \quad b^*_{r|s} = -\frac{b}{a}(1 - a^*_{r|s})
\]
and
\[
\log F(x) = [ax + b]^c \text{ at } m = -1
\]

**Proof:** We have
\[
E[\{X'(r,n,m,k)\} \mid X'(s,n,m,k) = y]
\]
\[
= \frac{(s-1)!(m+1)}{(r-1)!(s-r-1)!} \left[ \frac{1 - [F(x)]^{m+1}}{1 - [F(y)]^{m+1}} \right]^{r-1} \left[ \frac{1 - [F(x)]^{m+1}}{1 - [F(y)]^{m+1}} \right]^{s-r-1}
\]
\[
\times \frac{F^m(x)}{1 - F^{m+1}(y)} f(x) \, dx
\]

Let \(u = \frac{1 - [F(x)]^{m+1}}{1 - [F(y)]^{m+1}} = \left[ \frac{ax + b}{ay + b} \right]^c\)

then
\[
x = \frac{(ay + b)u^{1/c} - b}{a}
\]

Therefore,
\[ E[X'(r,n,m,k) \mid X'(s,n,m,k) = y] \]
\[ = \frac{(s-1)!}{(r-1)!(s-r-1)!} \int_b^1 (ay+b)u^{r/c} - b u^{r-1}(1-u)^{s-r-1} du \]
\[ = \frac{(s-1)!}{(r-1)!(s-r-1)!} \left[ B\left(\frac{1}{c} + r, s - r\right) y + \frac{b}{a} B\left(\frac{1}{c} + r, s - 1\right) - \frac{b}{a} B(r, s - r) \right] \]
and hence the result.

To prove that
\[ g_{r,s}^*(y) = a_{r,s}^*y + b_{r,s}^* \]
implies \[ 1 - [F(x)]^{m+1} = [ax + b]^c \]
we use Theorem 2.2.

Now
\[ A(y) = \frac{g_{r,s}^*(y)}{(s-1)[g_{r,s}^*(y) - g_{r,s-1}^*(y)]} = \frac{a_{r,s}^*}{(s-1)[a_{r,s}^* - a_{r,s-1}^*]} ay + b \]

Now
\[ a_{r,s-1}^* = \prod_{j=1}^{s-r-1} \frac{c(s-1-j)}{1+c(s-1-j)} = \prod_{j=1}^{s-r} \frac{c(s-j)}{1+c(s-j)} = \frac{c(s-1)+1}{c(s-1)} \prod_{j=1}^{s-r} \frac{c(s-j)}{c(s-1)} \]
\[ = a_{r,s}^* \]
implying
\[ A(y) = -\frac{ac}{ay + b} \]
and thus
\[ [F(x)]^{m+1} = \left[ 1 - \exp\left( \int_a^x \frac{ac}{ay + b} \, dy \right) \right], \quad m \neq -1 \]

and hence the result as \([a \alpha + b]^c = 1\).

For \(m = -1\) (lower records)

\[
F(x) = \exp\left\{ \int_p^x \frac{ac}{ax + b} \, dy \right\} = \exp[ax + b]^c
\]

and hence the result.

**Remark 2.3:** As noted in Remark 2.2, for

\[
F(x) = [a \xi(x) + b]^c
\]

\[
E[\xi(X'_r:n) \mid X'_s:n = y] = a_{r|s}^* \xi(y) + b_{r|s}^* = E[\xi(X_{n-r+1:n}) \mid X_{n-s+1:n} = y]
\]

where \(a_{r|s}^* = \prod_{j=1}^{s-r} \frac{c(s-j)}{c(s-j)+1}\) and \(b_{r|s}^* = -\frac{b}{a} (1 - a_{r|s}^*)\)

Therefore replacing \(n - r + 1\) by \(s\), we get

\[
E[\xi(X_s:n) \mid X_r:n = x] = a_1^* \xi(x) + b_1^*
\]

if and only if

\[
F(x) = [a \xi(x) + b]^c
\]

where \(a_1^* = \prod_{j=1}^{s-r} \frac{c(n-r+1-j)}{c(n-r+1-j)+1} = \prod_{l=r+1}^{s} \frac{c(n-l+1)}{c(n-l+1)+1}\)

and \(b_1^* = -\frac{b}{a} (1 - a_1^*)\)

as obtained by Khan and Abouammoh (2000).

Further,
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if and only if

\[ F(y) = [a \xi(y) + b]^c \]

as given by Khan and Alzaid (2004).

3. Characterization of distributions when \( \gamma_i \neq \gamma_j \)

The conditional distribution of \( X(s, n, m, k) \) given \( X(r, n, m, k) = x \),

\( 1 \leq r < s \leq n \), is

\[
 f_{s \mid r}(y \mid x) = \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \frac{[F(y)]^{y_i-1}}{[F(x)]^{y_i}} f(y), \quad x > y
\]

where,

\[
 C_r = \prod_{i=1}^{r+1} \gamma_i = \gamma_{r+1} C_{r-1}
\]

and \( a_i^{(r+1)}(s) = (\gamma_{r+1} - \gamma_i) a_i^{(r)}(s) \)

**Theorem 3.1:** Let \( \xi(x) \) be a monotonic and continuous function of \( x \). If for \( 1 \leq r < s \leq n \),

\[
 E[\xi(X'(s, n, m, k)) \mid X'(r, n, m, k) = x] = g_{s \mid r}(x), \quad \text{then for two consecutive values } r \text{ and } r + 1,
\]

\[
 F(x) = e^{-\int_0^x A(t) \, dt}
\]

where

\[
 A(t) = \frac{g'_{s \mid r}(t)}{\gamma_{r+1} [g_{s \mid r+1}(t) - g_{s \mid r}(t)]}
\]

and \( g(.) \) is a finite and differentiable function of \( x \).
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Proof: \( g_{s|r+1}(x) = \frac{C_{s-1}}{C_r} \sum_{i=r+2}^{s} a_i^{(r+1)}(s) \int_{x}^{\xi} \frac{\xi(y)[F(y)]^{\gamma_i-1} f(y)}{[F(x)]^{\gamma_i}} dy \)

\[ = \frac{C_{s-1}}{\gamma_{r+1}} \sum_{i=r+1}^{s} (\gamma_{r+1} - \gamma_i) a_i^{(r)}(s) \int_{x}^{\xi} \frac{\xi(y)[F(y)]^{\gamma_i-1} f(y)}{[F(x)]^{\gamma_i}} dy \]

\[ = g_{s|r}(x) - \frac{C_{s-1}}{C_r} \sum_{i=r+1}^{s} \gamma_i a_i^{(r)}(s) \int_{x}^{\xi} \frac{\xi(y)[F(y)]^{\gamma_i-1} f(y)}{[F(x)]^{\gamma_i}} dy \]  (3.1)

Now

\[ g_{s|r}(x) = \frac{C_{s-1}}{C_r} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_{x}^{\xi} \frac{\xi(y)[F(y)]^{\gamma_i-1} f(y)}{[F(x)]^{\gamma_i}} dy \]  (3.2)

Differentiating (3.2) both sides with respect to \( x \), we get

\[ g'_{s|r}(x) = \frac{C_{s-1}}{C_r} \sum_{i=r+1}^{s} a_i^{(r)}(s) \]

\[ \times \left[ \frac{\xi(x)[F(x)]^{\gamma_i-1} f(x)}{[F(x)]^{\gamma_i}} - \gamma_i \int_{x}^{\xi} \frac{\xi(y)[F(y)]^{\gamma_i-1} f(y)}{[F(x)]^{\gamma_i+1}} f(x) f(y) dy \right] \]

\[ = \frac{C_{s-1} \xi(x)f(x)}{C_r} \sum_{i=r+1}^{s} a_i^{(r)}(s) - \frac{f(x) \gamma_{r+1} C_{s-1}}{C_r} \sum_{i=r+1}^{s} \gamma_i a_i^{(r)}(s) \]

\[ \times \int_{x}^{\xi} \frac{\xi(y)[F(y)]^{\gamma_i-1}}{[F(x)]^{\gamma_i}} f(y) dy \]

in view of result \( \sum_{i=r+1}^{s} a_i^{(r)}(s) = 0, 1 \leq r < s \)  (Khan and Alzaid, 2004) and

(3.1), we have

\[ g_{s|r+1} = g_{s|r} + \frac{1}{\gamma_{r+1}} \frac{F(x)}{f(x)} g'_{s|r} \]

Therefore,
\[ \frac{f(x)}{F(x)} = \frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x) - g_{s|r}(x)]} = A(x) \]

and hence the Theorem.

Comparing Theorems 2.1 and 3.1, we notice that both the results are identical.

Thus proceeding as in Corollary 2.1 and Remark 2.1 we can show that

\[ F(x) = [a_1 x + b]^c \]

if and only if

\[ g_{s|r}(x) = a_{s|r}^* x + b_{s|r}^* \]  \hspace{1cm} (3.3)

and

\[ F(x) = [a_2 \xi(x) + b]^c \]

if and only if

\[ g_{s|r}(x) = a_{s|r}^* \xi(x) + b_{s|r}^* \]  \hspace{1cm} (3.4)

where \( a_{s|r}^* \) and \( b_{s|r}^* \) are as given in Corollary 2.1.

**Theorem 3.2:** Let for \( 1 \leq r < s \leq n \)

\[ g_{r|s}(y) = E[\xi \{X'(r, n, \tilde{m}, k)\} | X'(s, n, \tilde{m}, k) = y] \]  \hspace{1cm} (3.5)

then for two consecutive values \( s - 1 \) and \( s \),

\[ \frac{\gamma_s f(y)}{F(y)} \frac{B'_s(y)}{B_s(y)} = \frac{g'_{r|s}(y)}{[g_{r|s}(y) - g_{r|s-1}(y)]} = D(y) \]

and

\[ \sum_{i=1}^s a_i(s)[F(x)]^{\gamma_i - \gamma_s} = a_s(s)e^{-\frac{F}{\alpha} D(y) dy} \]
where \( g(.) \) is a finite and differentiable function of \( y \) and

\[
\sum_{i=1}^{s} a_i(s) = 0
\]

\[
B_s(y) = \sum_{i=1}^{s} a_i(s)[F(y)]^{\gamma_i}
\]

and

\[
a_s(s) = \frac{1}{\prod_{j=1, j \neq s}^{s-1} (\gamma_j - \gamma_s)} = \frac{1}{\prod_{j=1}^{s-1} (\gamma_j - \gamma_s)}
\]

(3.6)

**Proof:** We have

\[
f_{r|s}(x|y) = f[(X'(r,n,\bar{m},k)|X'(s,n,\bar{m},k) = y]
\]

\[
= \frac{\sum_{i=r+1}^{s} a_i^{(r)}(s) \left( \frac{F(y)}{F(x)} \right)^{\gamma_i} \left\{ \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} \right\} f(x)}{\left\{ \sum_{i=1}^{s} a_i(s)[F(y)]^{\gamma_i} \right\} F(x)}
\]

Therefore,

\[
\sum_{i=r+1}^{s} a_i^{(r)}(s) \int_{y}^{\theta} \xi(x) \left( \frac{F(y)}{F(x)} \right)^{\gamma_i} \left\{ \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} \right\} \frac{f(x)}{F(x)} dx
\]

\[
= g_{r|s}(y) \left\{ \sum_{i=1}^{s} a_i(s)[F(y)]^{\gamma_i} \right\}
\]

or

\[
\sum_{i=r+1}^{s} a_i^{(r)}(s) \int_{y}^{\theta} \xi(x) A(x,y) dx = g_{r|s}(y) B_s(y)
\]

(3.7)

where
\begin{equation}
A(x, y) = \left( \frac{F(y)}{F(x)} \right)^{\gamma_i} \left\{ \sum_{i=1}^{r} a_i(r)(F(x))^{\gamma_i} \right\} \frac{f(x)}{F(x)}
\end{equation}

and \[ \frac{\partial A(x, y)}{\partial y} = \gamma_i \frac{f(y)}{F(y)} A(x, y) \]

Differentiating (3.7) both sides with respect to \( y \), we get

\[ - \sum_{i=r+1}^{s} a_i(r)(s) \xi(y) A(y, y) + \frac{f(y)}{F(y)} \sum_{i=r+1}^{s} a_i(r)(s) \gamma_i \int_{y}^{\beta} \xi(x) A(x, y) \, dx \]

\[ = g_{r|s}(y) B_{s}(y) + g_{r|s}(y) B'_{s}(y). \]

Thus we have

\[ g_{r|s}(y) B_{s}(y) + g_{r|s}(y) B'_{s}(y) \]

\[ = \frac{f(y)}{F(y)} \sum_{i=r+1}^{s} a_i(r)(s) \gamma_i \int_{y}^{\beta} \xi(x) A(x, y) \, dx \]

\[ \text{as} \quad \sum_{i=r+1}^{s} a_i(r)(s) = 0. \]

Now

\[ \sum_{i=r+1}^{s-1} a_i(r)(s-1) \int_{y}^{\beta} \xi(x) A(x, y) \, dx = g_{r|s-1}(y) B_{s-1}(y) \]

and since

\[ a_i(r)(s-1) = (\gamma_s - \gamma_i) a_i(r)(s) \quad \text{and} \quad a_i(s-1) = (\gamma_s - \gamma_i) a_i(s) \]

Therefore \( \text{LHS} \) of (3.10) is

\[ \gamma_s g_{r|s}(y) B_{s}(y) - \sum_{i=r+1}^{s} a_i(r)(s) \gamma_i \int_{y}^{\beta} \xi(x) A(x, y) \, dx \]

\[ = g_{r|s-1}(y) B_{s-1}(y) \]

(3.11)
Thus in view of (3.9) and (3.11)

\[ g'_{r|s}(y)B_{s}(y) + g_{r|s}(y)B'_{s}(y) \]

\[ = \frac{f(y)}{F(y)}[\gamma_{s} g_{r|s}(y)B_{s}(y) - g_{r|s-1}(y)B_{s-1}(y)] \]

But

\[ B_{s-1}(y) = \sum_{i=1}^{s-1} a_{i}(s-1)[F(y)]^{\gamma_{i}} \]

\[ = \sum_{i=1}^{s} (\gamma_{s} - \gamma_{i})a_{i}(s)[F(y)]^{\gamma_{i}} = \gamma_{s}B_{s}(y) - \frac{F(y)}{f(y)}B'_{s}(y) \]

Therefore

\[ g'_{r|s}(y)B_{s}(y) + g_{r|s}(y)B'_{s}(y) \]

\[ = \frac{f(y)}{F(y)}[\gamma_{s} g_{r|s}(y)B_{s}(y) - \gamma_{s}g_{r|s-1}(y)B_{s}(y) + g_{r|s-1}B'_{s}(y) \]

which on simplification yields

\[ \frac{\gamma_{s}f(y)}{F(y)} - \frac{B'_{s}(y)}{B_{s}(y)} = \frac{g'_{r|s}(y)}{[g_{r|s}(y) - g_{r|s-1}(y)]} = D(y) \] (3.12)

Integrating (3.12) w.r.t. \( y \) from \( \alpha \) to \( x \), we get

\[ \log[F(y)]^{\gamma} - \log B_{s}(y) = -\log \left( \sum_{i=1}^{s} a_{i}(s)[F(y)]^{\gamma_{i} - \gamma_{s}} \right)_{\alpha}^{x} = \int_{\alpha}^{x} D(y) dy \]

\[ \log \left( \sum_{i=1}^{s} a_{i}(s)[F(x)]^{\gamma_{i} - \gamma_{s}} \right) - \log a_{s}(s) = -\int_{\alpha}^{x} D(y) dy \]

\[ \sum_{i=1}^{s} a_{i}(s)[F(x)]^{\gamma_{i} - \gamma_{s}} = a_{s}(s)e^{-\int_{\alpha}^{x} D(y) dy} \] (3.13)

proving the result.
Remark 3.1: It may be noted that for \( \gamma_i \neq \gamma_j \) but \( m_i = m_j = m \),

\[
a_t(r) = \frac{1}{(m+1)^{r-1}} (-1)^{r-t} \frac{1}{(i-1)! (r-i)!}
\]

and

\[
a_t^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}} (-1)^{s-t} \frac{1}{(i-r-1)! (s-i)!}
\]

and therefore the pdf of \( X'(r,n,\tilde{m},k) \) reduces to the pdf of \( X'(r,n,m,k) \) and the joint pdf of \( X'(r,n,\tilde{m},k) \) and \( X'(s,n,\tilde{m},k) \) reduces to the joint pdf of \( X'(r,n,m,k) \) and \( X'(s,n,m,k) \).

Further at \( m_i = m_j = m \),

\[
a_s(s) = \frac{1}{(m+1)^{s-1}} \frac{1}{(s-1)!}
\]

and

\[
\sum_{i=1}^{s} a_i(s)[F(x)]^{\gamma_i - \gamma_s} = \sum_{i=1}^{s} \frac{1}{(m+1)^{s-1}} (-1)^{s-t} \frac{1}{(i-1)! (s-i)!} [(F(x))^{m+1}]^{s-t}
\]

\[= \frac{1}{(m+1)^{s-1}} \sum_{i=1}^{s} (-1)^{s-t} \binom{s-1}{s-i} [(F(x))^{m+1}]^{s-t}
\]

\[= \frac{1}{(m+1)^{s-1}} \frac{1}{(s-1)!} [1 - (F(x))^{m+1}]^{s-1}
\]

implying that

\[
[1 - (F(x))^{m+1}] = \exp \left( -\frac{1}{s-1} \int_{x}^{y} D(y) dy \right)
\]

as obtained in Theorem 2.2.